J. Korean Math. Soc. **35** (1998), No. 4, pp. 1019–1043

ASYMPTOTICS FOR SOLUTIONS OF THE GINZBURG-LANDAU EQUATIONS WITH DIRICHLET BOUNDARY CONDITIONS

JONGMIN HAN

ABSTRACT. In this paper we study some asymptotics for solutions of the Ginzburg-Landau equations with Dirichlet boundary conditions. We consider the solutions (u_ϵ, A_ϵ) which minimize the Ginzburg-Landau energy functional $E_\epsilon(u, A)$. We show that the solutions (u_ϵ, A_ϵ) converge to some (u_*, A_*) in various norms as the coupling parameter $\epsilon \to 0$.

1. Introduction

Let Ω be a smooth bounded simply connected domain in $\mathbf{R}^2$. We consider two smooth functions

$$u_0 : \Omega \to \mathbf{C} \quad \text{and} \quad A_0 : \Omega \to \mathbf{R}$$

satisfying $|u_0| = 1$ on $\partial\Omega$, $\text{div}A_0 = 0$ and $\deg(g, \partial\Omega) = 0$ where $g = u_0|_{\partial\Omega}$. Here $\deg(g, \partial\Omega)$ is the winding number of g considered as a map from $\partial\Omega$ into S^1. Consider the following Ginzburg-Landau equations with Dirichlet boundary conditions, denoted by (P),

$$D_A^2 u + \frac{1}{\epsilon^2}(1 - |u|^2)u = 0 \quad \text{in } \Omega \tag{1}$$

$$\text{curl}^2 A + \frac{i}{2}(\overline{u}D_A u - u\overline{D_A u}) = 0 \quad \text{in } \Omega \tag{2}$$

$$u = g \ , \ A = A_0 \quad \text{on } \partial\Omega \tag{3}$$

where $\epsilon > 0$, $(D_A)_j = \partial_j - iA_j$ and $\text{curl}^2 A = -\Delta A + \nabla(\text{div}A)$. Here $u : \Omega \to \mathbf{C}$ is the complex order parameter and $A : \Omega \to \mathbf{R}^2$ is the magnetic vector potential.

Received April 3, 1998.
1991 Mathematics Subject Classification: 35B40, 35J65.
Key words and phrases: Ginzburg-Landau equations, asymptotic behavior.

The Ginzburg-Landau model has been proposed in 1950 to give phenomenological descriptions on superconductivity at low temperature. Physically $|u|$ represents the density of the superconducting electron pairs, so called Cooper pairs, in the superconducting material. When $|u| = 0$, the material remains in normal conducting state. When $|u| = 1$, the material remains in superconducting state.

For physical reasoning, it is natural that the Ginzburg-Landau equations be solved under a Neumann boundary condition (See [4]). But one can consider boundary value problems with the boundary data simply imposed on the gauge potential itself. In other words we can consider directly the influence of the external gauge potential instead of the external magnetic field to the superconductor. Such a problem has arisen in [7] and some existence results have been established. In this paper we shall consider the Ginzburg-Landau equations constraint to Dirichlet boundary condition.

The Ginzburg-Landau equations are the Euler-Lagrange equations of the following energy functional

$$E_\epsilon(u,A) = \frac{1}{2}\int_\Omega |D_A u|^2 + |F_A|^2 + \frac{1}{2\epsilon^2}(1-|u|^2)^2. \tag{4}$$

Here $F_A = \partial_1 A_2 - \partial_2 A_1$ is the magnetic field. The functional $E_\epsilon(u,A)$ has an important property; it is gauge invariant. This means that

$$(u,A) \to (ue^{i\chi}, A + \nabla\chi).$$

for any smooth functions $\chi : \Omega \to \mathbf{R}$. In Neumann boundary value problem, it is possible to fix the gauge as $\text{div}A = 0$ (see [4]). In that context, we shall consider the problem (P) under the condition $\text{div}A = 0$.

For simplicity of analysis one may consider the case that the gauge field vanishes identically, i.e., $A \equiv 0$. In this simpler form the equations (P) are reduced to

$$\begin{cases} \Delta u + \frac{1}{\epsilon^2}(1-|u|^2)u = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{5}$$

and the functional $E_\epsilon(u,A)$ is reduced to

$$\tilde{E}_\epsilon(u,A) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{2\epsilon^2}(1-|u|^2)^2.$$

The equation (5) has many interesting properties and has been widely studied in recent years (See [6]). One of remarkable results was obtained by Bethuel, Brezis and Hélein [1,2], a very detailed asymptotic characterization in the limit $\epsilon \to 0$ for the solutions to (5) minimizing $\tilde{E}_\epsilon(u, A)$. The Brouwer degree $d = \deg(g, \partial\Omega)$ plays a crucial role in the asymptotic analysis for the solutions u_ϵ corresponding to $(5)_\epsilon$.

When $d = 0$, u_ϵ converges to a harmonic map which minimizes

$$\int_\Omega |\nabla u|^2$$

over the space $H^1_g(\Omega) = \{u \in W^{1,2}(\Omega, S^1) : u = g \text{ on } \partial\Omega\}$. When $d \neq 0$, the analysis is more complicated because $H^1_g(\Omega) = \emptyset$. If Ω is a starshaped region u_ϵ converges the canonical map u_* defined as follows; there are exactly d points $a_1, \cdots, a_d$ such that

$$u_* : \Omega \backslash \{a_1, \cdots, a_d\} \to S^1$$

is a harmonic map. The location of $a_1, \cdots, a_d$ can be shown to be the set which minimizes the renormalized energy (See [2] for the definition of the renormalized energy).

In this paper we are concerned with the solutions (u_ϵ, A_ϵ) of (P) which minimize the functional $E_\epsilon(u, A)$ in an appropriate function space and consider a generalization of the works [1], the case $d = 0$, when the gauge field is nontrivial. In section 2 we state our main results; (u_ϵ, A_ϵ) converges to some (u_*, A_*) in various norms. The existence and uniqueness of (u_*, A_*) is proved. In section 3 we show that it converges in $C^{1,\alpha}(\Omega)$. In section 4 we show convergences in higher derivatives and compute convergence speeds.

REMARK. After finishing his work, the author found out that the asymptotic analysis of solutions was established in more general case $\deg g \neq 0$ with Neumann type boundary conditions by F. Bethuel and T. Riviére [3].

2. Statements of the main Theorem

We define function spaces by

$$H = \{u \in W^{1,2}(\Omega, \mathbf{C}) : u|_{\partial\Omega} = g\},$$
$$V = \{A \in W^{1,2}(\Omega, \mathbf{R}^2) : A|_{\partial\Omega} = A_0\, , \, \mathrm{div} A = 0\},$$
$$H_0 = W_0^{1,2}(\Omega, \mathbf{C})\, , \;\; V_0 = \{A \in W_0^{1,2}(\Omega, \mathbf{R}^2) : \mathrm{div} A = 0\},$$
$$X = H \times V\, , \;\; X_0 = H_0 \times V_0.$$

In what follows we shall often write $W^{k,p}(\Omega)$ instead of $W^{k,p}(\Omega, \mathbb{C})$ or $W^{k,p}(\Omega, \mathbf{R}^2)$ if there is no risk of confusion. We observe that for $B \in V_0$,

$$\|B\|_{V_0}^2 = \int_\Omega |\nabla B|^2 = \int_\Omega |F_B|^2.$$

DEFINITION 1. We say that (u, A) is a *weak solution* to (P) if $v = u - u_0 \in H_0$ and $B = A - A_0 \in V_0$ satisfy

$$\int_\Omega D_{A_0+B}(u_0 + v)\overline{D_{A_0+B}w} + \frac{1}{\epsilon^2}(|u_0 + v|^2 - 1)(u_0 + v)\overline{w} = 0$$
$$\int_\Omega F_{A_0+B}F_K - \mathrm{Im}(\overline{(u_0 + v)}D_{A_0+B}(u_0 + v)) \cdot K = 0$$

for all $(w, K) \in X_0$.

