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PHRAGMÉN-LINDELÖF AND CONTINUOUS DEPENDENCE TYPE RESULTS IN GENERALIZED DISSIPATIVE HEAT CONDUCTION

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ABSTRACT. This paper is concerned with investigating the asymptotic behavior of end effects for a generalized heat conduction problem with an added dissipation term defined on a three-dimensional semi-infinite cylinder. With homogeneous Dirichlet conditions on the lateral surface of the cylinder it is shown that solutions either grow exponentially or decay exponentially in the distance from the finite end of the cylinder. In particular, to render decay estimate explicit, we pattern after the analysis of Payne and Song [13, 15]. The continuous dependence effect of perturbing the equations parameters is also investigated.

1. Introduction

There has been an enormous explosion of activity in the field of heat conduction at low temperature, see, i.e., [4], [16], [12] and the references therein. In fact the model studied in [4, 16] which has been proposed for heat conduction makes use of the following relation between heat flux and temperature, i.e.,

$$\tau u_{i,t} = -u_i - \kappa T_{,i} + \mu \Delta u_i + \nu u_{j,ji} \quad \text{in} \quad R \times (0, \infty), \tag{1.1}$$

where u_i denotes the heat flux and T the temperature. The coefficients τ, κ, μ and ν are positive constants, Δ is the Laplace operator, and a comma has been used to indicate partial differentiation with respect to the corresponding coordinate. In addition we have adopted the summation convention in which a repeated Latin index in any term indicates

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summation over that index from one to three and a repeated Greek index indicates summation from one to two. Associated with this system (1.1) is the temperature equation

$$cT_{,t} = -u_{i,i}, \tag{1.2}$$

where c is a positive constant. In [12], the authors investigated questions of uniqueness of the forward and backward in time problems for (1.1), (1.2) as well as decay and growth (in time) properties of solution. For the backward in time problem Franchi and Staughan [4] investigated how the solution varied with the reaction time τ, and studied, in particular, perturbation of $\tau = 0$. Recently Payne and Song [13, 14] investigated decay (or growth) and continuous dependence type results of (1.1) and (1.2) defined on a semi-infinite cylinder with homogeneous data on the lateral surface and examined the question of spatial decay. In this paper we investigate the asymptotic behavior of solutions to a generalized heat conduction with an added dissipation term, defined on a three-dimensional semi-infinite cylinder, i.e.,

$$cT_{,t} = -u_{i,i} + \zeta \Delta T, \tag{1.3}$$

where ζ is a positive constant. In [16], the authors investigated questions of continuous dependence on initial-time and spatial geometry for (1.1) and (1.3).

Decay results of Saint-Venant type, particularly for elliptic boundary value problems are numerous (see, e.g., Horgan and Knowles [7] and Horgan [5, 6]). Similar results for solutions of time-dependent systems are of more recent origin beginning perhaps with the work of Boley [2]. Some of these latter results are cited in [7], but later results for nonlinear problems have been obtained by Song [17], Ames et al. [1], Song and Payne [15], Lin and Payne [10, 11], Horgan [5, 6], and Payne and Horgan [9]. More recently these Saint-Venant type and continuous dependence type results have been established as a consequence of more general Phragmén-Lindelöf type alternative results (see, e.g., Horgan and Payne [8], Flavin et al. [3], Lin and Payne [10]). It is this Phragmén-Lindelöf type alternative and continuous dependence on the parameter ζ that we derive in this paper. In particular, to render decay estimate explicit, we pattern after the analysis of Payne and Song [15, 13], and Horgan and Payne [9] in Section 4.

2. Statement of the problem

We are interested in heat conduction in a semi-infinite cylinder R with arbitrary cross section D. The boundary ∂D of D is assumed to be smooth enough to permit application of the divergence theorem. To make the geometry specific we assume that the generators of R are parallel to the x_3-axis and that the finite end lies in the plane $x_3 = 0$. Thus

$$R = \{(x_1, x_2, x_3) : (x_1, x_2) \in D, \quad x_3 > 0\}. \tag{2.1}$$

We also use the notation

$$R_z = \{(x_1, x_2, x_3) : (x_1, x_2) \in D, \quad x_3 > z \geq 0\}, \tag{2.2}$$

