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EFFICIENT PLOTTING OF CLOSED POLAR CURVES WITH MATHEMATICA

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ABSTRACT. A simple mathematical theory is developed on the periodicity of elementary polar functions. The periodicity plays an important role in efficient plotting of some closed polar curves, without the excessive use of plotting devices and materials. An efficient plotting algorithm utilizing the periodicity is proposed and its implementation by a Mathematica program is introduced for a family of closed polar functions.

0. Notations and Symbols

R : set of real numbers
Z : set of integers
$(p, q) = 1$: integers p and q are relatively prime
gcd (m, n) : greatest common divisor of integers m and n
(m, n) : least common multiple of integers m and n

1. Introduction

Many interesting geometric curves have been introduced with the aid of powerful computer by Fay [3] and Chamberlain [2] and other researchers. With a conventional drawing method, it is not easy to draw some complicated geometric curves, which often provides no educational interest. In this paper, we present some educationally interesting computer-generated geometric closed curves by a Mathematica programming for a family of polar functions in the form

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$$r = a\sin(p\theta/q) + b\cos(v\theta/w) + c, \tag{1}$$

where $\theta \in R$ is a polar angle and $a, b, c \in R$ that are not all zero and $p, q, v, w \in Z$ such that $(p, q) = 1$ and $(v, w) = 1$.

Since the rectangular coordinates (x, y) are given by $x = r\cos\theta$ and $y = r\sin\theta$, both x and y are periodic in θ. Their locus traces a closed curve in the plane. If the range of the polar angle θ is arbitrarily chosen, then the trace may not complete the closed curve or may excessively wrap around the closed curve. An excessive wrapping around the closed curve from the large range of the polar angle θ might cause breaking plotter pens, puncturing plot papers and early shortage of drawing ink or printing toner. It is desirable to plot the closed curves only within the range of a fundamental period. The main aim of this paper is to develop the theory on the fundamental period and an efficient plotting algorithm of the polar curves defined by Eqn. (1). Table 1 shows the variety of parameter values for a, b, c, p, q, v and w chosen in the plotting of the polar functions. Figure 1 shows a flow chart of the plotting algorithm developed on the basis of Theorem 1. Finally source codes of the computer programming with Mathematica are included along with the successful computer-generated plots shown in Figure 2.

2. The Periodicity of Closed Polar Functions.

An elementary number-theoretic approach is used to develop the following Lemma 1 and Theorem 1 which give the fundamental period of a family of closed polar functions described by Eqn. (1).

Lemma 1. *Let $(p, q) = 1$ and $d = gcd(p+q, p-q)$ for $p, q \in Z$. Then*

$$\begin{cases} 1 & if\ p+q\ is\ odd \\ 2 & if\ p+q\ is\ even. \end{cases}$$

Proof. If $p+q$ is an odd integer, we write $p+q = 2m_1 + 1$ for some $m_1 \in Z$. Then we have $p - q = (p+q) - 2q = 2(m_1 - q) + 1 = 2m_2 + 1$ for some $m_2 \in Z$.

Suppose that $d \neq 1$. Since $2p = (p+q) + (p-q)$ and $2q = (p+q) - (p-q)$, d is also a divisor of $2p$ and $2q$. But d must be an odd integer due to the fact that both $p+q$ and $p-q$ are odd integers. Hence d must be a divisor of p and q, contradicting the assumption that $(p, q) = 1$. This contradiction proves that $d = 1$. Similarly if

$p+q$ is even, let $p+q=2k_1$ and $p-q=p+q-2q=2(k_1-q)=2k_2$ for some $k_1,k_2\in Z$. Suppose that $d\neq 2$. Since d is also a divisor of $2p$ and $2q$, d must be a divisor of p and q, contradicting the assumption that $(p,q)=1$. This contradiction proves that $d=2$.

Definition 1. A function $r=f(\theta)$ is said to have a *period* T if $x(\theta)=r\cos\theta$ and $y(\theta)=r\sin\theta$ satisfy the relations $x(\theta)=x(\theta+T)$ and $y(\theta)=y(\theta+T)$. Among such periods T, the smallest positive T is called the *fundamental* period.

