J. Korean Math. Soc. **35** (1998), No. 4, pp. 981–997

OPTIMAL GEVREY EXPONENTS FOR SOME DEGENERATE ELLIPTIC OPERATORS

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ABSTRACT. We shall show first general Métivier operators $D_y^2 + (x^{2l} + y^{2k})D_x^2,\ l, k = 1, 2, \cdots$, have $G_{x,y}^{\{\theta,d\}}$-hypoellipticity in the vicinity of the origin $(0,0)$, where $\theta = \frac{l(1+k)}{l(1+k)-k}, d = \frac{\theta+k}{1+k}\ (> 1)$, and finally the optimality of these exponents $\{\theta, d\}$ will be shown.

0. Introduction

The aim of this paper is to determine the optimal non-isotropic exponents of Gevrey hypoellipticity for the general Métivier operators $D_y^2 + (x^{2l} + y^{2k})D_x^2, l, k = 1, 2, \cdots$, in the vicinity of the origin $(0,0) \in \mathbf{R}^2$. We shall give the precise definition of the Gevrey spaces at the beginning of §1.

In the paper [11], we have considered Gevrey hypoellipticity for a class of degenerate elliptic operators now called Grushin operators. We treated them essentially dividing into three groups. Those operators in the first group are analytic hypoelliptic in the space of hyperfunctions, and the operators in the second and the third group are Gevrey hypoelliptic in the ultradistribution spaces in a neighborhood of the origin. The typical examples in the first group are given by $D_y^2 + y^{2k}D_x^2,\ l, k, = 1, 2, \cdots$. The second group is represented by the general Métivier operators $D_y^2 + (x^{2l} + y^{2k})D_x^2,\ l, k = 1, 2, \cdots$, which have $G^{\{\theta\}}$-hypoellipticity with $\theta = \frac{l(1+k)}{l(1+k)-k} > 1$ and the third group is represented by the general Baouendi-Goulaouic operators $D_y^2 + y^{2k}D_x^2 + y^{2l}D_z^2,\ k > l \geq 0$, which have $G^{\{\mu\}}$-hypoellipticity with $\mu = \frac{1+k}{1+l} > 1$ by the results of [11].

Received May 20, 1998.
1991 Mathematics Subject Classification: 34, 35, 46.
Key words and phrases: Gevrey hypoellipticity, non-isotropic Gevrey exponents.
The author was partially supported by The Nitto Foundation.
This work is based on the invited talk at the annual meeting of the Korean Mathematical Society, 1997.

On the other hand, in the paper [2], A. Bove and D. Tartakoff proved that the above Baouendi-Goulaouic operators have non-isotropic Gevrey hypoellipticity in the space $G^{\{d_1,d_2,d_3\}}_{x,y,z}$, where $d_1 = \frac{1+k}{1+l} \equiv \mu, d_2 = \frac{\mu+k}{1+k}, d_3 = 1$ in a neighborhood of the origin. They also proved the optimality of the above exponents $\{d_1, d_2, d_3\}$. Their result means that Baouendi-Goulaouic operators are analytic hypoelliptic only in the z-direction. Their method of the proof is very elementary using L^2-estimates.

In this paper, in §1, we shall show the non-isotropic Gevrey hypoellipticity for the above Métivier operators in the spaces $G^{\{\theta,d\}}_{x,y}$ with $\theta = \frac{l(1+k)}{l(1+k)-k}$ and $d = \frac{\theta+k}{1+k}$ by applying the method of [2] and the result of [11], (see Theorem 1.3). We notice that $1 < d < \theta$. In §2 and §3, we shall consider the optimality of these exponents by applying the original idea of G. Métivier, [13], which means that Métivier operators are not analytic-hypoelliptic in all the direction. The proof of the optimality will be done by the contradiction generated by the assumption that the operator$D_y^2 + (x^{2l} + y^{2k})D_x^2$ has $G^{\{\theta',d'\}}_{x,y}$-hypoellipticity, $1 < \theta' < \theta, 1 < d' < d$ in a neighborhood of the origin $(0,0) \in \mathbf{R}^2$.

1. Non-isotropic hypoellipticity of the Métivier operators

DEFINITION 1.1. Let Ω be an open set in $\mathbf{R}^n$ and $\varphi \in C^\infty(\Omega)$. Then we say that $\varphi \in G^{\{\theta\}}(\Omega), \theta > 0$, if for any compact subset K of Ω there are positive constants C_0 and C_1 such that

$$\sup_{\substack{x\in K\\ \alpha}} |D^\alpha \varphi(x)| \le C_0 C_1^{|\alpha|} \alpha!^\theta, \ \alpha \in \mathbf{Z}^n_+. \tag{1.1}$$

We say that $\varphi \in G^{\{d_1,d_2,\cdots,d_n\}}(\Omega)$, $0 < d_1, d_2, \cdots, d_n$, if for any compact subset K of Ω there are positive constants C_0 and C_1 such that

$$\sup_{\substack{x\in K\\ \alpha}} |D^\alpha \varphi(x)| \le C_0 C_1^{|\alpha|} \alpha_1!^{d_1} \alpha_2!^{d_2} \cdots \alpha_n!^{d_n}, \ \alpha \in \mathbf{Z}^n_+. \tag{1.2}$$

PROPOSITION 1.1. *Let $\varphi \in C^\infty(\Omega)$. If for any compact subset K of Ω there are positive constants C_0 and C_1 such that*

$$\sup_{x\in K} |D_j^k \varphi| \le C_0 C_1^k k!^{d_j}, \ j = 1, 2, \cdots, n, \ k \in \mathbf{Z}_+, \tag{1.3}$$

then we have $\varphi \in G^{\{d_1,d_2,\cdots,d_n\}}(\Omega)$.