and

$$D_z = \{(x_1, x_2) \in D, \quad x_3 = z\}. \tag{2.3}$$

The initial and boundary conditions associated with (1.1) and (1.3) are

$$u_i(x_1, x_2, x_3, t) = 0 \quad \text{on} \quad \partial D, \quad 0 \leq t, \ x_3 < \infty, \tag{2.4}$$

$$T(x_1, x_2, x_3, t) = 0 \quad \text{on} \quad \partial D, \quad 0 \leq t, \ x_3 < \infty, \tag{2.5}$$

$$u_i(x_1, x_2, 0, t) = f_i \quad \text{on} \quad \overline{D} \times [0, \infty), \tag{2.6}$$

$$T(x_1, x_2, 0, t) = g \quad \text{on} \quad \overline{D} \times [0, \infty), \tag{2.7}$$

$$u_i(x_1, x_2, x_3, 0), \ T(x_1, x_2, x_3, 0) = 0 \quad \text{in} \quad R \times \{0\}. \tag{2.8}$$

We assume that the prescribed vector function f_i and the function g are differentiable and vanish on ∂D.

To study the decay or growth of solutions to (1.1), (1.3), (2.4)-(2.8), we introduce a cross sectional energy function

$$E(z, t) = -\int_0^t \int_{D_z} (\mu u_i u_{i,3} + \nu u_3 u_{j,j} - \kappa u_3 T + \kappa \zeta T T_{,3})\, dA d\eta. \tag{2.9}$$

We show that as $z \longrightarrow \infty$, either $E(z, t)$ decays to zero or $-E(z, t)$ becomes unbounded. We derive the rate of decay (or growth) explicitly.

We note that for $z > z_0$, $E(z,t)$ may be expressed, on using the divergence theorem and the initial-boundary conditions of (2.4)–(2.5) as

$$
\begin{aligned}
E(z,t) &= E(z_0,t) \\
&\quad - \int_0^t \int_{z_0}^z \int_{D_\xi} \left(\mu u_{i,j} u_{i,j} + \nu u_{i,i}^2 + u_i u_i + \kappa\zeta T_{,i} T_{,i}\right) dA d\xi d\eta \\
&\quad - \frac{1}{2} \int_{z_0}^z \int_{D_\xi} \left(\tau u_i u_i + \kappa c T^2\right) dA d\xi.
\end{aligned} \tag{2.10}
$$

In (2.10) the final integral on the right-hand side is evaluated at time t. On the other hand if $E(z,t) \to 0$ as $z \to \infty$, then

$$
\begin{aligned}
E(z,t) &= \int_0^t \int_{z_0}^z \int_{D_\xi} \left(\mu u_{i,j} u_{i,j} + \nu u_{i,i}^2 + u_i u_i + \kappa\zeta T_{,i} T_{,i}\right) dA d\xi d\eta \\
&\quad + \frac{1}{2} \int_{z_0}^z \int_{D_\xi} \left(\tau u_i u_i + \kappa c T^2\right) dA d\xi.
\end{aligned} \tag{2.11}
$$

Clearly if we show that $E(z,t)$ decays to zero and bound $E(z,t)$ in terms of explicit decaying function of z, then from (2.11) we observe that this is in fact an energy bound. On the other hand if $-E(z,t)$ is bounded below by an unbounded function $\varkappa(z,t)$, then (2.10)

$$
\begin{aligned}
&\int_0^t \int_{R/R_z} \left(\mu u_{i,j} u_{i,j} + \nu u_{i,i}^2 + u_i u_i + \kappa\zeta T_{,i} T_{,i}\right) dA d\eta \\
&\qquad + \frac{1}{2} \int_{R/R_z} \left(\tau u_i u_i + \kappa c T^2\right) dx \geq \varkappa(z,t) + E(0,t).
\end{aligned} \tag{2.12}
$$

The aim of the first part of this paper is to derive differential inequalities for $E(z,t)$ which will imply that $E(z,t)$ either grows exponentially in z or decays exponentially in z as $z \to \infty$. Note that $E(z,t)$ is indefinite while

$$
\begin{aligned}
\frac{\partial E}{\partial z} &= - \int_0^t \int_{D_z} \left(\mu u_{i,j} u_{i,j} + \nu u_{i,i}^2 + u_i u_i + \kappa\zeta T_{,i} T_{,i}\right) dA d\eta \\
&\quad - \frac{1}{2} \int_{D_z} \left(\tau u_i u_i dA + \kappa c T^2\right) dA,
\end{aligned} \tag{2.13}
$$

is negative.