THEOREM 2. *There exists a minimizer (u_ϵ, A_ϵ) of E_ϵ over X which is a weak solution to (P) in X. Moreover, this solution is smooth up to boundary.*

Proof. Existence of a weak solution was shown in [7]. We rewrite (1) and (2) as

$$\Delta u = 2iA \cdot \nabla u + |A|^2 u - \frac{1}{\epsilon^2}(1 - |u|^2)u \tag{6}$$

$$\Delta A = \frac{i}{2}(\overline{u}D_A u - u\overline{D_A u}) \tag{7}$$

Since $W^{1,2}(\Omega, \mathbf{C}) \subset L^p(\Omega, \mathbf{C})$ and $W^{1,2}(\Omega, \mathbf{R}^2) \subset L^p(\Omega, \mathbf{R}^2)$ for all $p \geq 1$, standard elliptic arguments show that u and A are smooth. $\square$

Let $H_1 = \{u \in H : |u| = 1\}$. Since $\deg(g, \partial\Omega) = 0$, H_1 is nonempty. We consider the minimization problem of the following functional

$$J(u) = \int_\Omega |\nabla u|^2.$$

over the space H_1. It is well known (see [1]) that J has a minimizer in H_1 which is the unique smooth solution of

$$\begin{cases} \Delta u + u|\nabla u|^2 = 0 & \text{in } \Omega \\ |u| = 1 & \text{in } \Omega \\ u = g & \text{on } \Omega. \end{cases} \tag{8}$$

Next we consider the minimization problem of the following functional

$$I(u, A) = \frac{1}{2}\int_\Omega |D_A u|^2 + |F_A|^2$$

over the space $X_1 = H_1 \times V$.

THEOREM 3. *The variational equations of the functional $I(u, A)$ are given by*

$$\Delta u + u|\nabla u|^2 = 0 \quad \textit{in } \Omega \tag{9}$$

$$curl^2 A + \frac{i}{2}(\overline{u}D_A u - u\overline{D_A u}) = 0 \quad \textit{in } \Omega \tag{10}$$

$$u = g \ , \ A = A_0 \quad \textit{on } \partial\Omega.$$

These equations admit a unique smooth solution (u_*, A_*) which minimizes $I(u, A)$ over X_1 and u_* is the unique solution of (8).

Proof. Let (u, A) be a critical point of the functional I over X_1. For given $v \in H_0$ and $B \in V_0$, if we let $w(t) = (u + tv)/|u + tv|$, then for all sufficiently small $|t|$, $w(t) \in H_1$ so that $w(0) = u$ and $w'(0) = (v - u^2\overline{v})/2$. Differentiating $|u|^2 = 1$, we obtain

$$\begin{cases} \overline{u}\nabla u + u\nabla\overline{u} = 0 \\ 2|\nabla u|^2 + \overline{u}\Delta u + u\Delta\overline{u} = 0. \end{cases} \tag{11}$$

Let $\alpha(t) = I(w(t), A + tB)$. Then

$$\begin{aligned}
2\alpha'(0) &= \int_\Omega \mathrm{Re}[\nabla(v - u^2\overline{v})\nabla\,\overline{u} + i\overline{u}\nabla(v - u^2\overline{v})\cdot A + i(\overline{v} - \overline{u}^2 v)\nabla u \cdot A \\
&\quad + 2i\overline{u}\nabla u \cdot B] + 2A\cdot B + 2F_A F_B \\
&= \int_\Omega \mathrm{Re}[-\Delta\overline{u}(v - u^2\overline{v}) + 2i\nabla u \cdot A(\overline{v} - \overline{u}^2 v)] \\
&\quad + 2B\cdot[\mathrm{Re}(i\overline{u}\nabla u) + A - \Delta A] \\
&= \int_\Omega \frac{1}{2} v(-\Delta\overline{u} + \overline{u}^2\Delta u - 2i\overline{u}^2\nabla u\cdot A - 2i\nabla\overline{u}\cdot A) \\
&\quad + \frac{1}{2}\overline{v}(-\Delta u + u^2\Delta\overline{u} + 2iu^2\nabla\overline{u}\cdot A + 2i\nabla u \cdot A) \\
&\quad + 2B\cdot[\mathrm{Re}(i\overline{u}\nabla u) + A - \Delta A] \\
&= 0.
\end{aligned}$$

Hence the Euler-Lagrange equations are

$$\begin{cases} -\Delta u + u^2\Delta\overline{u} + 2iu^2\nabla\overline{u}\cdot A + 2i\nabla u\cdot A = 0. \\ -\Delta A + A + \mathrm{Re}(i\overline{u}\nabla u) = 0. \end{cases}$$

By means of (11) and the fact that $|u| = 1$, these equations are equivalent to (9) and (10). Since (9) is equal to (8), (9) has a unique solution. Since (10) is linear in A, it has a unique solution.

Next we show the existence of minimizer of $I(u, A)$ over X_1 following [7]. Let (u_n, A_n) be a minimizing sequence of I over X_1, and let us write $u_n = u_0 + v_n$ and $A_n = A_0 + B_n$ for $(v_n, B_n) \in X_0$. Then

$$2I(u_n, A_n) \geq \int_\Omega |F_{A_0} + F_{B_n}|^2 \geq \frac{1}{2}\int_\Omega |F_{B_n}|^2 - \int_\Omega |F_{A_0}|^2.$$

Thus $\|B_n\|_{V_0}^2 \leq 4I(u_n, A_n) + 2\|A_0\|_V^2$. Since $|u_0 + v_n| = 1$,

$$\begin{aligned}
&2I(u_n, A_n) \\
&\geq \int_\Omega |D_{A_0+B_n}(u_0 + v_n)|^2 \\
&= \int_\Omega |\nabla u_0 + \nabla v_n|^2 + 2\mathrm{Re}[i(\overline{u}_0 + \overline{v}_n)\nabla(u_0 + v_n)\cdot(A_0 + B_n)] \\
&\quad + |A_0 + B_n|^2|u_0 + v_n|^2 \\
&\geq \int_\Omega \frac{1}{4}|\nabla v_n|^2 - C\int_\Omega |\nabla u_0|^2 + |B_n|^2 + |A_0|^2.
\end{aligned}$$

Hence (v_n, B_n) is uniformly bounded in X_0 so that passing to a subsequence, if necessary, we see that there is $(v_*, B_*) \in X_0$

$$\begin{cases} (v_n, B_n) \rightharpoonup (v_*, B_*) & \text{weakly in } X_0 \\ (v_n, B_n) \to (v_*, B_*) & \text{strongly in } L^p \text{ for all } 1 \le p < \infty. \end{cases}$$

Let $(u_*, A_*) = (u_0 + v_*, A_0 + B_*)$. On the other hand, we observe that

$$2I(u_n, A_n) = \int_\Omega (|\nabla v_n|^2 + |F_{B_n}|^2) + \Lambda(v_n, B_n),$$

where

$$\begin{aligned} \Lambda(v, B) = \int_\Omega & |\nabla u_0|^2 + 2\nabla u_0 \cdot \nabla v \\ & + 2\mathrm{Re}[i(\overline{u}_0 + \overline{v})\nabla(u_0 + v) \cdot (A_0 + B)] \\ & + |A_0 + B|^2 |u_0 + v|^2 + |F_{A_0}|^2 + 2F_{A_0} \cdot F_B. \end{aligned}$$

Since $\Lambda(v_n, B_n) \to \Lambda(v_*, B_*)$, from the weakly lower semicontinuity of norms

$$2I(u_n, A_n) \ge \int_\Omega (|\nabla v_*|^2 + |F_{B_*}|^2) + \Lambda(v_*, B_*) = 2I(v_*, B_*),$$

which implies that (u_*, A_*) is a minimizer of I over X_1. □

We are now in a position to state our main results. We establish

THOEREM 4. *Let (u_ϵ, A_ϵ) be any solutions of (1) and (2) which are minimizers of $E_\epsilon(u, A)$ over the space X. Then we have*
(i)

$$u_\epsilon \to u_* \quad \textit{in} \quad C^{1,\alpha}(\overline{\Omega}, \mathbf{C})$$

and

$$A_\epsilon \to A_* \quad \textit{in} \quad C^{1,\alpha}(\overline{\Omega}, \mathbf{R}^2)$$

for all $0 < \alpha < 1$.
(ii) *We have*

$$\|u_\epsilon - u_*\|_{L^\infty(\Omega)} \le C\epsilon^2 \quad \textit{and} \quad \|A_\epsilon - A_*\|_{L^\infty(\Omega)} \le C\epsilon^{3/2}.$$

(iii) *We have*

$$\|u_\epsilon - u_*\|_{C^k_{loc}(\Omega)} \le C\epsilon^2 \quad \textit{and} \quad \|A_\epsilon - A_*\|_{C^k_{loc}(\Omega)} \le C\epsilon^2$$

for every nonnegative integer k.

Proof. Theorems 11, 12 and 14. □

We will prove this Theorem in the next two sections following [1].

3. Convergence in $C^{1,\alpha}(\Omega)$

From now on let us denote by (u_ϵ, A_ϵ) the solutions to (P) which are minimizers of $E_\epsilon(u, A)$ over the space X.