Now we consider the operator, (general Métivier operator)

$$P = -\{\frac{\partial^2}{\partial y^2} + (x^{2l} + y^{2k})\frac{\partial^2}{\partial x^2}\},\ l, k \in \mathbf{N}.$$

We assume $(0,0) \in \Omega \subset \mathbf{R}^2_{x,y}$ and Ω is a bounded open set in the following. We denote by $||u|| = ||u||_{L^2(\Omega)}$.

THEOREM 1.1. *If the diameter of* Ω *is sufficiently small, there is a positive constant* C *such that*

$$(1.4) \qquad ||v_{yy}|| + ||v_y|| + ||v|| + ||x^{2l}v_{xx}|| + ||y^{2k}v_{xx}|| + ||y^k v_x|| + ||y^k x^l v_{xx}|| + ||x^l v_x|| + ||y^{k-1}v_x|| \leq C||Pv||,\ v \in C_0^\infty(\Omega).$$

Proof. First we have

$$(Pv, v) = ||v_y||^2 + ||y^k v_x||^2 + ||x^l v_x||^2 + 2l(x^l v_x, x^{l-1}v),$$

from where we get the estimate of type

$$||v_y|| + ||y^k v_x|| + ||x^l v_x|| - C_1||x^{l-1}v|| \leq C_2||Pv||,\ v \in C_0^\infty(\Omega).$$

On the other hand, by the Poincaré inequality, we have for any positive number ϵ

$$\epsilon||v|| \leq ||v_y||,\ v \in C_0^\infty(\Omega),$$

if we take the diameter of Ω sufficiently small. Then we have the estimate of the kind

$$||v_y|| + ||y^k v_x|| + ||x^l v_x|| + ||v|| \leq C||Pv||,\ v \in C_0^\infty(\Omega).$$

Next, considering (Pv, v_{yy}), $(Pv, x^{2l}v_{xx})$ and $(Pv, y^{2k}v_{xx})$, we arrive at the estimate of the form

$$||v_{yy}|| + ||x^l v_{xy}|| + ||y^k v_{xy}|| + ||x^{2l}v_{xx}|| + ||x^l y^k v_{xx}|| + ||y^{2k}v_{xx}|| + ||v_y|| + ||x^l v_x|| + ||y^k v_x|| + ||v|| - C'||y^{k-1}v_x|| \leq C''||Pv||,\ v \in C_0^\infty(\Omega).$$

We can see that for some constants $A > 0$ and $B > 0$ the following inequality holds:

$$||y^{k-1}v_x|| \leq A||v_{yy} + y^{2k}v_{xx}|| \leq B(||v_{yy}|| + ||y^{2k}v_{xx}||),\ v \in C_0^\infty(\Omega),$$

(see [4]).

Hence we have with some constant C

$$||y^{k-1}v_x|| \leq C(||Pv|| + ||x^{2l}v_{xx}||),\ v \in C_0^\infty(\Omega).$$

If we take the diameter of Ω sufficiently small again, considering with $(Pv, x^{2l}v_{xx})$, we have the inequality of the kind

$$||x^{2l}v_{xx}|| \leq C(||v_{yy}|| + ||Pv||),\ v \in C_0^\infty(\Omega),$$

and we have finally the estimate (1.4). □

As a particular case of the results of [10], we have the following theorem.

THEOREM 1.2. *Let* $u \in C^\infty(\Omega)$ *and* $f \in G^{\{\theta\}}(\Omega)$ *satisfying* $Pu = f$ *in* Ω. *Then* $u \in G^{\{\theta\}}, \theta = \frac{l(1+k)}{l(1+k)-k}$.

The purpose of this section is to prove the following theorem.

THEOREM 1.3. *Let* $u \in C^\infty(\Omega)$ *and* $f \in G^{\{\theta,d\}}(\Omega)$ *satisfying* $Pu = f$ *in* Ω, *where* $\theta = \frac{l(1+k)}{l(1+k)-k}$ *and* $d = \frac{\theta+k}{1+k}$. *Then* $u \in G^{\{\theta,d\}}(\Omega)$.

Proof. First notice that $1 < d < \theta$ for $l, k = 1, 2, \cdots$. We take a function $\psi \in C_0^\infty(\Omega) \cap G^{\{d\}}$ such that $\psi \equiv 1$ in a small open set $\omega \subset \bar{\omega} \subset \Omega$. We assume that $0 \in \omega$ and we may assume $j \geq 2k$. By the inequality (1.4) we have

$$\begin{aligned}
&||D_y^2\psi D_y^j u|| \leq C||P\psi D_y^j u|| \\
&\leq C\{||\psi D_y^j Pu|| + ||[P, \psi D_y^j]u||\} \\
&\leq C\{||\psi D_y^j Pu|| + ||\psi_y'' D_y^j u|| + 2||\psi_y' D_y^{j+1}u|| + ||x^{2l}\psi_x'' D_y^j u|| \\
&\quad + 2||x^{2l}\psi_x' D_y^j D_x u|| + 2||\psi_x' y^{2k} D_y^j D_x u|| + ||\psi_x'' y^{2k} D_y^j u|| \\
&\quad + \sum_{r=1}^{2k} \binom{j}{r} \frac{(2k)!}{(2k-r)!} ||\psi y^{2k-r} D_y^{j-r} D_x^2 u||\}.
\end{aligned} \tag{1.5}$$