In the next section we will be using the following Poincaré inequality. Let $\psi(x_1, x_2)$ be differentiable in D and vanish on ∂D. Then

$$\int_D \psi^2 dA \leq \frac{1}{\lambda} \int_D \psi_{,\alpha}\psi_{,\alpha} dA, \tag{2.14}$$

where λ is the first eigenvalue in the fixed membrane problem for D. In what follows, lower bounds for λ will suffice. For instance, λ is a monotone function of domain; also

$$\lambda \geq \pi j_0^2 A^{-1}, \tag{2.15}$$

where j_0 is the first zero of the Bessel function $J_0(x)$, and A denotes the area of D.

3. An energy inequality

We first write

$$E(z,t) = I_1(z,t) + I_2(z,t) + I_3(z,t), \tag{3.1}$$

where

$$I_1(z,t) = -\int_0^t \int_{D_z} (\mu u_i u_{i,3} + \nu u_3 u_{j,j})\, dA d\eta, \tag{3.2}$$

$$I_2(z,t) = \kappa \int_0^t \int_{D_z} u_3 T dA d\eta, \tag{3.3}$$

and

$$I_3(z,t) = -\kappa\zeta \int_0^t \int_{D_z} TT_{,3} dA d\eta. \tag{3.4}$$

Using Schwarz's inequality and rearranging we have

$$\begin{aligned} |I_1| \leq \int_0^t \int_{D_z} \Big\{ &\frac{\mu\alpha}{2}(1-\sigma) u_i u_i + \frac{\mu\alpha}{2}\sigma u_i u_i \\ &+ \frac{\mu}{2\alpha} u_{i,3}u_{i,3} + \frac{\nu}{2\beta} u_3^2 + \frac{\nu\beta}{2} u_{j,j}^2 \Big\} dA d\eta, \end{aligned} \tag{3.5}$$

for positive constants $\alpha, \beta, \sigma (\leq 1)$ to be chosen. Choosing

$$\alpha = 1, \quad \beta = \nu^{1/2}, \quad \sigma = 1, \tag{3.6}$$

we are led to

$$|I_1| \leq -\frac{\nu^{1/2}}{2}\frac{\partial E}{\partial z}. \tag{3.7}$$

Now for I_2, upon using Schwarz's inequality and the arithmetic-geometric mean inequality we have for arbitrary positive β_1

$$\begin{aligned} |I_2| &\leq \frac{\kappa}{\lambda^{1/2}}\left(\int_0^t\int_{D_z} u_3^2 dA d\eta\right)^{1/2}\left(\int_0^t\int_{D_z} T^2 dA d\eta\right)^{1/2} \\ &\leq \frac{\kappa}{2\lambda^{1/2}}\left(\beta_1\int_0^t\int_{D_z} u_3^2 dA d\eta + \beta_1^{-1}\int_0^t\int_{D_z} T_{,\alpha}T_{,\alpha} dA d\eta\right). \end{aligned} \tag{3.8}$$

Similarly for $|I_3|$, we find for arbitrary positive β_2

$$|I_3| \leq \frac{\kappa\zeta}{2\lambda^{1/2}}\left(\beta_2\int_0^t\int_{D_z} T_{,\alpha}T_{,\alpha} dA d\eta + \beta_2^{-1}\int_0^t\int_{D_z} T_{,3}^2 dA d\eta\right). \tag{3.9}$$

Combining I_2 and I_3 yields

$$\begin{aligned} |I_2| + |I_3| \leq{}& \frac{\kappa\beta_1}{2\lambda^{1/2}}\int_0^t\int_{D_z} u_3^2 dA d\eta + \frac{\kappa}{2\lambda^{1/2}}\beta_2^{-1}\int_0^t\int_{D_z} T_{,3}^2 dA d\eta \\ &+ \frac{\kappa}{2\lambda^{1/2}}(\beta_1^{-1} + \zeta\beta_2)\int_0^t\int_{D_z} T_{,\alpha}T_{,\alpha} dA d\eta. \end{aligned} \tag{3.10}$$

Choosing $\beta_1 = \sqrt{2}/\zeta$, $\beta_2 = 1/\sqrt{2}$, we find

$$|I_2| + |I_3| \leq -\frac{\kappa}{2\lambda^{1/2}}\left(\frac{\sqrt{2}}{\zeta} + \frac{1}{\sqrt{2}}\right)\frac{\partial E}{\partial z}. \tag{3.11}$$

