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$\mathcal{I}$-IDEALS GENERATED BY A SET IN IS-ALGEBRAS

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ABSTRACT. We consider a generalization of [1, Theorem 2.5]. We give a description of the element of $\langle A \cup B\rangle^l_{\mathcal{I}}$ (resp. $\langle A \cup B\rangle^r_{\mathcal{I}}$), where A and B are left (resp. right) $\mathcal{I}$-ideals of an **IS**-algebra X. For a nonempty left (resp. right) stable subset A of an **IS**-algebra, we obtain a condition for $\langle A\rangle^l_{\mathcal{I}}$ (resp. $\langle A\rangle^r_{\mathcal{I}}$) to be closed. We give a characterization of a closed $\mathcal{I}$-ideal in an **IS**-algebra, and show that, in a finite **IS**-algebra, every $\mathcal{I}$-ideal is closed.

1. Introduction

The notion of BCK-algebras was proposed by Y. Imai and K. Iséki in 1966. In the same year, K. Iséki [2] introduced the notion of a BCI-algebra which is a generalization of a BCK-algebra. In [5], Y. B. Jun et al. established the notion of an **IS**-algebra which is a generalization of the ring (see also [4]). For the general development of BCK/BCI/**IS**-algebras, the ideal theory plays an important role.

In this paper, we consider a generalization of [1, Theorem 2.5]. We give a description of the element of $\langle A \cup B\rangle^l_{\mathcal{I}}$ (resp. $\langle A \cup B\rangle^r_{\mathcal{I}}$), where A and B are left (resp. right) $\mathcal{I}$-ideals of an **IS**-algebra X. For a nonempty left (resp. right) stable subset A of an **IS**-algebra, we obtain a condition for $\langle A\rangle^l_{\mathcal{I}}$ (resp. $\langle A\rangle^r_{\mathcal{I}}$) to be closed. We give a characterization of a closed $\mathcal{I}$-ideal in an **IS**-algebra, and show that, in a finite **IS**-algebra, every $\mathcal{I}$-ideal is closed.

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2. Preliminaries

By a *BCI-algebra* we mean an algebra $(X, *, 0)$ of type (2,0) satisfying the following conditions for all $x, y, z \in X$:

(I) $((x * y) * (x * z)) * (z * y) = 0$,

(II) $(x * (x * y)) * y = 0$,

(III) $x * x = 0$,

(IV) $x * y = 0$ and $y * x = 0$ imply $x = y$.

In any BCI-algebra X one can define a partial order "$\leq$" by putting $x \leq y$ if and only if $x * y = 0$.

A BCI-algebra X has the following properties for any $x, y, z \in X$:

(1) $x * 0 = x$,

(2) $(x * y) * z = (x * z) * y$,

(3) $x \leq y$ implies that $x * z \leq y * z$ and $z * y \leq z * x$.

A nonempty subset I of a BCI-algebra X is called an *ideal* of X if it satisfies

(i) $0 \in I$,

(ii) $x * y \in I$ and $y \in I$ imply $x \in I$ for all $x, y \in X$.

Any ideal I has the property: $y \in I$ and $x \leq y$ imply $x \in I$.

In general, an ideal I of a BCI-algebra X need not be a subalgebra. If I is also a subalgebra of a BCI-algebra X, we say that I is a *closed ideal*, equivalently, an ideal I is closed if and only if $0 * x \in I$ whenever $x \in I$.

DEFINITION 1 (Jun et al. [5]). An **IS**-*algebra* is a non-empty set X with two binary operations "$*$" and "$\cdot$" and constant 0 satisfying the axioms

(V) $I(X) := (X, *, 0)$ is a BCI-algebra.

(VI) $S(X) := (X, \cdot)$ is a semigroup.

(VII) the operation "$\cdot$" is distributive (on both sides) over the operation "$*$", that is, $x \cdot (y * z) = (x \cdot y) * (x \cdot z)$ and $(x * y) \cdot z = (x \cdot z) * (y \cdot z)$ for all $x, y, z \in X$.

If an **IS**-algebra X contains an element 1_X such that $1_X \cdot x = x \cdot 1_X = x$ for all $x \in X$, then X is called an **IM**-*algebra*, and we call 1_X the *multiplicative identity*. If every non-zero element of an **IM**-algebra X has a multiplicative inverse, then X is called an **IG**-*algebra*. We shall write the multiplication $x \cdot y$ by xy, for convenience.