By assumption, the first term on the right is estimated by the quantity of the kind $C_0 C_1^j j!^d$ and we may consider the next six terms are also estimated by the same quantity since the operator P is elliptic outside of the origin and ψ' and ψ'' vanish on $\bar{\omega}$. Therefore, we have to consider each term of the last summation on the right. First take the term with $r = 2k$ which is essentially estimated by

$$\begin{aligned} &j(j-1)\cdots(j-2k+1)||\psi D_y^{j-2k}D_x^2u|| \\ &= j(j-1)\cdots(j-2k+1)||\psi D_y^2 D_y^{j-2k-2}D_x^2u|| \\ &\leq j(j-1)\ldots(j-2k+1)\{||D_y^2\psi D_y^{j-2k-2}D_x^2u|| \\ &\quad + 2||\psi' D_y^{j-2k-1}D_x^2u|| + ||\psi'' D_y^{j-2k-2}D_x^2u||\}. \end{aligned}$$

Again there is no problem for the last two terms. For the first term we apply the estimate (1.4) with j to $j-2k-2 = j-2(1+k)$ as in (1.5). Repeating this cycle $j/2(1+k)$ times, we may consider this is bounded by

$$C_u^j j!^{k/(1+k)}||\psi D_x^{j/(1+k)}u|| \leq C_0 C_1^j j!^{(\theta+k)/(1+k)},$$

where we have used the result of Theorem 1.2.
Next take the term with $r = k$ in (1.5) which is essentially estimated by

$$j(j-1)\cdots(j-k+1)||y^k D_y D_x \psi D_y^{j-k-1}D_x u||.$$

Now we can apply the estimate (1.4) with j to $j-(1+k)$. Repeating this cycle $j/(1+k)$ times, we may consider this is estimated by

$$C_u^j j!^{k/(1+k)}||\psi D_x^{j/(1+k)}u|| \leq C_0 C_1^j j!^{(\theta+k)/(1+k)}.$$

When $k = 1$ the proof is finished. So we shall consider the case where $k \geq 2$ in the followimg. □

The principal new disturbing term in the right-hand side of (1.5) is the term with $r = 1 : j(2k)||\psi y^{2k-1}D_y^{j-1}D_x^2u||$. This is essentially bounded by

$$j||y^k D_y D_x \psi y^{k-1}D_y^{j-2}D_x u|| \leq Cj||P\psi y^{k-1}D_y^{j-2}D_x u||,$$

where we used the estimate (1.4). The right hand-side of the above inequality is bounded by

$$j\{||\psi y^{k-1}D_y^{j-2}D_x Pu|| + ||[P, \psi y^{k-1}D_y^{j-2}D_x]u||\}.$$

The term we need to estimate in the right-hand side is

$$\begin{aligned} & j[y^{2k}D_{x'}^2, \psi y^{k-1}D_y^{j-2}D_x]u \\ & = 2j\psi_x{}' y^{3k-1}D_y^{j-2}D_x^2 u + j\psi_x{}'' y^{3k-1}D_y^{j-2}D_x u \\ & \quad + j\sum_{r=1}^{2k}\binom{j-2}{r}\frac{(2k)!}{(2k-r)!}\psi y^{3k-r-1}D_y^{j-2-r}D_x^3 u. \end{aligned}$$

Again the principal disturbing is the term with r = 1 which is

$$j(j-2)2k\psi y^{3k-2}D_y^{j-3}D_x^3 u.$$

This is essentially bounded by

$$j(j-2)||y^{2k}D_x^2\psi y^{k-2}D_y^{j-3}D_x u|| \leq Cj(j-2)||P\psi y^{k-2}D_y^{j-3}D_x u||.$$

After $k-1$ times of these steps we will have the bound of the form

$$C^k j^k ||P\psi D_y^{j-1-k}D_x u||.$$

Repeating this cycle $j/(1+k)$ times, we may consider that we have the bound of the form

$$C_u^j j^{jk/(1+k)}||P\psi D_x^{j/(1+k)}|| \leq \tilde{C}_0\tilde{C}_1^j j!^{(\theta+k)/(1+k)}.$$

We can see the terms with $r \geq 2$ in the right-hand side of (1.5) are not so harmful by the method of the consideration similar as above.

2. Formal solutions

We shall construct formal solutions to the equation

$$PU = -\left\{\frac{\partial^2}{\partial y^2} + (x^{2l} + y^{2k})\frac{\partial^2}{\partial x^2}\right\}U(x,y) = 0, \ (l,k \in \mathbf{N}), \tag{2.1}$$

in a neighborhood of the origin $(0,0) \in \mathbf{R}^2$.
Putting $\theta = l(1+k)/\{l(1+k)-k\}$, we consider the integral of the form

$$A(u) = \int e^{i\varrho^\theta x}\varrho^r u(\varrho, y\varrho^{\theta/(1+k)})d\varrho, \tag{2.2}$$

where r is a parameter determined later and $u(\varrho, t)$ is an infinitely differentiable function in $\mathbf{R}_\varrho \times \mathbf{R}_t$ with support in $\varrho > 0$ and rapidly decreasing as $\varrho \longrightarrow \infty$.