Inserting back (3.7) and (3.11) into (3.1), we obtain

$$|E(z,t)| \leq -\gamma^{-1}\frac{\partial E}{\partial z}, \tag{3.12}$$

where

$$\gamma = \left[\frac{\nu^{1/2}}{2} + \frac{\kappa}{2\lambda^{1/2}}\left(\frac{\sqrt{2}}{\zeta} + \frac{1}{\sqrt{2}}\right)\right]^{-1}. \tag{3.13}$$

To deal with this inequality of (3.12), we follow the procedure used in [3] and [8]. We recall that $\partial E/\partial z$ is negative. Thus for some value of z, for instance $z = z_0$, $E(z_0, t) < 0$, then $E(z, t) < 0$ for all subsequent values of z_0. In this case

$$-\frac{\partial E}{\partial z} \geq \gamma(-E). \tag{3.14}$$

Integrating this, we have

$$-E(z,t)e^{-\gamma(z-z_0)} \geq -E(z_0,t). \tag{3.15}$$

Clearly $-E(z,t)$ increases asymptotically at least exponentially fast, i.e.,

$$\lim_{z\to\infty} e^{-\gamma z}[-E(z,t)] \geq M \tag{3.16}$$

where M is a positive constant. Thus if there is a $z_0 \in (0,\infty)$ such that $E(z_0,t) < 0$, then $|E(z,t)|$ cannot remain bounded for all z.

We next consider the case in which there is no such z_0, i.e., $E(z,t) \geq 0$ for all $z > 0$. In this case, we write (3.12) as

$$\frac{\partial E}{\partial z} + \gamma E \leq 0, \tag{3.17}$$

An integration furnishes

$$E(z,t) \leq E(0,t)e^{-\gamma z} \tag{3.18}$$

We note that the decay rate γ is conservative since we do not pursue the optimal choices for arbitrary constants in (3.5) and (3.10). Clearly if $\lim_{z\to\infty} E(z,t) = 0$, then $E(z,t)$ may be rewritten as

$$\begin{aligned} E(z,t) = &\int_0^t \int_{R_z} \left(\mu u_{i,j}u_{i,j} + \nu u_{i,i}^2 + u_i u_i + \kappa\zeta T_{,i}T_{,i}\right) dx d\eta \\ &+ \frac{1}{2}\int_{R_z} \left(\tau u_i u_i + \kappa c T^2\right) dx. \end{aligned} \tag{3.19}$$

Thus we have established

THEOREM 3.1. *Let (u_i, T) be a solution of* (1.1), (1.2), (2.4)-(2.8), *then for fixed t either*

$$\begin{aligned} \lim_{z\to\infty} \Bigg[e^{-\gamma z} \Bigg\{ &\int_0^t \int_{R/R_z} \left(\mu u_{i,j}u_{i,j} + \nu u_{i,i}^2 + u_i u_i + \kappa\zeta T_{,i}T_{,i}\right) dx d\eta \\ &+ \frac{1}{2}\int_{R/R_z} \left(\tau u_i u_i + \kappa c T^2\right) dx \Bigg\} \Bigg] \geq Const, \end{aligned} \tag{3.20}$$

or

$$
\begin{aligned}
&\int_0^t \int_{R_z} \left(\mu u_{i,j} u_{i,j} + \nu u_{i,i}^2 + u_i u_i + \kappa\zeta T_{,i} T_{,i}\right) dx d\eta \\
&\quad + \frac{1}{2} \int_{R_z} \left(\tau u_i u_i + \kappa c T^2\right) dx \leq E(0,t) e^{-\gamma z}.
\end{aligned} \tag{3.21}
$$

Here γ *is given by* (3.13).

Assuming that $E(z,t)$ does not grow exponentially at infinity it follows that (3.21) holds, but to make this inequality explicit we need a bound for the total energy $E(0,t)$. This is provided in the next section.