PROPOSITION 5. *$(u_\epsilon, A_\epsilon) \to (u_*, A_*)$ strongly in X.*

Proof. Let $v_\epsilon = u_\epsilon - u_0$ and $B_\epsilon = A_\epsilon - A_0$. Since $(u_*, A_*) \in X$,

$$(12)\quad |D_{A_\epsilon} u_\epsilon|^2 + |F_{A_\epsilon}|^2 + \frac{1}{2\epsilon^2}(1-|u_\epsilon|^2)^2 \le \int_\Omega |D_{A_*}u_*|^2 + |F_{A_*}|^2 \equiv C_*.$$

As in the proof of Theorem 3, we have

$$C_* \ge \int_\Omega |F_{A_0} + F_{B_\epsilon}|^2 \ge \frac{1}{2}\int_\Omega |F_{B_\epsilon}|^2 - \int_\Omega |F_{A_0}|^2,$$

and

$$C_* \ge \int_\Omega \frac{1}{2\epsilon^2}(1-|u_\epsilon|^2)^2 \ge \frac{1}{4\epsilon^2}\int_\Omega |u_\epsilon|^4 - \frac{1}{2\epsilon^2}|\Omega|.$$

Thus (B_ϵ) is uniformly bounded in V_0 and (u_ϵ) is uniformly bounded in $L^4(\Omega, \mathbf{C})$. On the other hand,

$$\begin{aligned} C_* &\ge \int_\Omega |D_{A_0+B_\epsilon}(u_0 + v_\epsilon)|^2 \\ &= \int_\Omega |\nabla u_0 + \nabla v_\epsilon|^2 + 2\mathrm{Re}[i\overline{u}_\epsilon \nabla(u_0 + v_\epsilon)\cdot(A_0 + B_\epsilon)] \\ &\quad + |A_0 + B_\epsilon|^2|u_\epsilon|^2 \\ &\ge \int_\Omega \frac{1}{4}|\nabla v_\epsilon|^2 - C\int_\Omega |\nabla u_0|^2 + |u_\epsilon|^4 + |A_0|^4 + |B_\epsilon|^4. \end{aligned}$$

Hence (v_ϵ) is uniformly bounded in H_0. Now there exists a subsequence $(u_{\epsilon_n}, A_{\epsilon_n}) = (v_{\epsilon_n} + u_0, B_{\epsilon_n} + A_0)$ and $(\tilde{u}, \tilde{A}) \in X$ so that

$$\begin{cases} (u_{\epsilon_n}, A_{\epsilon_n}) \rightharpoonup (\tilde{u}, \tilde{A}) & \text{weakly in } X \\ (u_{\epsilon_n}, A_{\epsilon_n}) \to (\tilde{u}, \tilde{A}) & \text{strongly in } L^p \text{ for all } 1 \le p < \infty. \end{cases}$$

Since

$$\int_\Omega (1 - |u_{\epsilon_n}|^2)^2 \le 2C_*\epsilon_n^2\ ,$$

we get $|\tilde{u}| = 1$ and hence $(\tilde{u}, \tilde{A}) \in X_1$. We also find as in the proof of Theorem 3 that

$$\int_\Omega |D_{\tilde{A}}\tilde{u}|^2 + |F_{\tilde{A}}|^2 \leq C_*,$$

which implies that $(\tilde{u}, \tilde{A})$ is a minimizer of I over X_1. By the uniqueness of minimizer we have $(\tilde{u}, \tilde{A}) = (u_*, A_*)$.

We observe by (12) that

$$\begin{aligned}
&\int_\Omega |D_{A_{\epsilon_n}} u_{\epsilon_n} - D_{A_*} u_*|^2 + |F_{A_{\epsilon_n}} - F_{A_*}|^2 \\
&= \int_\Omega |D_{A_{\epsilon_n}} u_{\epsilon_n}|^2 + |D_{A_*} u_*|^2 + |F_{A_{\epsilon_n}}|^2 + |F_{A_*}|^2 \\
&\qquad - 2\mathrm{Re}(D_{A_{\epsilon_n}} u_{\epsilon_n} \cdot \overline{D_{A_*} u_*}) - 2F_{A_{\epsilon_n}} F_{A_*} \\
&\leq 2 \int_\Omega |D_{A_*} u_*|^2 + |F_{A_*}|^2 - \mathrm{Re}(D_{A_{\epsilon_n}} u_{\epsilon_n} \cdot \overline{D_{A_*} u_*}) - F_{A_{\epsilon_n}} F_{A_*}.
\end{aligned} \tag{13}$$

Since $(u_{\epsilon_n}, A_{\epsilon_n})$ converges weakly to (u_*, A_*) in X, the right hand side of (13) goes to zero. Thus $(u_{\epsilon_n}, A_{\epsilon_n})$ converges strongly to (u_*, A_*) in X. The convergence of whole sequence follows from the uniqueness of (u_*, A_*). □

LEMMA 6. (i) $|u_\epsilon| \leq 1$ *in* $\overline{\Omega}$.
(ii) (A_ϵ) *is uniformly bounded in* $W^{2,2}(\Omega, \mathbf{R}^2)$.
(iii) $\|\nabla u_\epsilon\|_{L^\infty(\Omega)} \leq C_0/\epsilon$, C_0 *is independent of* ϵ.
(iv) $|u_\epsilon| \to 1$ *uniformly on* Ω.

Proof. (i) If we let $w_\epsilon = (1 - |u_\epsilon|^2)/\epsilon^2$, then using (1) we find that

$$\epsilon^2 \Delta w_\epsilon = -2|D_{A_\epsilon} u_\epsilon|^2 + 2|u_\epsilon|^2 w_\epsilon \leq 2|u_\epsilon|^2 w_\epsilon. \tag{14}$$

Then the maximum principle implies that $w_\epsilon \geq 0$.

(ii) Since $\Delta A_\epsilon \in L^2(\Omega)$ by (7) and (12), standard elliptic estimates give

$$\begin{aligned}
\|A_\epsilon\|_{W^{2,2}(\Omega)} &\leq C(\Omega)(\|A_\epsilon\|_{L^2(\Omega)} + \|\Delta A_\epsilon\|_{L^2(\Omega)} + \|A_0\|_{W^{2,2}(\Omega)}) \\
&\leq C(\Omega, C_*, \|A_0\|_{W^{2,2}(\Omega)}).
\end{aligned}$$

(iii) Let

$$v_\epsilon = u_\epsilon - q_\epsilon,$$

where q_ϵ is a solution

$$\begin{aligned} \Delta q_\epsilon &= 0 \text{ in } \Omega \\ q_\epsilon &= g \text{ on } \partial\Omega. \end{aligned}$$

Then by (6) v_ϵ satisfies

$$\begin{aligned} \Delta v_\epsilon = 2iA_\epsilon \cdot \nabla u_\epsilon + |A_\epsilon|^2 u_\epsilon - \frac{1}{\epsilon^2}(1-|u_\epsilon|^2)u_\epsilon \equiv h_\epsilon \text{ in } \Omega \\ v_\epsilon = 0 \text{ on } \partial\Omega. \end{aligned}$$

Since $|u_\epsilon| \leq 1$ and (A_ϵ) is uniformly bounded in $L^\infty(\Omega, \mathbf{R}^2)$ by Sobolev imbedding, we observe that

$$\|v_\epsilon\|_{L^\infty(\Omega)} \leq \|u_\epsilon\|_{L^\infty(\Omega)} + \|q_\epsilon\|_{L^\infty(\Omega)} \leq C,$$

and

$$\|h_\epsilon\|_{L^\infty(\Omega)} \leq C\left(\|\nabla u_\epsilon\|_{L^\infty(\Omega)} + \frac{1}{\epsilon^2}\right) \leq C\left(\|\nabla v_\epsilon\|_{L^\infty(\Omega)} + \frac{1}{\epsilon^2}\right).$$

It follows from (i), (ii) and Lemma A.2 that

$$\|\nabla v_\epsilon\|^2_{L^\infty(\Omega)} \leq C\left(\|\nabla v_\epsilon\|_{L^\infty(\Omega)} + \frac{1}{\epsilon^2}\right) \leq \frac{1}{2}\|\nabla v_\epsilon\|^2_{L^\infty(\Omega)} + \frac{C}{\epsilon^2},$$

and thus

$$\|\nabla v_\epsilon\|^2_{L^\infty(\Omega)} \leq \frac{C}{\epsilon^2}.$$

This achieves the desired estimate

$$\|\nabla u_\epsilon\|_{L^\infty(\Omega)} \leq \|\nabla v_\epsilon\|_{L^\infty(\Omega)} + \|\nabla q_\epsilon\|_{L^\infty(\Omega)} \leq C/\epsilon.$$