Applying the operator P for (2.2) with $t = y\varrho^{\theta/(1+k)}$, we have

$$\begin{aligned} PA(u) = &- \int e^{i\varrho^\theta x} \varrho^{r+2\theta/(1+k)} \partial_t^2 u(\varrho, t) d\varrho \\ &+ \int e^{i\varrho^\theta x} \varrho^{r+2\theta} x^{2l} u(\varrho, t) d\varrho + \int e^{i\varrho^\theta x} \varrho^{r+2\theta} y^{2k} u(\varrho, t) d\varrho. \end{aligned} \tag{2.3}$$

For the third integral in the right-hand side, since we have

$$\begin{aligned} \varrho^{r+2\theta} y^{2k} &= \varrho^{r+2\theta-2k\theta/(1+k)} (y\varrho^{\theta/(1+k)})^{2k} \\ &= \varrho^{r+2\theta/(1+k)} t^{2k}, \end{aligned}$$

it holds that

$$\begin{aligned} PA(u) = &\int e^{i\varrho^\theta x} \varrho^{r+2\theta/(1+k)} \{-\partial_t^2 + t^{2k}\} u(\varrho, t) d\varrho \\ &+ \int e^{i\varrho^\theta x} \varrho^{r+2\theta} x^{2l} u(\varrho, t) d\varrho. \end{aligned} \tag{2.4}$$

We shall deal with the second integral in the right-hand side of (2.4) to replace x^{2l} by a differential operator in ϱ as will be seen in (2.7). We shall need the following formula of Faà di Bruno for the derivatives of a composition of two functions:

LEMMA 2.1. (cf. [9]) *Let I be an open interval in $\mathbf{R}$ and suppose that $f \in C^\infty(I)$. Assume that f takes real values in an open interval J and $g \in C^\infty(J)$. Then the derivatives of $h = g(f(t))$ are given by*

$$h^{(n)}(t) = \sum \frac{n!}{k_1! k_2! \cdots k_n!} g^{(\mu)}(f(t)) \left(\frac{f^{(1)}(t)}{1!}\right)^{k_1} \cdots \left(\frac{f^{(n)}(t)}{n!}\right)^{k_n},$$

where $\mu = k_1 + k_2 + \cdots + k_n$ and the sum is taken over all $k_1, \cdots, k_n$ for which $k_1 + 2k_2 + \cdots + nk_n = n$.

We apply Lemma 2.1 for $g = e^{f(\varrho)}$, $f(\varrho) = i\varrho^\theta x$, then we have

$$\begin{aligned} &\partial_\varrho^{2l} e^{i\varrho^\theta x} \\ = &\sum \frac{(2l)!}{k_1! k_2! \cdots k_{2l}!} \left(\frac{\theta}{1!}\right)^{k_1} \left(\frac{\theta(\theta-1)}{2!}\right)^{k_2} \cdots \left(\frac{\theta(\theta-1)\cdots(\theta-2l)}{(2l)!}\right)^{k_{2l}} \\ &\cdot (ix)^\mu \varrho^{\mu\theta-2l} e^{i\varrho^\theta x} \\ \equiv &\sum_\mu C(\mu) (ix)^\mu \varrho^{\mu\theta-2l} e^{i\varrho^\theta x}, \end{aligned}$$

where $\mu = k_1 + k_2 + \cdots + k_{2l}$ and $k_1 + 2k_2 + \cdots + 2lk_{2l} = 2l$. We see that $\mu = 2l$ only when $k_1 = 2l$ and $k_2 = k_3 = \cdots = k_{2l} = 0$. Hence we have

$$(-1)^l(\theta x)^{2l}\varrho^{2l(\theta-1)}e^{i\varrho^\theta x} = \partial_\varrho^{2l}e^{i\varrho^\theta x} - \sum_{1\leq\mu<2l} C(\mu)(ix)^\mu\varrho^{\mu\theta-2l}e^{i\varrho^\theta x}.$$

Multiplying both sides by $(-1)^l\theta^{-2l}\varrho^{r+2\theta/(1+k)}$, we have

$$\begin{aligned}\varrho^{r+2\theta}x^{2l}e^{i\varrho^\theta x} =&\varrho^{r+2\theta/(1+k)}(-1)^l\theta^{-2l}\partial_\varrho^{2l}e^{i\varrho^\theta x}\\ &+\sum_{1\leq\mu\leq 2l-1} C_1(\mu)(ix)^\mu\varrho^{r+(2+\mu-2l)\theta}e^{i\varrho^\theta x},\end{aligned}\tag{2.5}$$

where we use the equality

$$2\theta/(1+k) + 2l(\theta-1) = 2\theta/(1+k) + 2l\theta - 2l = 2\theta.$$

The highest degree with respect to x in the right-hand side is $2l-1$ and of which term is given by

$$I \equiv C_1(2l-1)(ix)^{2l-1}\varrho^{r+\theta}e^{i\varrho^\theta x}.$$

Again by using the formula of Faà di Bruno, we have

$$\partial_\varrho^{2l-1}e^{i\varrho^\theta x} = \sum_\mu C_2(\mu)(ix)^\mu\varrho^{\mu\theta-2l+1}ei\varrho^\theta x,$$

where $\mu = k_1 + k_2 + \cdots + k_{2l-1}$ and $k_1 + k_2 + \cdots + (2l-1)k_{2l-1} = 2l-1$. The highest degree with respect to x in the right-hand side is $2l-1$ and we have

$$\begin{aligned}&C_2(2l-1)(ix)^{2l-1}\varrho^{(2l-1)\theta-2l+1}e^{i\varrho^\theta x}\\ &= \partial_\varrho^{2l-1}e^{\varrho^\theta x} - \sum_{\mu\leq 2l-2} C_2(\mu)(ix)^\mu\varrho^{\mu\theta-2l+1}e^{i\varrho^\theta x}.\end{aligned}$$

Multiplying both sides by $\{C_1(2l-1)/C_2(2l-1)\}\varrho^{r+2\theta/(1+k)-1}$, we have

$$\begin{aligned}C_1(2l-1)(ix^{2l-1})\varrho^{r+\theta}e^{i\varrho^\theta x} =&\varrho^{r+2\theta/(1+k)-1}C_3(2l-1)\partial_\varrho^{2l-1}e^{i\varrho^\theta x}\\ &-\sum_{\mu\leq 2l-2} C_3(\mu)(ix)^\mu\varrho^{r+(2+\mu-2l)\theta}e^{i\varrho^\theta x}.\end{aligned}$$

By using the formula of Faà di Bruno recursively, considering with the expression (2.5), we arrive at the formula:

$$\varrho^{r+2\theta} x^{2l} e^{i\varrho^\theta x} = \sum_{j=0}^{2l} C_j \varrho^{r+2\theta/(1+k)-j} \partial_\varrho^{2l-j} e^{i\varrho^\theta x}, \tag{2.6}$$

where $C_0 = (-1)^l \theta^{-2l}$, $C_j \in \mathbf{R}$, $j = 1, \cdots, 2l$.