4. A bound for $E(0,t)$

In this section we indicate how we can bound the total energy assuming that (3.21) holds. We introduce functions of w_i and ϕ which satisfy the conditions (2.4)-(2.8) imposed on u_i and ϕ, and tend to zero as $x_3 \longrightarrow \infty$. Then from (3.1), upon integration by parts using the divergence theorem we have

$$
\begin{aligned}
E(0,t) = &-\mu \int_0^t \int_{D_0} w_i u_{i,3} dA d\eta - \nu \int_0^t \int_{D_0} w_3 u_{j,j} dA d\eta \\
&+ \kappa \int_0^t \int_{D_0} w_3 \phi dA d\eta - \kappa\zeta \int_0^t \int_{D_0} \phi T_{,3} dA d\eta \\
= &\ \mu \int_0^t \int_R w_{i,j} u_{i,j} dx d\eta + \nu \int_0^t \int_R w_{i,i} u_{j,j} dx d\eta - \kappa \int_0^t \int_{D_0} w_3 \phi dA d\eta \\
&+ \int_0^t \int_R w_i \left(\mu \Delta u_i + \nu u_{j,ji} - \kappa T_{,i}\right) dx d\eta \\
&+ \kappa\zeta \int_0^t \int_R \phi_{,i} T_{,i} dx d\eta + \kappa\zeta \int_0^t \int_R \phi \Delta T dx d\eta.
\end{aligned}
$$

Using the differential equations for u_i and T, and integrating further with time, one finds

$$\begin{aligned}
E(0,t) &= \mu \int_0^t \int_R w_{i,j} u_{i,j} dx d\eta + \nu \int_0^t \int_R w_{i,i} u_{j,j} dx d\eta \\
&+ \kappa \int_0^t \int_{D_0} w_3 \phi dA d\eta + \kappa\zeta \int_0^t \int_R \phi_{,i} T_{,i} dx d\eta \\
&+ + \int_0^t \int_R w_i(\tau u_{i,\eta} + u_i + \kappa T_{,i}) dx d\eta \\
&+ \kappa \int_0^t \int_R \phi(c T_{,\eta} + u_{i,i}) dx d\eta \\
&= \mu \int_0^t \int_R w_{i,j} u_{i,j} dx d\eta + \nu \int_0^t \int_R w_{i,i} u_{j,j} dx d\eta \\
&+ \kappa \int_0^t \int_{D_0} w_3 \phi dA d\eta - \int_0^t \int_R u_i(\tau w_{i,\eta} - w_i) dx d\eta \\
&+ \kappa \int_0^t \int_R w_i T_{,i} dx d\eta - \kappa \int_0^t \int_R \phi_{,\eta} T dx d\eta \\
&- \kappa \int_0^t \int_R u_i \phi_{,i} dx d\eta + \tau \int_R w_i u_i dx + \kappa c \int_R \phi T dx.
\end{aligned}$$

Using Schwarz's inequality, the arithmetic-geometric mean inequality, we obtain for arbitrary positive constants ϵ_i's

$$\begin{aligned}
E(0,t) &\le \frac{\mu}{2\epsilon_1} \int_0^t \int_R u_{i,j} u_{i,j} dx d\eta + \frac{\epsilon_1 \mu}{2} \int_0^t \int_R w_{i,j} w_{i,j} dx d\eta \\
&+ \frac{\nu}{2\epsilon_2} \int_0^t \int_R u_{i,i}^2 dx d\eta + \frac{\epsilon_2 \nu}{2} \int_0^t \int_R w_{j,j}^2 dx d\eta + \kappa \int_0^t \int_{D_0} w_3 \phi dA d\eta \\
&+ \frac{\epsilon_3}{2} \int_0^t \int_R u_i u_i dx d\eta + \frac{1}{2\epsilon_3} \int_0^t \int_R (\tau w_{i,\eta} - w_i)(\tau w_{i,\eta} - w_i) dx d\eta \\
&+ \frac{\epsilon_4}{2} \int_0^t \int_R w_i w_i dx d\eta + \frac{1}{2\epsilon_4} \int_0^t \int_R T_{,i} T_{,i} dx d\eta \\
&+ \frac{\epsilon_5}{2} \int_0^t \int_R \phi_{,\eta}^2 dx d\eta + \frac{1}{2\epsilon_5} \int_0^t \int_R T^2 dx d\eta \\
&+ \frac{\epsilon_6}{2} \int_0^t \int_R u_i u_i dx d\eta + \frac{1}{2\epsilon_6} \int_0^t \int_R \phi_{,i} \phi_{,i} dx d\eta \\
&+ \frac{\tau \epsilon_7}{\int_R} w_i w_i dx + \frac{\tau}{2\epsilon_7} \int_R u_i u_i dx \\
&+ \frac{\kappa c \epsilon_8}{2} \int_R \phi^2 dx + \frac{\kappa c}{2\epsilon_8} \int_R T^2 dx.
\end{aligned}$$