Theorem 1. *Let r be a family of polar functions in the form*

$$r=a\sin\bigl(p\theta/q\bigr)+b\cos\bigl(v\theta/w\bigr)+c,$$

where $\theta\in R$ is a polar angle and $a,\ b,\ c\in R$ that are not all zero and $p,\ q,\ v,\ w\in Z$ such that $(p,q)=1$ and $(v,w)=1$. Let $d_1=gcd\ (p+q,\ p-q)$ and $d_2=gcd(v+w,\ v-w)$. Let $\alpha=2q/d_1$ and $\beta=2w/d_2$. Then r has a fundamental period $\lambda\pi$ with $\lambda=lcm(s_1,\ s_2,\ s_3)$, where

$$s_1=\begin{cases}1 & if\ a=0\\ \alpha & if\ a\neq 0,\end{cases}$$

$$s_2=\begin{cases}1 & if\ b=0\\ \beta & if\ b\neq 0,\end{cases}$$

$$s_3=\begin{cases}1 & if\ c=0\\ 2 & if\ c\neq 0.\end{cases}$$

Proof. Let X_1,Y_1,X_2 and Y_2 be defined as follows:

$$X_1=\sin\bigl(\frac{p}{q}\theta\bigr)\cos\theta=\frac{1}{2}\left[\sin\bigl(\frac{p+q}{q}\theta\bigr)+\sin\bigl(\frac{p-q}{q}\theta\bigr)\right]$$
$$Y_1=\sin\bigl(\frac{p}{q}\theta\bigr)\sin\theta=\frac{1}{2}\left[\cos\bigl(\frac{q-p}{q}\theta\bigr)-\cos\bigl(\frac{p+q}{q}\theta\bigr)\right]$$
$$X_2=\cos\bigl(\frac{v}{w}\theta\bigr)\cos\theta=\frac{1}{2}\left[\cos\bigl(\frac{v+w}{w}\theta\bigr)+\cos\bigl(\frac{v-w}{w}\theta\bigr)\right]$$
$$Y_2=\cos\bigl(\frac{v}{w}\theta\bigr)\sin\theta=\frac{1}{2}\left[\sin\bigl(\frac{v+w}{w}\theta\bigr)-\sin\bigl(\frac{v-w}{w}\theta\bigr)\right].$$

Then we obtain the following relations

$$x = aX_1 + bX_2 + c\cos\theta \text{ and } y = aY_1 + bY_2 + c\sin\theta.$$

Define periods $T_1,\ T_2,\ T_3,\ T_4$ for $n_1,\ n_2,\ k_1,\ k_2 \in Z$ as follows:

$$T_1 = q\frac{2n_1}{p+q}\pi,\quad T_2 = q\frac{2n_2}{p-q}\pi,\quad T_3 = w\frac{2k_1}{v+w}\pi,\quad T_4 = w\frac{2k_2}{v-w}\pi.$$

It remains to seek the least common period between 2π, T_1, T_2, T_3 and T_4. Since $q/(p+q)$, $q/(p-q)$, $w/(v+w)$ and $w/(v-w)$ are not integers, the determination of the fundamental period requires a proper selection of $n_1,\ n_2,\ k_1,\ k_2$ such that $2n_1/(p+q)$, $2n_2/(p-q)$, $2k_1/(v+w)$ and $2k_2/(v-w)$ are the smallest integers. This selection can be made easily in view of Lemma 1. Let $d_1 = gcd(p+q,\ p-q)$ and $d_2 = gcd(v+w,\ v-w)$. Then $n_1 = (p+q)/d_1$, $n_2 = (p-q)/d_1$, $k_1 = (v+w)/d_2$ and $k_2 = (v-w)/d_2$ give us the smallest integers $2/d_1$, $2/d_1$, $2/d_2$ and $2/d_2$ respectively. Thus it suffices to consider only three integers $2q/d_1$, $2w/d_2$ and 2 for the fundamental period. Utilizing Lemma 1 we obtain $\alpha = 2q/d_1$, $\beta = 2w/d_2$ and $\lambda = lcm(s_1,\ s_2,\ s_3)$, which completes the proof.