Thus, we have obtained the formula:

$$\begin{aligned} PA(u) &= \int e^{i\varrho^\theta x} \varrho^{r+2\theta/(1+k)} \{-\partial_t^2 + t^{2k}\} u(\varrho, t) d\varrho \\ &+ \sum_{j=0}^{2l} \int \partial_\varrho^{2l-j} \left(e^{i\varrho^\theta x}\right) \varrho^{r+2\theta/(1+k)-j} C_j u(\varrho, t) d\varrho. \end{aligned} \tag{2.7}$$

There remains to consider the second summation by the integration by parts which will be equal to

$$\sum_{j=0}^{2l} \sum_{\nu=0}^{2l=j} \sum_{\nu=0}^{2l-j-\mu} \int e^{i\varrho^\theta x} \varrho^{r+2\theta/(1+k)-j-\mu} C_{j,\mu,\nu} \{\partial_\varrho^{2l-j-\mu-\nu} t^\nu \partial_t^\nu\} u(\varrho, t) d\varrho.$$

We know that $C_0 = C_{0,0,0} = (-1)^l \theta^{-2l}$, $C_{1,0,0} = 2l\{r + 2\theta/(1+k)\} + C_1 \equiv C(r)$, so that we have

$$PA(u) = \int e^{i\varrho^\theta x} \sum_{j=0}^{2zl} \varrho^{r+2\theta/(1+k)-j} \mathcal{P}_j u(\varrho, t) d\varrho, \tag{2.8}$$

where

$$\begin{aligned} \mathcal{P}_0 &= -\partial_t^2 + t^{2k} + (-1)^l \theta^{-2l} \partial_\varrho^{2l} \partial_\varrho^{2l}, \\ \mathcal{P}_1 &= C(r) \partial_\varrho^{2l-1} + C' \partial_\varrho^{2l-1} t \partial_t + C", \\ \mathcal{P}_j &= \sum_{\nu=0}^{j} C_{j,\nu} \partial_\varrho^{2l-j} t^\nu \partial_t^\nu + C_j', \quad j = 2, \cdots, 2l. \end{aligned}$$

Here we have seen that $C(r) = 2lr + 2\theta/(1+k) + C_1$ and C_1, C' and $C"$ are independent of r.

Now the differential operator

$$Q = -\frac{d^2}{dt^2} + t^{2k}, \quad -\infty < t < \infty,$$

is self-adjoint and positive definite in $L^2(\mathbf{R})$ and its eigen-values and eigen-functions are well known and denoted by λ_ν and $\varphi_\nu, \nu = 0, 1, \cdots$. We know that we have

$$\lambda_\nu \sim O(\nu^{2k/(1+k)}), \quad \nu \longrightarrow \infty, \quad (cf.[16]), \tag{2.9}$$

and

$$\varphi_\nu \in \mathcal{S}^{k/(1+k)}_{1/(1+k)}(\mathbf{R}), \quad (cf.[11]), \tag{2.10}$$

that is to say, there are positive constants C_0 and $C_{1,\nu}$ such that

$$\begin{aligned} \mid t^\alpha D_t^\beta \varphi_\nu(t) \mid \leq & C_0 C_{1,\nu}^{|\alpha+\beta|} \alpha!^{1/(1+k)} \beta!^{k/(1+k)} \|\varphi_\nu\|_{L^2(\mathbf{R})}, \\ & \alpha,\ \beta \in Z_+, t \in \mathbf{R}, \end{aligned} \tag{2.11}$$

or equivalently we have

$$\begin{aligned} \mid D_t^\beta \varphi_\nu(t) \mid \leq & C_0 C_{1,\nu}^\beta \beta!^{k/(1+k)} exp[-at^{1/(1+k)}] \|\varphi_\nu\|, \\ & t \in \mathbf{R},\ \beta \in Z_+, \quad (0 < a). \end{aligned} \tag{2.12}$$

We can see that $C_{1,\nu} = O(\lambda_\nu), \nu \longrightarrow \infty$, (cf. [11]). We assume that $\{\varphi_\nu\}_{\nu=0}^\infty$ is an orthonormal system in the following. We remark that (2.12) is also equivalent to the estimation of the following type:

$$\begin{aligned} |\varphi_\nu(t + i\tau)| \leq C \exp[-a_\nu |t|^{1+k} + b_\nu |\tau|^{1+k}] \|\varphi_\nu\|, \\ t + i\tau \in \mathbf{C},\ 0 < a_\nu \leq b_\nu < \infty, \quad (cf.[3]) \end{aligned} \tag{2.12$'$}$$

As in the paper [13], we denote by Π_0 the orthogonal projection on $L^2(\mathbf{R}_\varrho) \otimes \varphi_0(t)$. Then Π_0 and $\mathcal{P}_0$ are commutable and if we define H_j by the formula $\Pi_0 \mathcal{P}_j(f(\varrho)\varphi_0(t)) = (H_j f)(\varrho)\varphi_0(t)$, we obtain by the expression (2.8) that

$$\begin{aligned} H_0 &= \lambda_0 + (-1)^l \theta^{-2l} \partial_\varrho^{2l}, \\ H_1 &= (2lr + c)\partial_\varrho^{2l-1} + c', \end{aligned}$$

where c and c' are the constants independent of r. Let b_0 be a $2l - th$ power root of $-\theta^{2l}\lambda_0$ with the smallest positive imaginary part. Then we can see that $H_1 e^{i\varrho b_0} = 0$ if we take $r = \frac{-1}{2l}\{(c'/b_0)^{2l-1} - c\}$. This yields that $\Pi_0 \mathcal{P}_0 e^{i\varrho b_0} \cdot \varphi_0(t) = 0$.