EXAMPLE 1 (Jun et al. [4]). Let $\mathbb{Z}$ be the set of all integers. Then $(\mathbb{Z}, -, 0)$ is a BCI-algebra and that $\mathbb{Z} = (\mathbb{Z}, -, \cdot, 0, 1)$ is an **IS**-algebra.

EXAMPLE 2 (Jun et al. [5]). Let $X = \{0, a, b, c\}$. Define $*$-operation and multiplication "·" by the following tables

$*$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	a	b	c
c	0	0	0	0

Then, by routine calculations, we can see that X is an **IS**-algebra.

LEMMA 1 (Jun et al. [4, Proposition 1]). *Let X be an* **IS**-*algebra. Then we have*

(i) $0x = x0 = 0$,

(ii) $x \leq y$ *implies that* $xz \leq yz$ *and* $zx \leq zy$,

for all $x, y, z \in X$.

DEFINITION 2 (Ahn et al. [1]). A non-empty subset A of a semigroup $S(X) := (X, \cdot)$ is said to be *left* (resp. *right*) *stable* if $xa \in A$ (resp. $ax \in A$) whenever $x \in S(X)$ and $a \in A$. Both left and right stable is *two-sided stable* or simply *stable*.

It is clear that if A and B are left (resp. right) stable subsets of $S(X)$, then so is $A \cup B$.

DEFINITION 3 (Jun et al. [5]). A non-empty subset A of an **IS**-algebra X is called a *left* (resp. *right*) *$\mathcal{I}$-ideal* of X if

(i) A is a left (resp. right) stable subset of $S(X)$,

(ii) for any $x, y \in I(X)$, $x * y \in A$ and $y \in A$ imply that $x \in A$.

Both a left and right $\mathcal{I}$-ideal is called a *two-sided $\mathcal{I}$-ideal* or simply an *$\mathcal{I}$-ideal*. If A is a left (resp. right) $\mathcal{I}$-ideal of an **IS**-algebra X, then $0 \in A$. Thus A is an ideal of $I(X)$.

3. Main results

For convenience, in BCI-algebras, we denote

$$(\dots((x * y_1) * y_2) * \dots) * y_n = x * \prod_{i=1}^{n} y_i.$$

Especially, we write $x * \prod_{i=1}^{n} y_i = x * y^n$ if $y_i = y$ for $i = 1, ..., n$. If A is a subset of a BCI-algebra X, let $\langle A \rangle$ denote the ideal of X generated by A, that is, the smallest ideal of X containing A. If there exists $x \in A$ satisfying $x \geq 0$, then $\langle A \rangle$ can be described as the set of all $y \in X$ such that $y * \prod_{i=1}^{n} a_i = 0$ for some $a_1, a_2, ..., a_n \in A$.

In [1], S. S. Ahn et al. established the $\mathcal{I}$-ideal generated by a nonempty stable subset of a "$\cdot$"-commutative **IG**-algebra X.

DEFINITION 4 (Ahn et al. [1, Definition 2.3]). Let X be an **IS**-algebra. For any subset A of X, the intersection of all left (resp. right) $\mathcal{I}$-ideals of X containing A is said to be the *left* (resp. *right*) *$\mathcal{I}$-ideal generated by* A, and is denoted by $\langle A \rangle_{\mathcal{I}}^{l}$ (resp. $\langle A \rangle_{\mathcal{I}}^{r}$). Both a left and right $\mathcal{I}$-ideal generated by A is called the *$\mathcal{I}$-ideal generated by* A, and is denoted by $\langle A \rangle_{\mathcal{I}}$.

It is clear that if A and B are subsets of an **IS**-algebra X satisfying $A \subseteq B$, then $\langle A \rangle_{\mathcal{I}}^{l} \subseteq \langle B \rangle_{\mathcal{I}}^{l}$ (resp. $\langle A \rangle_{\mathcal{I}}^{r} \subseteq \langle B \rangle_{\mathcal{I}}^{r}$), and if A is a left (resp. right) $\mathcal{I}$-ideal of X, then $\langle A \rangle_{\mathcal{I}}^{l}$ (resp. $\langle A \rangle_{\mathcal{I}}^{r}$) $= A$.

