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A STUDY ON SOLUTIONS OF A CLASS OF HIGHER ORDER ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT. The main objective of this paper is to study the boundedness of solutions of the differential equation

$$L_n x + F(t,x) = f(t)\ , \quad n \geq 2 \qquad (*)$$

Necessary and sufficient conditions for boundedness of all solutions of $(*)$ will be obtainded. The asymptotic behavior of solutions of $(*)$ will also be studied.

1. Introduction

Define the n-th order differential operator L_n by

$$L_n = \frac{1}{h_n(t)}\frac{d}{dt}\frac{1}{h_{n-1}(t)}\frac{d}{dt}\cdots\frac{1}{h_1(t)}\frac{d}{dt}\frac{\cdot}{h_0(t)}\ , \quad n \geq 2$$

where $h_i(t), 0 \leq i \leq n$, are positive continuous functions on $[a,\infty)$.

Consider the differential equation

$$L_n x + F(t,x) = f(t) \qquad (*)$$

where $F(t,x)$ and $f(t)$ are continuous functions on $[a,\infty)\times R$ and $[a,\infty)$ respectively.

We introduce the quasi-derivatives of a function $x(t)$ by

$$D^0(x;h_0)(t) = \frac{x(t)}{h_0(t)},$$

$$D^i(x;h_0,\cdots,h_i)(t) = \frac{1}{h_i(t)}\frac{d}{dt}D^{i-1}(x;h_0,\cdots,h_{i-1})(t)\ , \quad i \leq i \leq n.$$

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Then the operator L_n can be rewritten as

$$L_n = D^n(\cdot, h_0, \cdots, h_n).$$

The domain $\mathcal{D}(L_n)$ of L_n is defined to be the set of all continuous functions $x : [T_x, \infty) \to R$ such that $D^i(x; h_0, \cdots, h_i)(t), 0 \le i \le n$, exist and are continuous on $[T_x, \infty)$. By a solution of Eq.$(*)$ we mean a function $x \in \mathcal{D}(L_n)$ which satisties $(*)$ at every point of $[T_x, \infty)$.

2. Preliminaries

Let $g_i(t)$, $1 \le i \le N$, be continuous functions on $[a, \infty)$. Generalizing upon notation introduced by Willett[6], we put for $t, s \in [a, \infty)$

$$I_0 = 1\ , \quad I_i(t, s; g_1, \cdots, g_i) = \int_s^t g_1(r) I_{i-1}(r, s; g_2, \cdots, g_i) dr\ , \quad 1 \le i \le N$$

The following identities hold:

$$\begin{aligned} I_i(t, s; g_1, \cdots, g_i) &= (-1)^i I_i(s, t; g_i, \cdots, g_1) \\ I_i(t, s; g_1, \cdots, g_i) &= \int_s^t g_i(r) I_{i-1}(t, r; g_1, \cdots, g_{i-1}) dr \end{aligned} \tag{1}$$

Lemma 2.1. *Suppose that $g_i(t), 1 \le i \le N$, are positive on $[a, \infty)$. If $I_N(t, a; g_1, \cdots, g_N)$ is bounded on $[a, \infty)$, then so are the functions $I_i(t, a; g_1, \cdots, g_i)$ for $1 \le i \le N-1$.*

Proof. Let $b > a$ be fixed. Then, by (1), we have for $t \ge b$

$$\begin{aligned} I_N(t, a; g_1, \cdots, g_N) &= \int_a^t g_N(r) I_{N-1}(t, r; g_1, \cdots, g_{N-1}) dr \\ &\ge \int_a^b g_N(r) I_{N-1}(t, r; g_1, \cdots, g_{N-1}) dr \\ &\ge I_{N-1}(t, b, g_1, \cdots, g_{N-1}) \int_a^b g_N(r) dr \end{aligned}$$

which implies that $I_{N-1}(t, b; g_1, \cdots, g_{N-1})$ is bounded on $[b, \infty)$. Hence $I_{N-1}(t, a; g_1, \cdots, g_{N-1})$ is bounded on $[a, \infty)$. The boundedness of $I_i(t, a; g_1, \cdots, g_i)$ for $1 \le i \le N-2$ follows by induction. □