3. Plotting Algorithm and Implementation by Mathematica

Figure 1 shows a flow chart describing a plotting algorithm based on the result of Theorem 1. A Mathematica program has been written utilizing the algorithm and successfully implemented for the various parameter values listed in Table 1 on a personal computer with Windows 95 operating system.

Table 1. Parameter values and computed λ's

#	a	b	c	p	q	v	w	λ
1	0.0	1.0	0.00	1	1	1	1	1
2	0.0	1.0	1.0	1	1	1	1	2
3	0.0	1.0	0.0	1	1	1	3	3
4	0.0	1.0	1/3	1	1	1	2	4
5	0.75	0.0	0.5	4	3	1	3	6
6	0.0	1.0	1/4	1	1	4	1	2
7	0.0	1.0	0.0	1	1	3	2	4

#	a	b	c	p	q	v	w	λ
8	0.0	2.5	1.0	5	3	2	3	6
9	1.0	0.0	0.375	5	3	3	2	6
10	0.0	0.5	0.25	7	5	7	2	4
11	1.0	0.625	0.05	5	3	4	3	6
12	1.0	1.0	1.0	9	4	1	2	8
13	0.5	0.5	0.175	1	3	2	1	6
14	1.0	0.5	1.0	2	3	2	3	6
15	0.5	0.45	0.5	5	4	2	1	8
16	0.0	1.0	0.25	5	6	5	3	6
17	0.0	0.75	0.5	4	3	2	3	6
18	1.5	0.5	0.3125	2	3	4	1	6
19	0.0	1.0	1/4	1	3	7	2	4
20	2.0	0.5	0.0125	1	4	2	1	8
21	1.0	1.0	0.025	2	3	5	3	6
22	0.0	1.0	0.0	1	1	9	10	20
23	1.0	0.4	−0.025	4	3	4	1	6
24	0.3	0.25	0.0	4	1	5	2	4
25	1.0	0.0	−0.75	1	3	2	1	6
26	1.0	1.0	1.0	4	3	2	3	6
27	0.0	1.0	3	5	3	7	1	2
28	1.0	0.0	0.625	7	5	1	2	10
29	0.75	1.0	0.005	7	5	6	5	10
30	0.5	0.1	0.2	2	1	2	1	2
31	0.75	0.0	−0.75	2	3	2	1	6
32	0.91	1.3	0.01	5	4	3	2	8
33	0.0	0.75	0.375	5	3	3	1	2
34	1.0	0.0	0.25	7	5	1	2	10
35	0.625	0.625	0.05	2	3	1	3	6
36	0.1	0.3	0.0625	4	1	3	2	4
37	0.6	0.2	0.0625	5	1	4	3	6
38	0.7	0.1	0.625	6	5	13	2	20
39	0.9	0.3	0.725	3	1	7	12	24
40	0.1	0.6	0.825	4	3	19	2	12