We shall define a formal solution $u = \sum_{j\geq 0} u_j$ of $PA(u) = 0$ starting with $u_0(\varrho, t) = e^{i\varrho b_0} \cdot \varphi_0(t)$ and resolving recursively the equations for $j \geq 1$:

$$
\begin{aligned}
\mathcal{P}_0(I-\Pi_0)u_j &\doteq - \sum_{n=1}^{\min(j,2l)} \varrho^{-n}(I-\Pi_0)\mathcal{P}_n u_{j-n}, \\
\mathcal{P}_0\Pi_0 u_j &= -\varrho^{-1}\Pi_0\mathcal{P}_1(I-\Pi_0)u_j - \varrho^{-1}\Pi_0\mathcal{P}_1\Pi_0 u_{j-1} \\
&\quad - \sum_{n=1}^{\min(j,2l-1)} \varrho_{-(n+1)}\Pi_0\mathcal{P}_{n+1}u_{j-n}.
\end{aligned} \tag{2.13}
$$

Since we start with $\mathcal{P}_0 u_0 = 0$ and $\Pi_0\mathcal{P}_1 u_0 = 0$, (2.13) will be nicely solved (see §3). Formally we have for $j \geq 2l$

$$
\begin{aligned}
&\mathcal{P}_0(u_0+u_1+u_2+\cdots+u_j) = \\
&-\varrho^{-1}\mathcal{P}_1(u_0+u_1+\cdots+u_{j-1}) \\
&-\varrho^{-2}\mathcal{P}_2(u_0+u_1+\cdots+u_{j-2}) \\
&\cdots\cdots\cdots \\
&-\varrho^{-2l}\mathcal{P}_{2l}(u_0+u_1+\cdots+u_{j-2l}) - \varrho^{-1}\Pi_0\mathcal{P}_j(I-\Pi_0)u_j \\
&-\varrho^{-2}\Pi_0\mathcal{P}_2 u_{j-1} - \varrho^{-3}\Pi_0\mathcal{P}_3 u_{j-2} - \cdots - \varrho^{-2l}\Pi_0\mathcal{P}_{2l}u_{j-2l+1}.
\end{aligned} \tag{2.14}
$$

Also we have formally for $j \longrightarrow \infty$

$$
\mathcal{P}_0 u_j \sim -\sum_{n=1}^{2l} \varrho^{-n}\mathcal{P}_n u_{j-n} \sim O(\varrho^{-j}). \tag{2.15}
$$

3. Optimality of the index $\{\theta, d\}$ for the operator P in (2.1).

Let $\lambda_\nu, \nu = 0, 1, \cdots$, be eigen-values for the operator $Q = -\frac{d^2}{dt^2} + t^{2k}$ in $L^2(\mathbf{R})$. We write $\mathcal{L}_0 = H_0$ and consider first the fundamental solution of the operator $\mathcal{L}_\nu = \lambda_\nu + (-1)^l\theta^{-2l}\partial_\varrho^{2l}, \nu = 0, 1, \cdots$:

$$
\begin{aligned}
E_\nu(\varrho) &= (2\pi)^{-1}\int_{-\infty}^{\infty} e^{i\varrho\xi}\frac{d\xi}{\lambda_\nu + \theta^{-2l}\xi^{2l}} \\
&= O(e^{-|\varrho|\Im b_\nu}), \quad -\infty < \varrho < \infty,
\end{aligned} \tag{3.1}
$$

where b_ν is one of the $2l$th roots of $-\theta^{2l}\lambda_\nu$ with the smallest positive imaginary part for each $\nu = 0, 1, 2, \ldots$. Then we have

$$0 < \Im b_0 < \Im b_1 < \cdots < \Im b_\nu < \cdots . \tag{3.2}$$

We shall construct the approximate solutions of (2.14) in the following form:

$$\begin{aligned} u_0(\varrho, t) &= e^{ib_o\varrho} \cdot \varphi_0(t), \\ u_j(\varrho, t) &= \sum_{\nu=0}^{(2l+1)j} f_{j,\nu}(\varrho) \cdot \varphi_\nu(t), \quad j = 1, 2, \ldots . \end{aligned} \tag{3.3}$$

Then we have

$$(I - \Pi_0)u_1 = \sum_{\nu=1}^{2l+1} f_{1,\nu}(\varrho) \cdot \varphi_\nu(t).$$

Therefore, by (2.14) the equation for $(I - Pi_0)u_1$ is given by

$$\begin{aligned} \mathcal{P}_0(I - \Pi_0)u_1 &= \sum_{\nu=1}^{2l+1}(\lambda_\nu + (-1)^l\theta^{-2l}\partial_\varrho^{2l}) f_{1,\nu}(\varrho) \cdot \varphi_\nu(t) \\ &= -\varrho^{-1}(I - \Pi_0)\mathcal{P}_1 u_0 \\ &= -\varrho^{-1}\sum_{\nu=1}^{2l+1}(\mathcal{P}_1 u_0, \varphi) \cdot \varphi, \end{aligned}$$

from where we have

$$\begin{aligned} \mathcal{L}_\nu f_{1,\nu}(\varrho) =& (\lambda_\nu + (-1)^l\theta^{-2l}\partial_\varrho^{2l}) f_{1,\nu}(\varrho) = -\varrho^{-1}(\mathcal{P}_1 u_0, \varphi_\nu) \\ \equiv& g_{1,\nu}(\varrho), \\ 0 < \varrho < \infty,& \quad \nu = 1, 2, \ldots, 2l + 1. \end{aligned} \tag{3.4}$$