PROPOSITION 1 (Ahn et al. [1, Theorem 2.5]). *Let X be a "$\cdot$"-commutative* **IG***-algebra and A a nonempty stable subset of $S(X)$. Then*

$$\langle A \rangle_{\mathcal{I}} = \{x \in X \,|\, \exists a_1, a_2, ..., a_n \in A \text{ and } \exists r_1, r_2, ..., r_n \in X \setminus \{0\} \text{ such that } r_n(\dots(r_2(r_1(x * a_1) * a_2) * \dots) * a_n) = 0\}.$$

In Proposition 1, the condition of X is too strong. So we feel the need to weaken the condition of X. In the following theorem, we will do it, and the result is a generalization of Proposition 1.

THEOREM 1. *Let X be an* **IS***-algebra and A a nonempty left (resp. right) stable subset of $S(X)$. Then*

$$\langle A\rangle_{\mathcal{I}}^{l} \text{ (resp. } \langle A\rangle_{\mathcal{I}}^{r}) = \{x \in X | x * \prod_{i=1}^{n} a_i = 0 \text{ for some } a_1, a_2, ..., a_n \in A\}.$$

Proof. Let A be a nonempty left (resp. right) stable subset of $S(X)$ and denote

$$B := \{x \in X | x * \prod_{i=1}^{n} a_i = 0 \text{ for some } a_1, a_2, ..., a_n \in A\}.$$

We first claim that B is a left (resp. right) stable subset of $S(X)$. Let $x \in S(X)$ and $b \in B$. Then there exist $a_1, a_2, ..., a_n \in A$ such that $b * \prod_{i=1}^{n} a_i = 0$. It follows that

$$xb * \prod_{i=1}^{n} (xa_i) = x(b * \prod_{i=1}^{n} a_i) = x0 = 0$$

$$(\text{resp. } bx * \prod_{i=1}^{n} (a_i x) = (b * \prod_{i=1}^{n} a_i)x = 0x = 0).$$

Since A is left (resp. right) stable, $xa_i \in A$ (resp. $a_i x \in A$) for $i = 1, 2, ..., n$. Hence $xb \in B$ (resp. $bx \in B$), which shows that B is left (resp. right) stable. Now let $x * y \in B$ and $y \in B$. Then there exist $a_1, ..., a_n, b_1, ...b_m \in A$ such that

$$(x * y) * \prod_{i=1}^{n} a_i = 0 \text{ and } y * \prod_{j=1}^{m} b_j = 0.$$

Using (2) n-times repeatedly in the first equation above, then

$$(x * \prod_{i=1}^{n} a_i) * y = 0 \text{ or } x * \prod_{i=1}^{n} a_i \leq y.$$

It follows from (3) that $(x * \prod_{i=1}^{n} a_i) * \prod_{j=1}^{m} b_j \leq y * \prod_{j=1}^{m} b_j = 0$ so that $(x * \prod_{i=1}^{n} a_i) * \prod_{j=1}^{m} b_j = 0$. Hence $x \in B$ and B is a left (resp. right)

$\mathcal{I}$-ideal of X. Obviously, $A \subseteq B$. Let C be a left (resp. right) $\mathcal{I}$-ideal containing A. To show $B \subset C$, let x be an element of B. Then $x * \prod_{i=1}^{n} a_i = 0$ for some $a_1, ..., a_n \in A$. Since $0 \in C$, we have $x * \prod_{i=1}^{n} a_i \in C$. Since $a_1, ..., a_n \in C$, it follows by using Definition 3(ii) repeatedly that $x \in C$. Thus $B \subset C$ and $B = \langle A \rangle_{\mathcal{I}}^{l}$ (resp. $\langle A \rangle_{\mathcal{I}}^{r}$), ending the proof. □

We know that, in the following example, the union of any left (resp. right) $\mathcal{I}$-ideals A and B may not be a left (resp. right) $\mathcal{I}$-ideal of an **IS**-algebra X.

EXAMPLE 3. Let $X = \{0, a, b, c\}$. Define $*$-operation and multiplication "$\cdot$" by the following tables

$*$	0	a	b	c
0	0	0	b	b
a	a	0	c	b
b	b	b	0	0
c	c	b	a	0

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
c	0	a	b	c

It is easy to prove that X is an **IS**-algebra, and $\{0, a\}$ and $\{0, b\}$ are $\mathcal{I}$-ideals of X, but $\{0, a\} \cup \{0, b\} = \{0, a, b\}$ is not an $\mathcal{I}$-ideal of X.