The Mathematica function package *Graphics'Graphics'* was called and then the function *PolarPlot* was effectively used in the implementation. The successful implementation results are shown in Figure 2. The programming source codes are included here for interested readers.

```
(* Program: Closed Polar Curves with Mathematica *)
Off[General::beta]
Needs["Graphics`Graphics`"]
(*_________________________________________________________________*)
(* list    a        b        c         p   q   v   w   *)
(*_________________________________________________________________*)
   t[1]  = {0.000,  1.0000,  0.0000,   1,  1,  1,  1};
   t[2]  = {0.000,  1.0000,  1.0000,   1,  1,  1,  1};
   t[3]  = {0.000,  1.0000,  0.0000,   1,  1,  1,  3};
   t[4]  = {0.000,  1.0000,  1/3,      1,  1,  1,  2};
   t[5]  = {0.750,  0.0000,  0.5000,   4,  3,  1,  3};
   t[6]  = {0.000,  1.0000,  1/4,      1,  1,  4,  1};
   t[7]  = {0.000,  1.0000,  0.0000,   1,  1,  3,  2};
   t[8]  = {0.000,  2.5000,  1.0000,   5,  3,  2,  3};
   t[9]  = {1.000,  0.0000,  0.3750,   5,  3,  3,  2};
   t[10] = {0.000,  0.5000,  0.2500,   7,  5,  7,  2};
   t[11] = {1.000,  0.6250,  0.0500,   5,  3,  4,  3};
   t[12] = {1.000,  1.0000,  1.0000,   9,  4,  1,  2};
   t[13] = {0.500,  0.5000,  0.1750,   1,  3,  2,  1};
   t[14] = {1.000,  0.5000,  1.0000,   2,  3,  2,  3};
   t[15] = {0.500,  0.4500,  0.5000,   5,  4,  2,  1};
   t[16] = {0.000,  1.0000,  0.2500,   5,  6,  5,  3};
   t[17] = {0.000,  0.7500,  0.5000,   4,  3,  2,  3};
   t[18] = {1.500,  0.5000,  0.3125,   2,  3,  4,  1};
   t[19] = {0.000,  1.0000,  1/4,      1,  3,  7,  2};
   t[20] = {2.000,  0.5000,  0.0125,   1,  4,  2,  1};
   t[21] = {1.000,  1.0000,  0.0250,   2,  3,  5,  3};
   t[22] = {0.000,  1.0000,  0.0000,   1,  1,  9,  10};
   t[23] = {1.000,  0.4000,  -0.0250,  4,  3,  4,  1};
   t[24] = {0.300,  0.2500,  0.0000,   4,  1,  5,  2};
   t[25] = {1.000,  0.0000,  -0.7500,  1,  3,  2,  1};
   t[26] = {1.000,  1.0000,  1.0000,   4,  3,  2,  3};
   t[27] = {0.000,  1.0000,  3.0000,   5,  3,  7,  1};
   t[28] = {1.000,  0.0000,  0.6250,   7,  5,  1,  2};
   t[29] = {0.750,  1.0000,  0.0050,   7,  5,  6,  5};
   t[30] = {0.500,  0.1000,  0.2000,   2,  1,  2,  1};
   t[31] = {0.750,  0.0000,  -0.7500,  2,  3,  2,  1};
   t[32] = {0.910,  1.3000,  0.0100,   5,  4,  3,  2};
   t[33] = {0.000,  0.7500,  0.3750,   5,  3,  3,  1};
   t[34] = {1.000,  0.0000,  0.2500,   7,  5,  1,  2};
   t[35] = {0.625,  0.6250,  0.0500,   2,  3,  1,  3};
   t[36] = {0.100,  0.3000,  0.0625,   4,  1,  3,  2};
   t[37] = {0.600,  0.2000,  0.0625,   5,  1,  4,  3};
```

```
      t[38] = {0.700,  0.1000,  0.6250,  6,  5,  13,  2};
      t[39] = {0.900,  0.3000,  0.7250,  3,  1,  7,   12};
      t[40] = {0.100,  0.6000,  0.8250,  4,  3,  19,  2};
   (*-------------------------------------------------------*)
ng = 0; (* Initialize the counter *)
For [i = 1, i <= 40, i++,
  a = t[i][[1]]; b = t[i][[2]]; c = t[i][[3]];
  p = t[i][[4]]; q = t[i][[5]]; v = t[i][[6]]; w = t[i][[7]];
  h1 = GCD[p,q]; h2 = GCD[v,w];
  If[h1 != 1, p = p/h1; q = q/h1]; If[h2 != 1, v = v/h2; w = w/h2];
  d1 = GCD[p+q,p-q]; d2 = GCD[v+w,v-w]; alpha = 2q/d1; beta = 2w/d2;
  If[a == 0.0, s1 = 1, s1 = alpha]; If[b == 0.0, s2 = 1, s2 = beta];
  If[c == 0.0, s3 = 1, s3 = 2];     per[i] = lambda = LCM[s1, s2, s3];
  ng = ng + 1;
  h[ng] = PolarPlot [a*Sin[p*t/q] + b*Cos[v*t/w] + c,{t,0,lambda*Pi},
                  Axes → None, PlotPoints → 75,
                  DisplayFunction → Identity,
              PlotLabel → FontForm ["#"TextForm[ng], {"Times", 7}]];
  ];
s = Table[h[i], {i, 1, ng}]; s = Partition[s,5];
Show[GraphicsArray[s], GraphicsSpacing → {0, 0.1}];
```

4. Conclusion

The use of the Mathematica built-in functions simplified many required symbolic calculations and shortened the length of the programming codes for the plotting algorithm. If other programming software had been used, a lot of unexpected errors would have occurred due to the lengthy codes.