(iv) We first show that $|u_\epsilon| \to 1$ uniformly on every compact subset of Ω. Let $K \subset\subset \Omega$. Assume the contrary. Then there would be a sequence $\epsilon_n \to 0$ and $x_n \in K$ so that $1 - |u_{\epsilon_n}(x_n)|^2 \geq \alpha$ for some fixed $\alpha > 0$. Since $(u_{\epsilon_n}, A_{\epsilon_n}) \to (u_*, A_*)$ in X, by (12)

$$\frac{1}{2\epsilon_n^2}\int_\Omega (1 - |u_{\epsilon_n}|^2)^2 \to 0.$$

Let $\delta_n = \alpha\epsilon_n/4C_0 < \operatorname{dist}(K, \partial\Omega)$ for all sufficiently small ϵ_n. Here C_0 is the constant in (iii). If $|x - x_n| \leq \delta_n$, then by (iii)

$$\big|\, |u_{\epsilon_n}(x)|^2 - |u_{\epsilon_n}(x_n)|^2 \,\big| \leq 2\delta_n \frac{C_0}{\epsilon_n} = \frac{\alpha}{2}$$

Now

$$\begin{aligned} 0 \leftarrow \frac{1}{\epsilon_n^2}\int_\Omega (1 - |u_{\epsilon_n}|^2)^2 &\geq \frac{1}{\epsilon_n^2}\int_{|x-x_n|\leq\delta_n} (1 - |u_{\epsilon_n}|^2)^2 \\ &\geq \frac{1}{\epsilon_n^2}\cdot\frac{\alpha^2}{4}\cdot\pi\delta_n^2 \\ &\geq \frac{\alpha^4}{64C_0^2}, \end{aligned}$$

a contradiction.

Since Ω is smooth, there is a constant C depending only on Ω such that

$$|\Omega \cap B(x,\delta)| \geq C\delta^2$$

for all $x \in \overline{\Omega}$ and for all sufficiently small $\delta > 0$. Using this fact, we can show that for any $x \in \partial\Omega$, $|u_\epsilon| \to 1$ uniformly on $\Omega \cap B(x,\delta)$ for some $\delta > 0$ as above, and the proof is completed. □

In the rest of this paper we assume by Lemma 6 that $|u_\epsilon| \geq 1/2$.

LEMMA 7. *Let* $f_\epsilon^2 = \sum_{j,k} |(u_\epsilon)_{x_j x_k}|^2$. *Then*

$$f_\epsilon^2 \leq \Delta(|\nabla u_\epsilon|^2) + C(1 + |\nabla A_\epsilon|^2 + |\nabla u_\epsilon|^4) \quad \text{in } \Omega \tag{15}$$

where C *is independent of* ϵ.

Proof. For simplicity, we drop the subscript ϵ. By direct calculation,

$$\Delta(|\nabla u|^2) = 2f^2 + 2\mathrm{Re}[\overline{u}_{x_j} \Delta u_{x_j}].$$

Let $A = (A_1, A_2)$. By (6)

$$\begin{aligned}\overline{u}_{x_j} \Delta u_{x_j} &= \overline{u}_{x_j} [2iA \cdot \nabla u + |A|^2 u - \frac{1}{\epsilon^2}(1 - |u|^2)u]_{x_j} \\ &= 2iu_{x_j}(\nabla A_j \cdot \nabla \overline{u}) + 2iA_j(\nabla u_{x_j} \cdot \nabla \overline{u}) \\ &\quad + 2uA_j(\nabla A_j \cdot \nabla \overline{u}) + |A|^2 |\nabla u|^2 \\ &\quad + \frac{1}{\epsilon^2}(|\nabla u|^2 |u|^2 + u^2 \nabla \overline{u} \cdot \nabla \overline{u}) - \frac{1}{\epsilon^2}(1 - |u|^2)|\nabla u|^2.\end{aligned}$$

Hence by (1)

$$\begin{aligned}\Delta(|\nabla u|^2) = 2f^2 &+ 2\mathrm{Re}[2iu_{x_j}(\nabla A_j \cdot \nabla \overline{u}) \\ &+ 2iA_j(\nabla u_{x_j} \cdot \nabla \overline{u}) + 2uA_j(\nabla A_j \cdot \nabla \overline{u})] \\ &+ 2|A|^2|\nabla u|^2 + \frac{1}{\epsilon^2}|\overline{u}\nabla u + u \nabla \overline{u}|^2 + \frac{2D_A^2 u}{u}|\nabla u|^2.\end{aligned}$$

Since $1/2 \le |u| \le 1$ and $A \in L^\infty(\Omega, \mathbf{R}^2)$ by Sobolev embedding theorem, we have

$$\begin{aligned}2f^2 &\le -2\mathrm{Re}[2iu_{x_j}(\nabla A_j \cdot \nabla \overline{u}) + 2iA_j(\nabla u_{x_j} \cdot \nabla \overline{u}) + 2uA_j(\nabla A_j \cdot \nabla \overline{u})] \\ &\quad + \Delta(|\nabla u|^2) + \frac{2|D_A^2 u|}{|u|}|\nabla u|^2 \\ &\le \Delta(|\nabla u|^2) + C(\,|\nabla A|\,|\nabla u|^2 + f|A|\,|\nabla u| + |u|\,|A|\,|\nabla A|\,|\nabla u|\,) \\ &\quad + C(\,|A|\,|\nabla u| + |A|^2\,|u| + |\Delta u|\,)\,|\nabla u|^2 \\ &\le f^2 + \Delta(|\nabla u|^2) + C(1 + |\nabla A|^2 + |\nabla u|^4)\ .\end{aligned}$$

This gives the desired inequality. □

PROPOSITION 8. *u_ϵ is uniformly bounded in $W^{2,2}_{loc}(\Omega, \mathbf{C})$.*

Proof. Given $\delta > 0$, there is a number $R > 0$ so that

$$\int_{B(x,R)\cap\Omega} |\nabla u_*|^2 < \frac{\delta}{4}$$

for each $x \in \Omega$. Since $u_\epsilon \to u_*$ in H, for all sufficiently small ϵ

$$\int_{B(x,R)\cap\Omega} |\nabla u_\epsilon|^2 \leq \int_{B(x,R)\cap\Omega} 2|\nabla u_*|^2 + 2|\nabla u_\epsilon - \nabla u_*|^2 \leq \delta \tag{16}$$

for each $x \in \Omega$. Let $K \subset\subset \Omega$. Fix a point $x \in K$ and choose $r < \min\{R,\ \mathrm{dist}(K, \partial\Omega)\}$. Let ζ be a smooth function with support in $B(x,r)$ satisfying $0 \leq \zeta \leq 1$ and $\zeta = 1$ on $B(x, r/2)$. Furthermore, we may assume that for each $h \in L^1(\Omega)$

$$\Big| \int_\Omega h|\nabla\zeta|^2 \Big| \ , \ \Big| \int_\Omega h(\Delta\zeta) \Big| \ < \ C_r \int_\Omega |h|.$$

Multiplying (15) by ζ^2, we obtain

$$\begin{aligned}
\int_\Omega f_\epsilon^2\zeta^2 &\leq C_r\Big(1 + \int_\Omega |\nabla u_\epsilon|^2\Delta\zeta^2 + \int_\Omega |\nabla A_\epsilon|^2\zeta^2 + \int_\Omega |\nabla u_\epsilon|^4\zeta^2\Big) \\
&\leq C_r\Big(1 + \int_\Omega 2|\nabla u_\epsilon|^2\zeta\Delta\zeta + \int_\Omega 2|\nabla u_\epsilon|^2|\nabla\zeta|^2 + \int_\Omega |\nabla u_\epsilon|^4\zeta^2\Big) \\
&\leq C_r\Big(1 + \int_\Omega |\nabla u_\epsilon|^4\zeta^2\Big).
\end{aligned} \tag{17}$$

Since $W^{1,1}(\Omega)$ is embedded in $L^2(\Omega)$, we see that for each $h \in W^{1,1}(\Omega)$

$$\Big(\int_\Omega h^2\Big)^{1/2} \leq C \int_\Omega |h| + |\nabla h|.$$

Let $h = |\nabla u_\epsilon|^2\zeta$. Then

$$\begin{aligned}
\int_\Omega |\nabla u_\epsilon|^4\zeta^2 &\leq C_r\Big(\int_\Omega |\nabla u_\epsilon|^2\zeta + |\nabla u_\epsilon|^2|\nabla\zeta| + |\nabla u_\epsilon| f_\epsilon\zeta\Big)^2 \\
&\leq C_r\Big(1 + \int_\Omega |\nabla u_\epsilon| f_\epsilon\zeta\Big)^2.
\end{aligned} \tag{18}$$

Consequently, by (16), (17), (18) and Hölder inequality we obtain

$$\int_\Omega f_\epsilon^2\zeta^2 \le C_r\Big(1+\delta\int_\Omega f_\epsilon^2\zeta^2\Big).$$

Therefore if δ is sufficiently small, we conclude that

$$\int_{B(x_0,r/2)} f_\epsilon^2 \le \int_\Omega f_\epsilon^2\zeta^2 \le C_r.$$

Hence u_ϵ is uniformly bounded in $W^{2,2}_{loc}(\Omega,\mathbf{C})$. □

LEMMA 9. *For all* $\epsilon > 0$,

$$\int_{\partial\Omega}\Big|\frac{\partial u_\epsilon}{\partial\nu}\Big|^2 \le C.$$

Here C *is independent of* ϵ *and* ν *is the outward unit normal vector field on* $\partial\Omega$.