Lemma 2.2. *If $x \in \mathcal{D}(L_n)$, then we have for $t, s \in [T_x, \infty)$*

$$D^0(x; h_0)(t) = \sum_{i=0}^{n-1} D^i(x; h_0, \cdots, h_i)(s) I_i(t, s; h_1, \cdots, h_i)$$
$$+ \int_s^t I_{n-1}(t, r; h_1, \cdots, h_{n-1}) h_n(r) D^n(x; h_0, \cdots, h_n)(r) dr \qquad (2)$$

This lemma is a generalization of Taylor's formula with remainder. The last integral in (2) may be rewritten as

$$I_n(t, s; h_1, \cdots, h_{n-1}, h_n D^n(x; h_0, \cdots, h_n)) = \int_s^t h_1(r_1) \int_s^{r_1} h_2(r_2) \int_s^{r_2}$$
$$\cdots \int_s^{r_{n-1}} h_{n-1}(r_{n-1}) \int_s^{r_{n-1}} h_n(r_n) D^n(x; h_0, \cdots, h_n)(r_n) dr_n dr_{n-1} \cdots dr_2 dr_1$$

3. Main Results

Theorem 3.1. *Suppose that there is a function $\phi \in \mathcal{D}(L_n)$ satisfying $L_n\phi(t) = f(t)$ on $[a, \infty)$ and such that $D^0(\phi; h_0)(t)$ is bounded on $[a, \infty)$. Suppose moreover that there are a number $\gamma \in (0, 1]$ and a positive continuous function $q(t)$ on $[a, \infty)$ such that*

$$|F(t, x)| \leq q(t)|x|^\gamma \quad \textit{for } (t, x) \in [a, \infty) \times R \qquad (3)$$

If

$$\lim_{t \to \infty} I_n(t, a; h_1, \cdots, h_{n-1}, h_n h_0^\gamma q) < \infty \qquad (4)$$

then, for every solution $x(t)$ of (*), $D^0(x; h_0)(t)$ is bounded.

Proof. We observe that condition (3) ensures that every soultion of (*) can be continued to $t = \infty$. Let $x(t)$ be an arbitrary solution of (*) defined on $[\alpha, \infty)$. From Lemma 2.2 we have

$$D^0(x - \phi; h_0)(t) = \sum_{i=0}^{n-1} D^i(x - \phi; h_0, \cdots, h_i)(\alpha) I_i(t, \alpha; h_1, \cdots, h_i)$$
$$- I_n(t, \alpha; h_1, \cdots, h_{n-1}, h_n F(\cdot, x)) \quad \text{for } t \geq \alpha.$$

Condition (4) implies that $I_i(t,\alpha;h_1,\cdots,h_i), 1 \le i \le n-1$, are bounded on $[\alpha,\infty)$. On the other hand, using (3) we find

$$|I_n(t,\alpha;h_1,\cdots,h_{n-1},h_nF(\cdot,x))| \le I_n(t,\alpha;h_1,\cdots,h_{n-1},h_nh_0^{\gamma}q|D^0(x;h_0)|^{\gamma})\ ,\ t\ge\alpha.$$

Taking these facts into account, we see that the function $u(t) = \max_{\alpha\le s\le t}|D^0(x;h_0)(s)|$ satisfies the following inequality for $t \ge \alpha$;

$$\begin{aligned}
u(t) &\le c + I_n(t,\alpha;h_1,\cdots,h_{n-1},h_nh_0^{\gamma}q|D^0(x;p_0)|^{\gamma})\\
&= c + \int_\alpha^t h_1(s_1)\int_\alpha^{s_1} h_2(s_2)\int_\alpha^{s_2}\cdots\int_\alpha^{s_{n-2}} h_{n-1}(s_{n-1})\\
&\int_\alpha^{s_{n-1}} h_n(s_n)h_0^{\gamma}(s_n)q(s_n)|D^0(x;h_0)(s_n)|^{\gamma}ds_n\cdots ds_2ds_1\\
&\le c + \int_\alpha^t u^{\gamma}(s_1)h_1(s_1)\int_\alpha^{s_1} h_2(s_2)\int_\alpha^{s_2}\cdots\int_\alpha^{s_{n-2}} h_{n-1}(s_{n-1})\\
&\int_\alpha^{s_{n-1}} h_n(s_n)h_0^{\gamma}(s_n)q(s_n)ds_n\cdots ds_2ds_1\\
&= c + \int_\alpha^t u^{\gamma}(s_1)h_1(s_1)I_{n-1}(s_1,\alpha;h_2,\cdots,h_{n-1},h_nh_0^{\gamma}q)ds_1,
\end{aligned}$$