We can see easily that there are positive constants C_0 and C_1 such that

$$| \partial_\varrho^h g_{1,\nu}(\varrho) | \leq C_0 C_1^h \varrho^{-1} e^{-\Im b_0 \varrho}, \quad 1 + h \leq \varrho < \infty, \nu = 1, 2, \ldots . \tag{3.5}$$

Now by virtue of (3.2) and by using the fundamental solution E_ν in (3.1), we can see that there is a solution $f_{1,\nu}(\varrho)$ of the equation (3.4) in $\varrho \geq R_0$

for sufficiently large R_0 such that

$$(3.6) \qquad |\,\partial_\varrho^h f_{1,\nu}(\varrho)\,| \le C_0 C_1^{h+2l} \varrho^{-1} e^{-\Im b_0 \varrho}, \quad R_0 + h \le \varrho < \infty,$$
$$\nu = 1, 2, \dots 2l+1, \quad h = 0, 1, 2, \dots,$$

where C_0 and C_1 are another couple of the constants.

We illustrate how to derive the estimate of the kind (3.6). We can take a solution $f_{1,\nu}(\varrho)$ to the equation (3.4) as an integral

$$f_{1,\nu}(\varrho) = \int_{R_0}^{\infty} E_\nu(\varrho - \tau) g_{1,\nu}(\tau) d\tau, \quad R_0 < \varrho < \infty$$

Then at first we have to estimate the integral of the kind

$$\int_{R_0}^{\infty} e^{-b_1' |\varrho - \tau|} \tau^{-1} e^{-b_0' \tau} d\tau, \quad 0 < b_0' < b_1',$$
$$= e^{-b_1' \varrho} \int_{R_0}^{\varrho} e^{(b_1' - b_0')\tau} \tau^{-1} d\tau + \int_{\varrho}^{\infty} e^{-b_1' |\varrho - \tau|} \tau^{-1} e^{-b_0' \tau} d\tau$$
$$\equiv I + II.$$

We can easily see $II \le {b_1'}^{-1} \varrho^{-1} e^{-b_0' \varrho}, 0 < \varrho < \infty$. As for I, if we take R_0 sufficiently large, we can see that

$$\int_{R_0}^{\infty} e^{(b_1' - b_0')\tau} \tau^{-1} d\tau \le \frac{2}{b_1' - b_0'} e^{(b_1' - b_0')\varrho} \varrho^{-1}, \quad R_0 < \varrho < \infty.$$

On the other hand, by (2.13), we have the equation of the form

$$\mathcal{P}_0 \Pi_0 u_1 = \mathcal{L}_0 f_{1,0}(\varrho) \cdot \varphi_0(t) = g_{1,0}(\varrho),$$

where we can see $g_{1,0}(\varrho)$ satisfies the estimates of the form

$$(3.7) \quad |\,\partial_\varrho^h g_{1,0}(\varrho)\,| \le C_0 C_1^h \varrho^{-2} e^{-\Im b_0 \varrho}, \quad 1 + h \le \varrho < \infty, \ h = 0, 1, \dots.$$

Here the condition $\Pi_0 \mathcal{P}_1 u_0 = 0$ is crucial. By using the fundamental solution E_0, we have the same estimates as in (3.6) for $f_{1,0}(\varrho)$ as follows. We know that $E_0(\varrho)$ is a sum of $e^{ib_j|\varrho|}, j = 1, 2, \dots, l$. And we may consider that $g_{1,0}(\varrho) \equiv 0, \varrho < R_0$ and $g_{1,0}(\varrho) = O(\varrho^{-1} e^{-\Im b_0 \varrho})$. Therefore, in order to estimate

$$\int_0^{\infty} E_0(\varrho - \tau) g_{1,0}(\tau) d\tau, \quad 0 < \varrho < \infty,$$

it's sufficient to consider

$$e^{b_0\varrho}\int_0^{\varrho} e^{ib_j\tau}g_{1,0}(\tau)d\tau + e^{ib_j\varrho}\int_{\varrho}^{\infty} e^{-ib_j\tau}g_{1,0}(\tau)d\tau$$
$$= e^{ib_0\varrho}\int_0^{\infty} e^{ib_j\tau}g_{1,0}(\tau)d\tau - e^{ib_0\varrho}\int_{\varrho}^{\infty} e^{ib_j\tau}g_{1,0}(\tau)d\tau$$
$$+ e^{ib_j\varrho}\int_{\varrho}^{\infty} e^{ib_j\tau}g_{1,0}(\tau)d\tau$$
$$e^{ib_j\varrho}\cdot Const. + O(\varrho^{-1}e^{-\Im b_0\varrho}).$$

On the right-hand side, $e^{ib_j\varrho}$ is a solution of the homogeneous equation $\mathcal{L}_0 u = 0, 0 < \varrho < \infty$, and we have the estimate of the kind (3.6) for $f_{1,0}$ in case $h = 0$ by subtracting such a part.