Proof. Let $U=(U_1,U_2)$ be a smooth vector field on Ω so that $U=\nu$ on $\partial\Omega$. We note that by the uniform boundedness of u_ϵ in H,

$$\begin{aligned}2\mathrm{Re}\int_\Omega \Delta u_\epsilon(U\cdot\nabla\overline{u}_\epsilon) &= 2\int_{\partial\Omega}\Big|\frac{\partial u_\epsilon}{\partial\nu}\Big|^2 - \int_\Omega U\cdot\nabla(|\nabla u_\epsilon|^2)+O(1)\\ &= 2\int_{\partial\Omega}\Big|\frac{\partial u_\epsilon}{\partial\nu}\Big|^2 - \int_{\partial\Omega}|\nabla u_\epsilon|^2+O(1).\end{aligned}$$

On the other hand, by (6), (12) and the fact that $|u_\epsilon|=1$ on $\partial\Omega$

$$\begin{aligned}&2\mathrm{Re}\int_\Omega \Delta u_\epsilon(U\cdot\nabla\overline{u}_\epsilon)\\ &= 2\mathrm{Re}\int_\Omega 2iA_\epsilon\cdot\nabla u_\epsilon(U\cdot\nabla\overline{u}_\epsilon)+|A_\epsilon|^2u_\epsilon(U\cdot\nabla\overline{u}_\epsilon)\\ &\quad -\frac{1}{\epsilon^2}u_\epsilon(1-|u_\epsilon|^2)(U\cdot\nabla\overline{u}_\epsilon)\\ &= O(1)-\frac{1}{\epsilon^2}\int_\Omega(1-|u_\epsilon|^2)(u_\epsilon\nabla\overline{u}_\epsilon\cdot U+\overline{u}_\epsilon\nabla u_\epsilon\cdot U)\\ &= O(1)+\frac{1}{2\epsilon^2}\int_\Omega U\cdot\nabla(1-|u_\epsilon|^2)^2\\ &= O(1)-\frac{1}{2\epsilon^2}\int_\Omega(1-|u_\epsilon|^2)^2\mathrm{div}U\\ &= O(1).\end{aligned}$$

Hence

$$\int_{\partial\Omega} 2\left|\frac{\partial u_\epsilon}{\partial \nu}\right|^2 - |\nabla u_\epsilon|^2 = O(1).$$

If we set $\tau = \nu^\perp = (-\nu_2, \nu_1)$, the tangent vector to $\partial\Omega$, then $|\nabla u_\epsilon|^2 = |\frac{\partial u_\epsilon}{\partial \nu}|^2 + |\frac{\partial u_\epsilon}{\partial \tau}|^2$ on $\partial\Omega$. Since $u_\epsilon = g$ on $\partial\Omega$, we conclude that

$$\int_{\partial\Omega} \left|\frac{\partial u_\epsilon}{\partial \nu}\right|^2 = \int_{\partial\Omega} \left|\frac{\partial g}{\partial \tau}\right|^2 + O(1) = O(1). \qquad \square$$

PROPOSITION 10. *u_ϵ is uniformly bounded in $W^{2,2}(\Omega, \mathbf{C})$.*

Proof. In view of Proposition 8, it is enough to compute uniform estimates of u_ϵ in $W^{2,2}(\Omega, \mathbf{C})$ near the boundary of Ω.

Let $x_0 \in \partial\Omega$. From the smoothness of $\partial\Omega$, changing coordinates if necessary, we can find a smooth function $h : \mathbf{R} \to \mathbf{R}$ with $h(0) = x_0$ such that for some $r > 0$, $\Omega \cap B(x_0, r) = \{x \in B(x_0, r) | x_2 > h(x_1)\}$. Furthermore, there is $r' > 0$ so that the image of $U = \{y \in B(0, r') | y_2 > 0\}$ lies in $\Omega \cap B(x_0, r)$ diffeomorphically under the correspondence $(x_1, x_2) \leftrightarrow (x_1, x_2 - h(x_1)) = (y_1, y_2) \in U$.

Let $\tilde{u}_\epsilon(y_1, y_2) = u_\epsilon(y_1, y_2 + h(y_1))$ and $\tilde{A}_\epsilon(y_1, y_2) = A_\epsilon(y_1, y_2 + h(y_1))$. It is easy to check that if (u_ϵ, A_ϵ) is a weak solution of (P), then $(\tilde{u}_\epsilon, \tilde{A}_\epsilon)$ is a weak solution of

(19)

$$\begin{cases} L\tilde{u}_\epsilon = 2ib \cdot \nabla \tilde{u}_\epsilon + |\tilde{A}_\epsilon|^2 \tilde{u}_\epsilon - \frac{1}{\epsilon^2}(1 - |\tilde{u}_\epsilon|^2)\tilde{u}_\epsilon & \text{in } U \\ L\tilde{A}_\epsilon = \frac{i}{2}(\bar{\tilde{u}}_\epsilon \tilde{D}_{\tilde{A}_\epsilon} \tilde{u}_\epsilon - \tilde{u}_\epsilon \overline{\tilde{D}_{\tilde{A}_\epsilon} \tilde{u}_\epsilon}) & \text{in } U \\ (\tilde{u}_\epsilon, \tilde{A}_\epsilon) = (\tilde{u}_0, \tilde{A}_0) & \text{on } \{x_2 = 0\} \cap \partial U, \end{cases}$$

where

$$\begin{cases} L = \sum_{i,j} \frac{\partial}{\partial y_j}(a_{ij} \frac{\partial}{\partial y_i}), \quad a_{11} = 1, \ a_{12} = a_{21} = -h', \ a_{22} = 1 + h'^2, \\ b = (b_1, b_2), \quad b_1 = \tilde{A}_1, \ b_2 = \tilde{A}_2 - h'\tilde{A}_1, \\ \tilde{D}_{\tilde{A}_\epsilon} = \tilde{\nabla} - i\tilde{A}_\epsilon, \\ \nabla = (\partial_{y_1}, \partial_{y_2}), \quad \tilde{\nabla} = (\partial_{y_1}, (1 - h')\partial_{y_2}). \end{cases}$$

Let $\tilde{f}_\epsilon^2 = \sum_{j,k} |(\tilde{u}_\epsilon)_{y_j y_k}|^2$. We want to get an inequality of type (15) for $\tilde{f}_\epsilon^2$. Since the right hand side of (19-2) belongs to $W^{2,2}(U, \mathbf{R}^2)$, by elliptic estimates we have $\|\tilde{A}_\epsilon\|_{W^{2,2}(U,R^2)} \leq C$ and hence $\|\tilde{A}_\epsilon\|_{L^\infty(U,R^2)} \leq$

C. For simplicity we drop the subscript ϵ and write u instead of $\tilde{u}$ and so on. By direct calculations, we see that

$$L(|\nabla u|^2) = 2a_{ij}u_{y_iy_k}\overline{u}_{y_jy_k} + 2\mathrm{Re}[\overline{u}_{y_k} \cdot L(u_{y_k})]. \tag{20}$$

Differentiating the first equation of (19), we find that

(21)

$$\begin{aligned}
&2\mathrm{Re}[\overline{u}_{y_k}L(u_{y_k})] \\
&= 2\mathrm{Re}[-\overline{u}_{y_k}((a_{ij})_{y_k}u_{y_i})_{y_j} + \overline{u}_{y_k}(2ib\cdot\nabla u + |A|^2u - \frac{1}{\epsilon^2}(1-|u|^2)u\ )_{y_k}] \\
&= -2\mathrm{Re}[\overline{u}_{y_k}((a_{ij})_{y_k}u_{y_i})_{y_j}] + 2\mathrm{Re}[2i(b_j)_{y_k}u_{y_j}\overline{u}_{y_k} + 2ib_ju_{y_jy_k}\overline{u}_{y_k}] \\
&\quad + 2|A|^2|\nabla u|^2 + 4\mathrm{Re}[uA_j\nabla A_j\cdot\nabla\overline{u}] + \frac{1}{\epsilon^2}|\overline{u}\nabla u + u\nabla\overline{u}|^2 \\
&\quad + \frac{2}{u}(Lu - 2ib\cdot\nabla u - |A|^2u)|\nabla u|^2.
\end{aligned}$$