As a result of efficient Mathematica programming, an interesting family of closed polar functions has been efficiently plotted with the optimal use of plotting materials and devices. The theory on computing the period of closed curves played an essential role in the construction of the efficient plotting algorithm. The computer results for the numerous parameter values clearly confirmed the basic theory developed in this paper.

The powerful Mathematica programming easily gave us the visualization of the polar curves, which certainly helped to improve our mathematical intuition and thought on the given problem. With the use of the Mathematica program, the underlying mathematical idea shown in this paper can be extended to other interesting problems which give an educational interest in many fields of mathematics and science.

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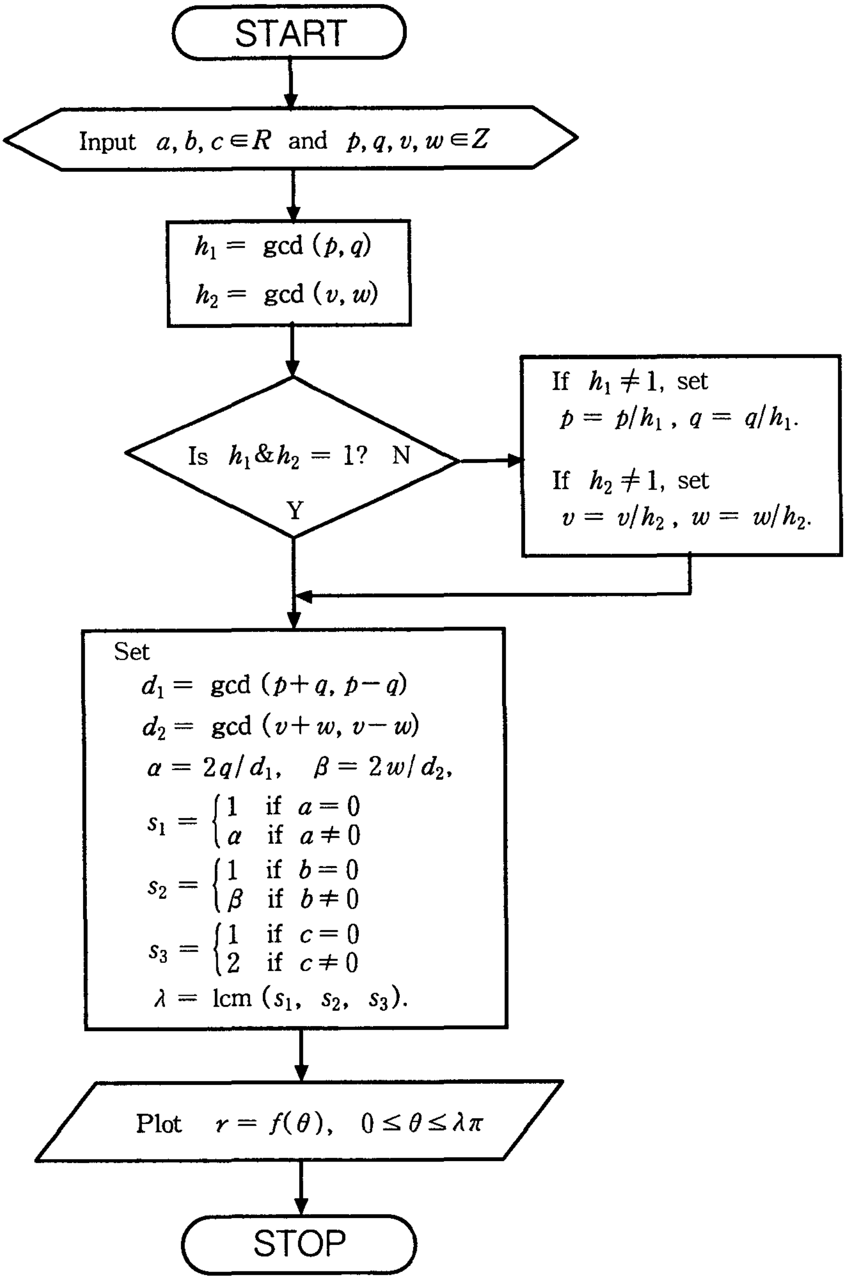