Next, by using the induction procedure, we can prove the estimate of the kind

$$(3.8)\qquad \begin{aligned} &|\partial_\varrho^h f_{j,\nu}(\varrho)| \le C_0C_1^{j+h}j!\varrho^{-j}e^{-\Im b_0\varrho},\\ &|\partial_\varrho^h H_1 f_{j,0}(\varrho)| \le C_0C_1^{j+h}(j+1)!\varrho^{-j-1}e^{-\Im b_0\varrho},\\ &R_0(j+1)+h \le \varrho < \infty, \nu = 0,1,\cdots,(2l+1)j, \quad j = 1,2,\cdots. \end{aligned}$$

Now we take cut-off function $\chi_j(\varrho) \in C^\infty(\mathbf{R})$ such that

$$(3.9)\qquad \begin{aligned} &\chi(\varrho) = 0 \quad \text{for} \quad \varrho \le 2(j+1)R_0,\\ &\chi(\varrho) = 1 \quad \text{for} \quad \varrho \ge 4(j+1)R_0,\\ &|\partial_\varrho^h \chi_j(\varrho)| \le C_0^h \text{ for } h \le\\ &\text{where } C_0 \text{ is a constant independent of } j \text{ and } h. \end{aligned}$$

We define

$$(3.10)\qquad u(\varrho,t) = \sum_{j=0}^{\infty}\chi_j(\varrho)u_j(\varrho,t).$$

Then we can see that there are a couple of the constants C_0 and C_1 such that

$$(3.11)\qquad |\partial_y^\alpha u(\varrho,t)| \le C_0C_1^\alpha \alpha!^{k/(1+k)}\varrho^{\theta\alpha/(1+k)}e^{-\Im b_0\varrho}, \quad \alpha = 0,1,2,\dots,$$
$$-\infty < y < \infty,\ (t = \varrho^{\theta/(1+k)}y),$$

from where we have

$$U(x,y) = A(u) \in G^{\{d,\theta\}}_{y,x}(\mathbf{R}^2), \quad d = \frac{\theta + k}{1+k}, \text{ (see (2.2))}.$$

Next we can see that there exists a constant M satisfying

$$(3.13) \qquad |u(\varrho,t) - u_0(\varrho,t)| \le \frac{M}{\varrho} exp[-\Im b_0 \varrho], \quad (\varrho,t) \in \mathbf{R}^2, \varrho > 0.$$

On the other hand, since $\varphi_0(0) > 0$, (cf. [16]), we have for sufficiently large ϱ

$$(3.14) \qquad \Re u(\varrho,t) \ge \frac{1}{2}\varphi_0(0)e^{-\Im b_0 \varrho}.$$

Furthermore, by virtue of the construction of $u(\varrho,t)$, we can write

$$(3.15) \qquad PA(u) = A(v + w),$$

where for $w(\varrho,t)$ for any ν there is a constant C_ν such that

$$|\partial_y^\alpha w(\varrho,t)| \le C_\nu^{\alpha+1}\alpha!^{k/(1+k)}\varrho^{\theta\alpha/(1+k)}e^{-\Im b_0\varrho}$$

for sufficiently large ϱ, from where we have the estimates of the kind

$$(3.16) \qquad |\partial_y^\alpha \partial_x^\beta A(w)| \le C_\varepsilon \varepsilon^{(\alpha+\beta)}\alpha!^d\beta!^\theta, \quad (\alpha,\beta) \in Z_+^2,$$

for any $\varepsilon > 0$. As for v, (cf. (2.15)), roughly speaking, for $\varrho \sim R_0(j+1)$ we have

$$\begin{aligned} |v(\varrho,t)| &\sim C_0 C_1^j j! \varrho^{-j} e^{-\Im b_0 \varrho} \\ &\sim C_0 \left(\frac{C_1}{eR_0}\right)^j e^{-\Im b_0 \varrho} \\ &\sim C_0 \left(\frac{C_1}{eR_0}\right)^\varrho e^{-\Im b_0 \varrho}, \end{aligned}$$

where C_1 is independent of R_0 and finally we see that for any $\delta > 0$ if we take R_0 sufficiently large we have the estimation of the kind

$$(3.17)\qquad \begin{aligned} |\partial_y^\alpha v(\varrho,t)| &\le C_0 C_1^\alpha \alpha!^{k/(1+k)} \varrho^{\theta\alpha/(1+k)} \delta^\varrho e^{-\Im b_0 \varrho} \\ &= C_0 C_1^\alpha \alpha!^{k/(1+k)} \varrho^{\theta\alpha/(1+k)} e^{\varrho(\log\delta - \Im b_0)}, \end{aligned}$$

from where we have finally the estimation of the form

$$|\partial_y^\alpha \partial_x^\beta PA(u)| \le C\varepsilon^{(\alpha+\beta)} \alpha!^d \beta!^\theta, \quad (\alpha,\beta) \in Z_+^2.$$

Here we can take $\varepsilon > 0$ arbitrarily depending on R_0.

The final step will rely upon the results of G. Métivier, [12]. Since the estimate of type (1.4) holds for P and P^* the hypothesis H_1 in [12] is satisfied. Then we can apply the slight modification of Théorème 3.1 in [12] for the case of Gevrey hypoellipticity instead of analytic hypoellipticity, which yields that if P is $G^{\{d',\theta'\}}$-hypoelliptic in a neighborhood ω (say bounded) and $0 < d' < d, 0 < \theta' < \theta$, then there are positive constants L and C satisfying

$$\sup_{\bar\omega} |\partial_y^\alpha \partial_x^\beta A(u)| \le C(L\varepsilon)^{(\alpha+\beta)} \alpha!^d \beta!^\theta, \quad (\alpha,\beta) \in Z_+^2.$$

This is impossible because of (3.14) and (2.12)$'$ with $\nu = 0$ if we take ε sufficiently small. Thus the optimality of the Gevrey index $\{d,\theta\}$ for the operator P is proved.

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