Since $\|A\|_{L^\infty(\Omega)} \le C$ and $|Lu| \le C(f + |\nabla u|)$, by (20) and (21)

$$\begin{aligned}
L(|\nabla u|^2) &\ge 2\theta f^2 - C|\nabla u|(|\nabla u| + f) - C(|\nabla b||\nabla u|^2 + f|b||\nabla u|) \\
&\qquad - C|u||A||\nabla A||\nabla u| - C(|b||\nabla u| + |A|^2|u| + |Lu|)|\nabla u|^2 \\
&\ge \theta f^2 - C(1 + |\nabla A|^2 + |\nabla u|^4).
\end{aligned}$$

Here θ is the ellipticity constant of L. Thus

$$f^2 \le C\Big(L(|\nabla u|^2) + 1 + |\nabla A|^2 + |\nabla u|^4\Big).$$

Choosing a test function ζ with support in $B(0,r')$ satisfying $0 \le \zeta \le 1$ and $\zeta = 1$ on $B(0, r'/2)$, we are led to

$$\int_U f^2\zeta^2 \le C\Big(1 + \int_U |\nabla u|^4\zeta^2 + L(|\nabla u|^2)\zeta^2\Big).$$

If the last term of the righthand side is uniformly bounded, then

$$\int_U f^2\zeta^2 \le C\int_U 1 + |\nabla u|^4\zeta^2.$$

Using the same arguments used in the proof of Proposition 8, we conclude that

$$\int_U f^2\zeta^2 \le C.$$

which proves the proposition.

It remains to show that

$$\int_U L(|\nabla u|^2)\zeta^2 \le C. \tag{22}$$

First, we observe by integration by parts that

$$\begin{aligned}\int_U L(|\nabla u|^2)\zeta^2 = \int_U |\nabla u|^2 L\zeta^2 + 2\int_{\{y_2=0\}} a_{12}|\nabla u|^2(\zeta^2)_{y_1} \\ + \int_{\{y_2=0\}} (a_{12})_{y_1}|\nabla u|^2\zeta^2 + \int_{\{y_2=0\}} a_{22}|\nabla u|^2(\zeta^2)_{y_2} \\ - \int_{\{y_2=0\}} a_{22}(|\nabla u|^2)_{y_2}\zeta^2.\end{aligned}$$

Invoking Proposition 5 and Lemma 9, we see that all the integrals on RHS are uniformly bounded except for the last term. On the other hand, it follows from integration by parts that

$$\begin{aligned}&\int_{\{y_2=0\}} a_{22}(|\nabla u|^2)_{y_2}\zeta^2 \\ &= 2\mathrm{Re}\int_{\{y_2=0\}} a_{22}\overline{u}_{y_1}u_{y_1y_2}\zeta^2 + a_{22}\overline{u}_{y_2}u_{y_2y_2}\zeta^2 \\ &= 2\mathrm{Re}\Big[-\int_{\{y_2=0\}}(a_{22}\zeta^2)_{y_1}\overline{g}_{y_1}u_{y_2} - \int_{\{y_2=0\}} a_{22}\overline{g}_{y_1y_1}u_{y_2}\zeta^2 \\ &\quad + \int_{\{y_2=0\}} a_{22}\overline{u}_{y_2}u_{y_2y_2}\zeta^2\Big].\end{aligned}$$

The first two terms are uniformly bounded by Lemma 9. To estimate the last term, we note that on $\{y_2 = 0\}$, by (19)

$$\begin{aligned}a_{22}u_{y_2y_2} = &-(a_{11}u_{y_1})_{y_1} - (a_{21}u_{y_2})_{y_1} \\ &-(a_{12}u_{y_1})_{y_2} - (a_{22})_{y_2}u_{y_2} + 2ib\cdot\nabla u + |A|^2u.\end{aligned}$$

Now using Lemma 9 and integrating by parts, we find that

$$
\begin{aligned}
&\operatorname{Re}\int_{\{y_2=0\}} a_{22}\overline{u}_{y_2}u_{y_2y_2}\zeta^2 \\
&= \operatorname{Re}\int_{\{y_2=0\}} -(a_{11}g_{y_1})_{y_1}\overline{u}_{y_2}\zeta^2 - (a_{21}u_{y_2})_{y_1}\overline{u}_{y_2}\zeta^2 - (a_{12}u_{y_1})_{y_2}\overline{u}_{y_2}\zeta^2 \\
&\qquad - (a_{22})_{y_2}|u_{y_2}|^2\zeta^2 + 2i(b\cdot\nabla u)\overline{u}_{y_2}\zeta^2 + |A|^2u\overline{u}_{y_2}\zeta^2 \\
&= O(1) - \operatorname{Re}\int_{\{y_2=0\}} (a_{21}u_{y_2})_{y_1}\overline{u}_{y_2}\zeta^2 + (a_{12}u_{y_1})_{y_2}\overline{u}_{y_2}\zeta^2 \\
&= O(1) - 2\operatorname{Re}\int_{\{y_2=0\}} a_{12}u_{y_1y_2}\overline{u}_{y_2}\zeta^2 \\
&= O(1) + \int_{\{y_2=0\}} (a_{12}\zeta^2)_{y_1}|u_{y_2}|^2 \\
&= O(1).
\end{aligned}
$$

Thus (22) is proved. □

THEOREM 11. *Let (u_ϵ, A_ϵ) be any solutions of* (1) *and* (2) *which are minimizers of $E_\epsilon(u, A)$ over the space X. Then we have*

$$u_\epsilon \longrightarrow u_* \quad in \ C^{1,\alpha}(\overline{\Omega}, \mathbf{C})$$

and

$$A_\epsilon \longrightarrow A_* \quad in \ C^{1,\alpha}(\overline{\Omega}, \mathbf{R}^2)$$

for all $0 < \alpha < 1$.

Proof. We note that by Proposition 10, (∇u_ϵ) is uniformly bounded in $L^p(\Omega, \mathbf{C}^2)$ for all $p > 2$ and so is (ΔA_ϵ) in $L^p(\Omega, \mathbf{R}^2)$ by (7). This verifies the assertion for A_ϵ. Let

$$w_\epsilon = \frac{1}{\epsilon^2}(1 - |u_\epsilon|^2).$$

Then, since $|u_\epsilon| \geq 1/2$, (14) reads

$$-\epsilon^2\Delta w_\epsilon + w_\epsilon \leq 2|D_{A_\epsilon}u_\epsilon|^2. \tag{23}$$

Multiplying (23) by w_ϵ^{p-1} and using Hölder inequality, we find that

$$\int_\Omega w_\epsilon^p \leq 2\int_\Omega w_\epsilon^{p-1}|D_{A_\epsilon}u_\epsilon|^2 \leq C_p\|w_\epsilon\|_{L^p(\Omega)}^{p-1}.$$

Consequently, by (6) we conclude that (u_ϵ) is uniformly bounded in $W^{2,p}(\Omega, \mathbf{C})$ for all $p > 2$, which completes the proof. □

4. Convergences in higher derivatives

Since $g : \partial\Omega \to S^1$ is of degree 0, there is a smooth function $\varphi_0 : \partial\Omega \to \mathbf{R}$ so that $g = e^{i\varphi_0}$. We also denote by φ_0 its harmonic extension in Ω. It is easily seen (BBH1) that $u_* = e^{i\varphi_0}$.

THEOREM 12. *We have*
(i) $\|u_\epsilon - u_*\|_{L^\infty(\Omega)} \leq C\epsilon^2$,
(ii) $\|A_\epsilon - A_*\|_{L^\infty(\Omega)} \leq C\epsilon^{3/2}$.