where c is a positive constant. Thus we have

$$u(t) \le c + \int_\alpha^t v(s)u^{\gamma}(s)ds\ , \quad t\ge\alpha \tag{5}$$

where $v(s) = h_1(c)I_{n-1}(s,\alpha;h_2,\cdots,h_{n-1},h_nh_0^{\gamma}q)$. Since $\gamma \le 1$ and $\int_\alpha^t v(s)ds = I_n(t,\alpha;h_1,\cdots,h_{n-1},h_nh_0^{\gamma}q)$ is bounded by (4), we are able to apply Bihari's lemma[2] to (5) to conclude that $u(t)$ is bounded on $[\alpha,\infty)$. If follows that $D^0(x;h_0)(t)$ is bounded, and the proof is complete. □

Theorem 3.2. *Suppose that, for every $t \ge a$, $F(t,x) \le 0$ for $x > 0$ and $F(t,x)$ is nonincreasing in x. Suppose that every solution of (*) can be continued to $t=\infty$. If $D^0(x;h_0)(t)$ is bounded for every solution x(t) of (*), then*

$$\lim_{t\to\infty} I_n(t,a;h_1,\cdots,h_{n-1},h_n|F(\cdot,ch_0)|) < \infty$$

for any constant $c > 0$.

Proof. There is a constant $M > 0$ such that $|D^0(\phi;h_0)(t)| \le M$ for $t \ge a$. Let $c > 0$ be given arbitrarily and let $x(t)$ be a solution of (*) satisfying the initial conditions

$$\begin{aligned}
&D^0(x-\phi,h_0)(a) \ge M + c,\\
&D^i(x-\phi,h_0,\cdots,h_i)(a) > 0\ , \quad \text{for } 1\le i\le n-1
\end{aligned} \tag{6}$$

By Lemma 2.2 we have

$$D^0(x-\phi,h_0)(t)=\sum_{i=0}^{n-1} D^i(x-\phi;h_0,\cdots,h_i)(a)I_i(t,a;h_1,\cdots,h_i) \\ +I_n(t,a;h_1,\cdots,h_{n-1},-h_nF(\cdot,x))\,,\quad t\geq a \qquad (7)$$

from which we see that $D^0(x-\phi;h_0)(t)>0$ and $D^1(x-\phi;h_0,h_1)(t)>0$ whenever $x(t)>0$. Therefore $D^0(x-\phi;h_0)(t)$ is positive and increasing on $[a,\infty)$, and so, with the use of the first condition of (6), we have

$$D^0(x,h_0)(t)\geq D^0(\phi,h_0)(t)+D^0(x-\phi;h_0)(a) \\ \geq -M+M+c=c\,,\quad t\geq a \qquad (8)$$

Since $D^0(x-\phi;h_0)(t)$ is bounded by hypothesis, it follows from (7), (8) and the second condition of (6) that

$$\lim_{t\to\infty} I_n(t,a;h_1,\cdots,h_{n-1},h_n|F(\cdot,ch_0)|)<\infty.$$

This completes the proof.

Theorem 3.3. *Suppose that there is a function $\phi\in\mathcal{D}(L_n)$ such that $L_n\phi(t)=f(t)$ on $[a,\infty)$ and $D^0(\phi;h_0)(t)$ tends to a finite limit as $t\to\infty$. If in addition to (4) $\lim_{t\to\infty} I_n(t,a;h_nh_0^\gamma q\cdot h_{n-1},\cdots,h_1)<\infty$ then, for every solution $x(t)$ of $(*)$, $D^0(x;h_0)(t)$ tends to a finite limit at $t\to\infty$. In particular, for every oscillatory solution $x(t)$ of $(*)$, $D^0(x;h_0)(t)$ tends to zero as $t\to\infty$.*