The following theorem gives a description of the element of $\langle A \cup B \rangle_{\mathcal{I}}^{l}$ (resp. $\langle A \cup B \rangle_{\mathcal{I}}^{r}$), where A and B are left (resp. right) $\mathcal{I}$-ideals of an **IS**-algebra X.

THEOREM 2. *Let A and B be left (resp. right) $\mathcal{I}$-ideals of an* **IS**-*algebra X. Then*

$$\langle A \cup B \rangle_{\mathcal{I}}^{l} \text{ (resp. } \langle A \cup B \rangle_{\mathcal{I}}^{r}) = \{x \in X \mid (x * a) * b = 0 \text{ for some } a \in A \text{ and } b \in B\}.$$

Proof. Denote

$$K := \{x \in X \mid (x * a) * b = 0 \text{ for some } a \in A \text{ and } b \in B\}.$$

Clearly, $K \subseteq \langle A \cup B \rangle_{\mathcal{I}}^{l}$ (resp. $\langle A \cup B \rangle_{\mathcal{I}}^{r}$). Let $x \in \langle A \cup B \rangle_{\mathcal{I}}^{l}$ (resp. $\langle A \cup B \rangle_{\mathcal{I}}^{r}$). Then, by Theorem 1, there exist $q_1, ..., q_n \in A \cup B$ such that $x * \prod_{i=1}^{n} q_i = 0$. If $q_i \in A$ (resp. B) for all $i = 1, ..., n$, then $x \in A$ (resp. B), and so $x \in K$ since $(x * x) * 0 = 0$ (resp. $(x * 0) * x = 0$). If some of $q_1, ..., q_n$ belong to A and others belong to B, we may assume that $q_1, ..., q_k \in A$ and $q_{k+1}, ..., q_n \in B$ for $1 \leq k < n$, without loss of generality. Let $b = x * \prod_{i=1}^{k} q_i$. Then

$$b * \prod_{j=k+1}^{n} q_j = (x * \prod_{i=1}^{k} q_i) * \prod_{j=k+1}^{n} q_j = x * \prod_{i=1}^{n} q_i = 0,$$

and so $b \in B$. Now let $a = x * (x * \prod_{i=1}^{k} q_i)$, i.e., $a = x * b$. Then

$$\begin{aligned} a * \prod_{i=1}^{k} q_i &= (x * (x * \prod_{i=1}^{k} q_i)) * \prod_{i=1}^{k} q_i \\ &= (x * \prod_{i=1}^{k} q_i) * (x * \prod_{i=1}^{k} q_i) && \text{[by (2)]} \\ &= 0, && \text{[by (III)]} \end{aligned}$$

and hence $a \in A$ because A is a left (resp. right) $\mathcal{I}$-ideal and $q_i \in A$ for $i = 1, ..., k$. Noticing that $(x * a) * b = (x * b) * a = a * a = 0$, we have $x \in K$, which proves that $\langle A \cup B \rangle_{\mathcal{I}}^{l}$ (resp. $\langle A \cup B \rangle_{\mathcal{I}}^{r}$) $\subseteq K$. This completes the proof. □

DEFINITION 5. Let X be an **IS**-algebra. A left (resp. right) $\mathcal{I}$-ideal A of X is said to be *closed* if $0 * a \in A$ whenever $a \in A$.

We note that, in Example 2, $\{0, c\}$ is a closed $\mathcal{I}$-ideal of X.

We give a characterization of a closed $\mathcal{I}$-ideal in an **IS**-algebra.

THEOREM 3. *Let A be a nonempty subset of an* **IS**-*algebra X. Then A is a closed left (resp. right) $\mathcal{I}$-ideal of X if and only if*

(i) *A is a left (resp. right) stable subset of $S(X)$,*

(ii) *if $x * z \in A$, $y * z \in A$ and $z \in A$, then $x * y \in A$.*

Proof. Necessity is obvious. Let A be a nonempty subset of X satisfying (i) and (ii). Let $x * y \in A$ and $y \in A$. Note that every left (resp. right) stable set contains the zero element 0. Since $0 * 0, y * 0 \in A$, it follows from (ii) that $0*y \in A$. By using (ii) again, we get $x = x*0 \in A$. This proves that A is a closed left (resp. right) $\mathcal{I}$-ideal of X. □

For any subset A of an **IS**-algebra X, denote

$$L(A) := \{0 * (0 * a) | a \in A\}.$$

Clearly, $L(A) \subseteq \langle A \rangle_{\mathcal{I}}^{l}$ (resp. $\langle A \rangle_{\mathcal{I}}^{r}$).