Proof. We investigate that by the maximum principle (see Theorem 3.7 [5]), (23) yields

$$\|w_\epsilon\|_{L^\infty} \leq C.$$

Hence if we let $u_\epsilon = \rho_\epsilon e^{i\varphi_\epsilon}$ with $\rho_\epsilon = |u_\epsilon|$ (this is well defined because $|u_\epsilon| \geq \frac{1}{2}$), then

$$\|1 - \rho_\epsilon\|_{L^\infty(\Omega)} \leq C\epsilon^2. \tag{24}$$

Take the imaginary part of (6) to obtain

$$\rho_\epsilon \Delta\varphi_\epsilon + 2\nabla\rho_\epsilon \cdot \nabla\varphi_\epsilon = 2A_\epsilon \cdot \nabla\rho_\epsilon. \tag{25}$$

Multiplying ρ_ϵ on both sides of (25) and using the fact that $\Delta\varphi_0 = 0$ and $\mathrm{div}A_\epsilon = 0$, we find that

$$\begin{cases} -\Delta(\varphi_\epsilon - \varphi_0) = \mathrm{div}((\rho_\epsilon^2 - 1)(\nabla\varphi_\epsilon - A_\epsilon)) & \text{in } \Omega \\ \varphi_\epsilon - \varphi_0 = 0 & \text{on } \partial\Omega. \end{cases} \tag{26}$$

Now elliptic estimates (see Chapter 8 [5]) applied to (26) yield

$$\begin{aligned} \|\varphi_\epsilon - \varphi_0\|_{L^\infty(\Omega)} &\leq C\|(\rho_\epsilon^2 - 1)(\nabla\varphi_\epsilon - A_\epsilon)\|_{L^\infty(\Omega)} \\ &\leq C\|\rho_\epsilon^2 - 1\|_{L^\infty(\Omega)} \leq C\epsilon^2. \end{aligned} \tag{27}$$

Hence

$$|u_\epsilon - u_*| \le |(\rho_\epsilon - 1)e^{i\varphi_\epsilon}| + |e^{i\varphi_\epsilon} - e^{i\varphi_0}|$$
$$\le |\rho_\epsilon - 1| + |\varphi_\epsilon - \varphi_0| \le C\epsilon^2,$$

and (i) is proved.

To prove (ii) we now apply Lemma A.2 to (6) and (9) to obtain

$$\|\nabla u_\epsilon - \nabla u_*\|^2_{L^\infty(\Omega)} \le C\|u_\epsilon - u_*\|_{L^\infty(\Omega)} \le C\epsilon^2.$$

Since

$$\nabla u_\epsilon - \nabla u_* = e^{i\varphi_\epsilon}(\nabla \rho_\epsilon + i\rho_\epsilon \nabla \varphi_\epsilon) - ie^{i\varphi_0}\nabla\varphi_0,$$

we have

(28)

$$|\nabla\rho_\epsilon| \le |\nabla u_\epsilon - \nabla u_*| + |(\rho_\epsilon - 1)\nabla\varphi_\epsilon| + |\varphi_\epsilon - \varphi_0|\,|\nabla\varphi_\epsilon| + |\nabla\varphi_\epsilon - \nabla\varphi_0|$$
$$\le C\epsilon + C\epsilon^2 + |\nabla\varphi_\epsilon - \nabla\varphi_0|.$$

Next, we rewrite (25) as

$$\Delta(\varphi_\epsilon - \varphi_0) = -\frac{2}{\rho_\epsilon}\nabla\rho_\epsilon \cdot \nabla\varphi_\epsilon + \frac{2}{\rho_\epsilon}A_\epsilon \cdot \nabla\rho_\epsilon. \tag{29}$$

Since $\varphi_\epsilon - \varphi_0 = 0$ on $\partial\Omega$, by Lemma A.2, (27) and (28) we have

$$\|\nabla\varphi_\epsilon - \nabla\varphi_0\|^2_{L^\infty(\Omega)} \le C\|\nabla\rho_\epsilon\|_{L^\infty(\Omega)}\|\varphi_\epsilon - \varphi_0\|_{L^\infty(\Omega)}$$
$$\le C\epsilon^2(C\epsilon + \|\nabla\varphi_\epsilon - \nabla\varphi_0\|_{L^\infty(\Omega)}).$$

Thus we obtain

$$\|\nabla\varphi_\epsilon - \nabla\varphi_0\|_{L^\infty(\Omega)} \le C\epsilon^{3/2}. \tag{30}$$

On the other hand, equations (7) and (10) read

$$\begin{cases} \Delta A_\epsilon &= \rho_\epsilon^2 A_\epsilon + \mathrm{Re}(i\overline{u}_\epsilon \nabla u_\epsilon) \\ \Delta A_* &= A_* + \mathrm{Re}(i\overline{u}_* \nabla u_*). \end{cases} \tag{31}$$

Since by (24) and (30)

$$|\mathrm{Re}(i\overline{u}_\epsilon \nabla u_\epsilon) - \mathrm{Re}(i\overline{u}_* \nabla u_*)| \le |(\rho_\epsilon^2 - 1)\nabla\varphi_\epsilon| + |\nabla\varphi_\epsilon - \nabla\varphi_0| \le C\epsilon^{3/2},$$

it follows from Lemma A.3 that

$$\|A_\epsilon - A_*\|_{L^\infty(\Omega)} \le C\epsilon^{3/2}. \qquad \square$$

LEMMA 13. *We have*
(i) $\|\nabla\varphi_\epsilon\|_{C^k_{loc}(\Omega)} \leq C$,
(ii) $\|A_\epsilon\|_{C^k_{loc}(\Omega)} \leq C$,
(iii) $\left\|\frac{1-\rho_\epsilon}{\epsilon^2}\right\|_{C^k_{loc}(\Omega)} \leq C$.

Proof. We use inductions on k. The case $k = 0$ follows from Theorem 11 and (24). We take real and imaginary parts of (6) and (7) to obtain

$$\Delta\rho_\epsilon = \rho_\epsilon|\nabla\varphi_\epsilon|^2 - 2\rho_\epsilon A_\epsilon \cdot \nabla\varphi_\epsilon + |A_\epsilon|^2\rho_\epsilon - \frac{1}{\epsilon^2}(1-\rho_\epsilon^2)\rho_\epsilon \tag{32}$$

$$\Delta\varphi_\epsilon = -\frac{2}{\rho_\epsilon}\nabla\rho_\epsilon \cdot \nabla\varphi_\epsilon + \frac{2}{\rho_\epsilon}A_\epsilon \cdot \nabla\rho_\epsilon \tag{33}$$

$$\Delta A_\epsilon = \rho_\epsilon^2 A_\epsilon - \rho_\epsilon^2 \nabla\varphi_\epsilon. \tag{34}$$

It comes from (32) that $(\Delta\rho_\epsilon)$ is uniformly bounded in $C^k_{loc}(\Omega)$ and hence

$$\|\rho_\epsilon\|_{W^{k+2,p}_{loc}(\Omega)} \leq C_p \quad \forall p < \infty$$

and

$$\|\nabla\rho_\epsilon\|_{C^k_{loc}(\Omega)} \leq C. \tag{35}$$

Similar arguments applied to (33) deduce

$$\|\varphi_\epsilon\|_{W^{k+2,p}_{loc}(\Omega)} \leq C_p \quad \text{and} \quad \|A_\epsilon\|_{W^{k+2,p}_{loc}(\Omega)} \leq C_p \quad \forall p < \infty.$$

Therefore

$$\|A_\epsilon\|_{C^{k+1}_{loc}(\Omega)} \leq C. \tag{36}$$

Now $(\Delta\varphi_\epsilon)$ is uniformly bounded in $W^{k+1,p}_{loc}(\Omega)$ so that

$$\|\varphi_\epsilon\|_{W^{k+3,p}_{loc}(\Omega)} \leq C_p \quad \forall p < \infty.$$

In particular,

$$\|\nabla\varphi_\epsilon\|_{C^{k+1}_{loc}(\Omega)} \leq C. \tag{37}$$

For the proof of (iii) let $w_\epsilon = (1-\rho_\epsilon)/\epsilon^2$ and $\Omega'' \subset\subset \Omega' \subset\subset \Omega$ be given. We rewrite (32) as

(38) $-\epsilon^2 \Delta w_\epsilon = \rho_\epsilon |\nabla \varphi_\epsilon|^2 - 2\rho_\epsilon A_\epsilon \cdot \nabla\varphi_\epsilon + |A_\epsilon|^2 \rho_\epsilon - \rho_\epsilon(1+\rho_\epsilon) w_\epsilon \equiv f_\epsilon.$

We take ∂^k on both sides and use Lemma A.1 to get

$$\|\partial^{k+1} w_\epsilon\|^2_{L^\infty(\Omega'')} \leq C\|\partial^k w_\epsilon\|_{L^\infty(\Omega')}(\|\Delta\partial^k w_\epsilon\|_{L^\infty(\Omega')} + \|\partial^k w_\epsilon\|_{L^\infty(\Omega')}).$$

By induction assumptions (i) - (iii), we have

$$\|\partial^k w_\epsilon\|_{L^\infty(\Omega')} \leq C$$

and

$$\|\Delta\partial^k w_\epsilon\|_{L^\infty(\Omega')} = \|\frac{1}{\epsilon^2}\partial^k f_\epsilon\|_{L^\infty(\Omega')} \leq \frac{C}{\epsilon^2}.$$