Figure 1. Flow chart of a plotting algorithm

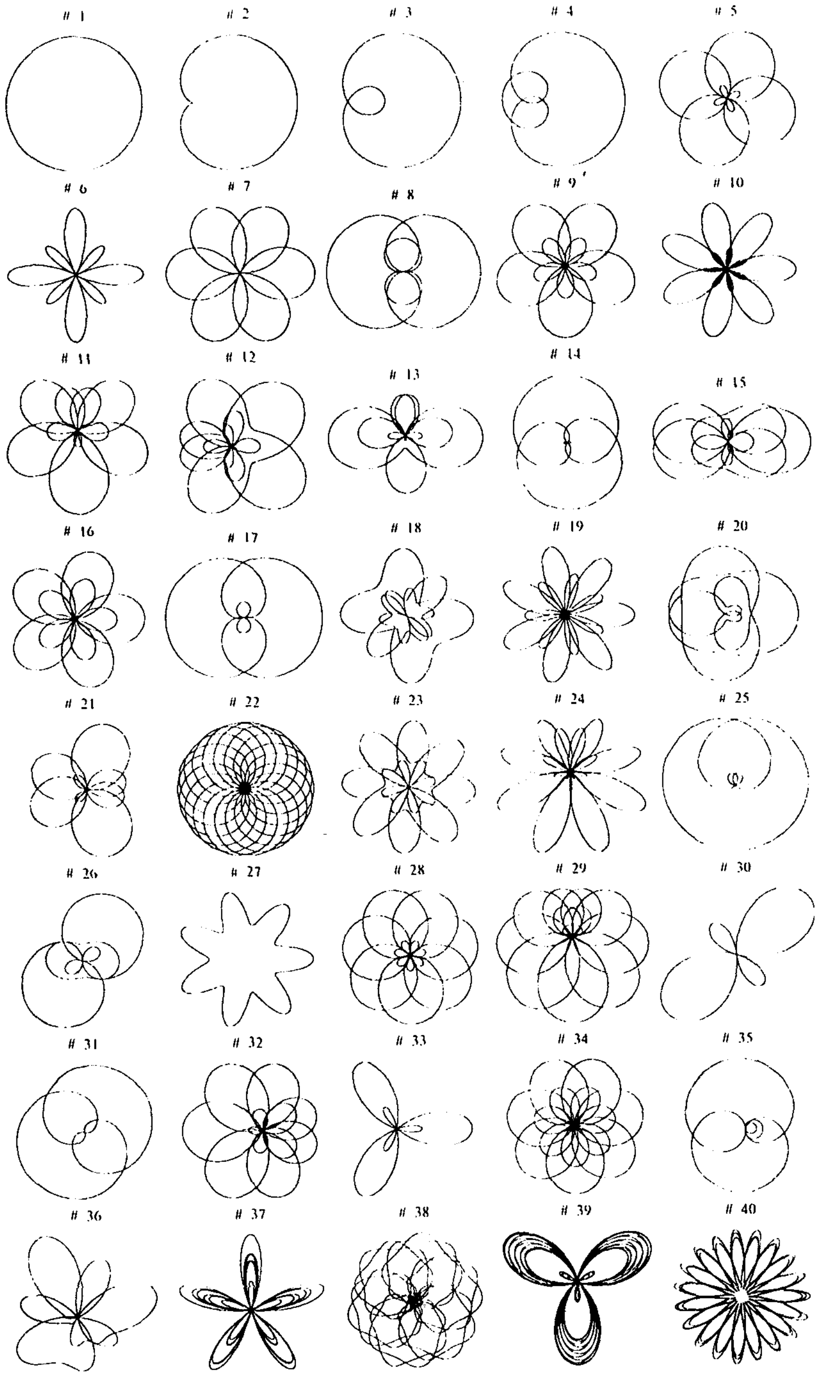

Figure 2. A family of closed polar curves