THEOREM 4. *Let A be a nonempty left (resp. right) stable subset of an* **IS***-algebra X. If $0 * a \in L(A)$ for all $a \in A$, then $\langle A \rangle_{\mathcal{I}}^{l}$ (resp. $\langle A \rangle_{\mathcal{I}}^{r}$) is a closed left (resp. right) $\mathcal{I}$-ideal of X.*

Proof. Let $x \in \langle A \rangle_{\mathcal{I}}^{l}$ (resp. $\langle A \rangle_{\mathcal{I}}^{r}$). Then there exists $a_1, a_2, ..., a_n \in A$ such that $x * \prod_{i=1}^{n} a_i = 0$. It follows that

$$\begin{aligned}(0 * x) * \prod_{i=1}^{n}(0 * a_i) &= ((x * \prod_{i=1}^{n} a_i) * x) * \prod_{i=1}^{n}(0 * a_i) \\ &= ((x * x) * \prod_{i=1}^{n} a_i) * \prod_{i=1}^{n}(0 * a_i) \quad \text{[by (2)]} \\ &= (0 * \prod_{i=1}^{n} a_i) * \prod_{i=1}^{n}(0 * a_i) \quad \text{[by (III)]} \\ &= 0,\end{aligned}$$

so that $0 * x \in \langle A \rangle_{\mathcal{I}}^{l}$ (resp. $\langle A \rangle_{\mathcal{I}}^{r}$) since $0 * a_i \in L(A) \subseteq \langle A \rangle_{\mathcal{I}}^{l}$ (resp. $\langle A \rangle_{\mathcal{I}}^{r}$) for $i = 1, 2, ..., n$. Hence $\langle A \rangle_{\mathcal{I}}^{l}$ (resp. $\langle A \rangle_{\mathcal{I}}^{r}$) is a closed left (resp. right) $\mathcal{I}$-ideal of X. □

THEOREM 5. *In a finite* **IS***-algebra, every $\mathcal{I}$-ideal is closed.*

Proof. Let X be an **IS**-algebra with $|X| = n$, where n is a positive integer, and let A be an $\mathcal{I}$-ideal of X. It is sufficient to show that $0 * a \in A$ whenever $a \in A$. For any $a \in A$, consider the following elements

$$0 * a^0, 0 * a^1, 0 * a^2, \ldots, 0 * a^n, \text{ where } 0 * a^0 = 0.$$

Since $|X| = n$, there exist integers s and t with $0 \leq s < t \leq n$ such that $0 * a^s = 0 * a^t$. It follows that

$$\begin{aligned} 0 &= (0 * a^t) * (0 * a^s) && \text{[by (III)]} \\ &= ((0 * a^s) * a^{t-s}) * (0 * a^s) \\ &= ((0 * a^s) * (0 * a^s)) * a^{t-s} && \text{[by (2)]} \\ &= 0 * a^{t-s}, && \text{[by (III)]} \end{aligned}$$

so that $(0 * a) * a^{t-s-1} = 0 * a^{t-s} = 0 \in A$. Since A is an $\mathcal{I}$-ideal, by Definition 3(ii) we have $0 * a \in A$. This completes the proof. □

COROLLARY 1. *Let A be a nonempty subset of a finite* **IS**-*algebra X. Then $\langle A \rangle_{\mathcal{I}}$ is a closed $\mathcal{I}$-ideal of X.*

Proof. Straightforward. □

THEOREM 6. *Let A be a left (resp. right) $\mathcal{I}$-ideal of an* **IS**-*algebra X such that $\forall a \in A$, $\exists a' \in A$ and $x \in X$ satisfying $a = xa'$ (resp. $a = a'x$). Then A is closed.*

Proof. Let A be a left (resp. right) $\mathcal{I}$-ideal of an **IS**-algebra X such that $\forall a \in A$, $\exists a' \in A$ and $x \in X$ satisfying $a = xa'$ (resp. $a = a'x$). Then, for any $a \in A$, $0 * a = 0 * xa' = (0 * x)a' \in A$ (resp. $0 * a = 0 * a'x = a'(0 * x) \in A$), since A is left (resp. right) stable. Thus A is a closed left (resp. right) $\mathcal{I}$-ideal of X. □

COROLLARY 2. *In an* **IM**-*algebra, every $\mathcal{I}$-ideal is closed.*

Proof. Straightforward. □

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