Thus we conclude that

$$\|\epsilon w_\epsilon\|_{C^{k+1}_{loc}(\Omega)} \leq C. \tag{39}$$

We rewrite (38) as

(40)

$$-\epsilon^2\Delta w_\epsilon + 2w_\epsilon = \rho_\epsilon|\nabla\varphi_\epsilon|^2 - 2\rho_\epsilon A_\epsilon \cdot \nabla\varphi_\epsilon + |A_\epsilon|^2\rho_\epsilon + 3\epsilon^2 w_\epsilon^2 - \epsilon^4 w_\epsilon^3 \equiv h_\epsilon.$$

Then (36), (37) and (39) give

$$\|\epsilon w_\epsilon\|_{C^{k+1}(\overline{\Omega'})} \leq C_0$$
$$\|h_\epsilon\|_{C^{k+1}(\overline{\Omega'})} \leq C_0,$$

where $C_0 = C_0(\Omega')$. Differentiating (40) $(k+1)$-times, we find that

$$-\epsilon^2\Delta\partial^{k+1}w_\epsilon + 2\partial^{k+1}w_\epsilon = \partial^{k+1}h_\epsilon.$$

Let $\tilde{w} = \partial^{k+1}w_\epsilon - C_0$. Then we have

$$-\epsilon^2\Delta\tilde{w} + 2\tilde{w} \leq 0 \quad \text{on } \Omega'$$
$$\tilde{w} \leq \frac{C_0}{2\epsilon} \quad \text{on } \overline{\Omega'}.$$

Applying Lemma A.4 we conclude that

$$\|\partial^{k+1}w_\epsilon\|_{L^\infty(\Omega'')} \leq C_0 + \frac{C_0}{2\epsilon}e^{-d/4\epsilon}$$

where $d = \mathrm{dist}(\Omega'', \partial\Omega')$. Consequently

$$\|w_\epsilon\|_{C^{k+1}_{loc}(\Omega)} \leq C,$$

and the proof is completed. □

THEOREM 14. *We have*
(i) $\|u_\epsilon - u_*\|_{C^k_{loc}(\Omega)} \le C\epsilon^2$,
(ii) $\|A_\epsilon - A_*\|_{C^k_{loc}(\Omega)} \le C\epsilon^2$.

Proof. We claim that

$$\|\varphi_\epsilon - \varphi_0\|_{C^k_{loc}(\Omega)} \le C\epsilon^2. \tag{41}$$

We use inductions on k. The case $k = 0$ follows from (27). From (29), (35), Lemma 13 and Lemma A.1, we find that

$$\begin{aligned}
&\|\varphi_\epsilon - \varphi_0\|^2_{C^{k+1}_{loc}(\Omega)} \\
&\le C\|\varphi_\epsilon - \varphi_0\|_{C^k_{loc}(\Omega)} \\
&\quad \times \Big(\|\varphi_\epsilon - \varphi_0\|_{C^k_{loc}(\Omega)} + \| - (\nabla\rho_\epsilon \cdot \nabla\varphi_\epsilon)/\rho_\epsilon + (A_\epsilon \cdot \nabla\rho_\epsilon)/\rho_\epsilon\|_{C^k_{loc}(\Omega)}\Big) \\
&\le C\epsilon^4.
\end{aligned}$$

It follows from Lemma 13 and (41) that

$$\|u_\epsilon - u_*\|_{C^k_{loc}(\Omega)} \le \|(\rho_\epsilon - 1)e^{i\varphi_\epsilon}\|_{C^k_{loc}(\Omega)} + \|e^{i\varphi_\epsilon} - e^{i\varphi_0}\|_{C^k_{loc}(\Omega)} \le C\epsilon^2.$$

On the other hand (41) says that

$$\begin{aligned}
&\|\mathrm{Re}(i\overline{u}_\epsilon \nabla u_\epsilon) - \mathrm{Re}(i\overline{u}_* \nabla u_*)\|_{C^k_{loc}(\Omega)} \\
&\le \|(\rho_\epsilon^2 - 1)\nabla\varphi_\epsilon\|_{C^k_{loc}(\Omega)} + \|\nabla\varphi_\epsilon - \nabla\varphi_0\|_{C^k_{loc}(\Omega)} \le C\epsilon^2 .
\end{aligned}$$

Now Lemma A.3 applied to (31) verifies (ii). □

APPENDIX. In this section we state and prove several lemmas used essentially in the proofs in the context. The proofs of lemma A.1 and A.2 can be found in [1]. Throughout this section Ω denotes a smooth bounded domain in $\mathbf{R}^N$.

LEMMA A.1. *Suppose that*

$$-\Delta u = f \quad in\ \Omega.$$

Then for each $K \subset\subset \Omega$

$$\|\nabla u\|^2_{L^\infty(K)} \le C_K(\|f\|_{L^\infty(\Omega)}\|u\|_{L^\infty(\Omega)} + \|u\|^2_{L^\infty(\Omega)})$$

where C_K *depends only on* N *and* K.

LEMMA A.2. *Suppose that*

$$-\Delta u = f \quad in \quad \Omega.$$
$$u = 0 \quad on \quad \partial\Omega$$

Then

$$\|\nabla u\|^2_{L^\infty(\Omega)} \leq C\|f\|_{L^\infty(\Omega)}\|u\|_{L^\infty(\Omega)}$$

where C depends only on N and Ω.

LEMMA A.3. *Let*

$$\Delta u_\epsilon + c_\epsilon u_\epsilon = f_\epsilon \quad in \;\; \Omega$$
$$\Delta u_0 + cu_0 = f \quad in \;\; \Omega$$
$$u_\epsilon - u_0 = 0 \quad on \;\; \partial\Omega.$$

Suppose that for some C independent of ϵ

$$\|c_\epsilon - c\|_{L^\infty(\Omega)} \leq C\epsilon$$
$$\|f_\epsilon - f\|_{L^\infty(\Omega)} \leq C\epsilon.$$

Then

$$\|u_\epsilon - u_0\|_{L^\infty(\Omega)} \leq C\epsilon.$$

Proof. We see that

$$\Delta(u_\epsilon - u_0) + c_\epsilon(u_\epsilon - u_0) = -(c_\epsilon - c)u_0 + (f_\epsilon - f).$$

By elliptic estimates

$$\|u_\epsilon - u_0\|_{L^\infty(\Omega)} \leq C(\|(c_\epsilon - c)u_0\|_{L^\infty(\Omega)} + \|f_\epsilon - f\|_{L^\infty(\Omega)}) \leq C\epsilon. \quad \square$$

LEMMA A.4. *Let $u(r)$ be the solution of*

$$-\epsilon^2\Delta u + 2u = 0 \quad in \; B(0,R)$$
$$u = 1 \quad on \; \partial B(0,R)$$

Then for $\epsilon < 2R$, we have

$$u(r) \leq e^{\frac{1}{4\epsilon R}(r^2-R^2)} \quad on \;\; B(0,R).$$

Proof. A direct computation shows that $e^{\frac{1}{4\epsilon R}(r^2-R^2)}$ is a supersolution. $\square$

ACKNOWLEDGEMENTS. The author would like to thank to Professor Dongho Chae for his hearty encouragements and many helpful discussions. This work was partially supported by GARC-KOSEF.

References

[1] F. Bethuel, H. Brezis and F. Hélein, *Asymptotics for the minimization of a Ginzburg-Landau functional*, Calc. Var. and PDE **1** (1993), 123-148.
[2] ———, *Ginzburg-Landau Vortices*, Birkhäuser, Boston, 1994.
[3] F. Bethuel and T. Riviére, *Vortices for a variational problem related to superconductivity*, Ann. Inst. Henri Poincaré -Anal. non Linéaire **12** (1995), 243-303.
[4] Q. Du, M. D. Gunzburger, and J. S. Peterson, *Analysis and Approximation of the Ginzburg-Landau Model of Superconductivity*, SIAM Review **34** (1992), 54-81.
[5] D. Gilbarg and N. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Second Ed., Springer-Verlag, Berlin, Heidelberg, New York, 1983.
[6] F.-H. Lin, *Static and Moving Vortices in Ginzburg-Landau Theories*, Nonlinear Partial Differential Equations in Geometry and Physics edited by G. Baker and A. Freire, Birkhäuser Verlag, Basel, 1997.
[7] Y. Yang, *Boundary value problems of the Ginzburg-Landau equations*, Proc. Royal Soc. Edingburg **114A** (1990), 355-365.

Department of Mathematics
Seoul National University
Seoul 151-742, Korea
E-mail: jmhan@math.snu.ac.kr