Proof. Let $x(t)$ be any solution of (*) defined on $[\alpha,\infty)$. It suffices to show that $D^0(x-\phi;h_0)(t)$ has a finite limit as $t\to\infty$. Suppose the contrary. Then, there are two constants ξ,η such that

$$\liminf_{t\to\infty} D^0(x-\phi;h_0)(t)<\xi<\eta<\limsup_{t\to\infty} D^0(x-\phi;h_0)(t) \qquad (9)$$

Let $T\geq\alpha$ be so large that

$$cI_n(t,T;h_nh_0^\gamma q\cdot h_{n-1},\cdots,h_1)<\frac{\eta-\xi}{2} \qquad (10)$$

for $t\geq T$, where $c=\sup_{t\geq T}|D^0(x;h_0)(t)|^\gamma$. Choose $A_0<B_0<A_1<B_1$ so that $T<A_0, D^0(x-\phi;h_0)(A_0)<\xi<\eta<D^0(x-\phi;h_0)(B_0)$ and $D^0(x-\phi;h_0)(A_1)<\xi<$

$\eta < D^0(x-\phi;h_0)(B_1)$. Let $[s_1,s_2]$ be the smallest interval containing B_0 such that $D^0(x-\phi;h_0)(s_1) = D^0(x-\phi;h_0)(s_2) = \xi$ and $\max\{D^0(x-\phi;h_0)(t); t \in [s_1,s_2]\} = D^0(x-\phi;h_0)(s') > \eta$. Clearly, $T < s_1 < s' < s_2$. Let $s_2 \le t_1 \le t_2 \le \cdots \le t_{n-1}$ be such that

$$D^i(x-\phi;h_0,\cdots,h_i)(t_i) = 0\,, \quad 1 \le i \le n-1 \tag{11}$$

such t_i exist, because $D^i(x-\phi;h_0,\cdots,h_i), 1 \le i \le n-1$, are oscillatory by (9). On repeated integration of $(*)$, we have in view of (11)

$$D^1(x-\phi;h_0,h_1)(t) = (-1)^n \int_t^{t_1} h_2(r_2) \int_{r_2}^{t_2} \cdots \int_{r_{n-2}}^{t_{n-2}} h_{n-1}(r_{n-1}) \int_{r_{n-1}}^{t_{n-1}} h_n(r_n) F(r_n, x(r_n)) dr_n \cdots d_{r2} \tag{12}$$

Multiplying both sides of (12) by $h_1(t)$ and integrating from s_1 to s', we obtain

$$\begin{aligned}\eta - \xi &< \int_{s_1}^{s'} h_1(r_1) \int_{r_1}^{t_1} h_2(r_2) \int_{r_2}^{t_2} \cdots \int_{r_{n-2}}^{t_{n-2}} h_{n-1}(r_{n-1}) \\ &\qquad \int_{r_{n-1}}^{t_{n-1}} h_n(r_n) q(r_n) h_0^\gamma(r_n) |D^0(x;h_0)(r_n)|^\gamma dr_n \cdots dr_1 \\ &\le \int_{s_1}^{t_{n-1}} h_1(r_1) \int_{r_1}^{t_{n-1}} h_2(r_2) \int_{r_2}^{t_{n-1}} \cdots \int_{r_{n-2}}^{t_{n-1}} h_{n-1}(r_{n-1}) \\ &\qquad \int_{r_{n-1}}^{t_{n-1}} h_n(r_n) q(r_n) h_0^\gamma(r_n) |D^0(x;h_0)(r_n)|^\gamma dr_n \cdots dr_1\end{aligned}$$

The last integral equals, in view of (1),

$$\begin{aligned}&\int_{s_1}^{t_{n-1}} I_{n-1}(r,s_1;h_{n-1},\cdots,h_1) h_n(r) q(r) h_0^\gamma(r) |D^0(x;h_0)(r)|^\gamma dr \\ &\qquad = I_n(t_{n-1}, s_1; h_n h_0^\gamma q |D^0(x;h_0)|^\gamma, h_{n-1}, \cdots, h_1),\end{aligned}$$

so that, making use of (10), we conclude that

$$\eta - \xi < c_n I_n(t_{n-1}, s_1; h_n h_0^\gamma q, h_{n-1}, \cdots, h_1) < \frac{\eta - \xi}{2}$$

a contradiction. Therefore $D^0(x-\phi;h_0)(t)$ must approach a finite limit as $t \to \infty$. This completes the proof. □

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