Now this may be rewritten as

$$\begin{aligned}E(0,t) \leq F(w_i,\phi) &+ \frac{\mu}{2\epsilon_1}\int_0^t\int_R u_{i,j}u_{i,j}dxd\eta + \frac{\nu}{2\epsilon_2}\int_0^t\int_R u_{i,i}^2 dxd\eta \\ &+ \left(\frac{\epsilon_3}{2}+\frac{\epsilon_6}{2}\right)\int_0^t\int_R u_iu_idxd\eta + \left(\frac{1}{2\epsilon_4}+\frac{1}{2\epsilon_5\lambda}\right)\int_0^t\int_R T_{,i}T_{,i}dxd\eta \\ &+ \frac{\tau}{2\epsilon_7}\int_R u_iu_idx + \frac{\kappa c}{2\epsilon_8}\int_R T^2dx,\end{aligned}$$

where

$$\begin{aligned}F(w_i,\phi) = \frac{\epsilon_1\mu}{2}&\int_0^t\int_R w_{i,j}w_{i,j}dxd\eta + \frac{\epsilon_2\nu}{2}\int_0^t\int_R w_{j,j}^2dxd\eta \\ &+ \frac{1}{2\epsilon_3}\int_0^t\int_R(\tau w_{i,\eta}-w_i)(\tau w_{i,\eta}-w_i)dxd\eta + \frac{\epsilon_4}{2}\int_0^t\int_R w_iw_idxd\eta \\ &+ \frac{\epsilon_5}{2}\int_0^t\int_R \phi_{,\eta}^2dxd\eta + \frac{1}{2\epsilon_6}\int_0^t\int_R \phi_{,i}\phi_{,i}dxd\eta + \kappa\int_0^t\int_{D_0} w_3\phi dAd\eta \\ &+ \frac{\tau\epsilon_7}{2}\int_R w_iw_idx + \frac{\kappa c\epsilon_8}{2}\int_R\phi^2dx.\end{aligned}$$

Choosing $\epsilon_1=\epsilon_2=\epsilon_3=1$, $\epsilon_4=1/\kappa\zeta$, $\epsilon_5=\epsilon_4/\lambda$, and $\epsilon_7=\epsilon_8=2$, we find

$$E(0,t) \leq F(w_i,\phi) + E(0,t)/2.$$

It follows from this that an upper bound for the total energy is given by

$$E(0,t) \leq 2F(w_i,\phi). \tag{4.1}$$

Modeling after the work of Payne and Song [13, 15], and Horgan and Payne [9], we set a special choice for w_i and ϕ satisfying (2.4)-(2.8)

$$\begin{aligned}w_i(x_1,x_2,x_3,t) &= f_i(x_1,x_2,t)e^{-kx_3},\\ \phi(x_1,x_2,x_3,t) &= g(x_1,x_2,t)e^{-kx_3},\end{aligned} \tag{4.2}$$

where k is an arbitrary positive constant at our disposal (to be determined later). It remains to compute the upper bound for $E(0,t)$. On computing (4.1) we find

$$F(w_i,\phi) = \frac{C_1}{k} + C_2 + C_3k, \tag{4.3}$$

where

$$C_1 = \frac{\mu}{4}\int_0^t\int_{D_0} f_{i,\beta}f_{i,\beta}dAd\eta + \frac{\nu}{4}\int_0^t\int_{D_0} f_{\alpha,\alpha}^2 dAd\eta$$
$$+\frac{1}{4}\int_0^t\int_{D_0}(\tau f_{i,\eta}-f_i)(\tau f_{i,\eta}-f_i)dAd\eta + \frac{\tau}{2}\int_{D_0} f_i f_i dA$$
$$+\quad \frac{1}{4}(\kappa\zeta\lambda)^{-1}\int_0^t\int_{D_0} g_{,\eta}^2 dAd\eta + \frac{1}{4}\int_0^t\int_{D_0} g_{,\alpha}g_{,\alpha}dAd\eta$$
$$+\frac{\kappa c}{2}\int_{D_0} g^2 dA,$$

$$C_2 = -\frac{\nu}{2}\int_0^t\int_{D_0} f_{\alpha,\alpha}f_3 dAd\eta + \kappa\int_0^t\int_{D_0} f_3 g dAd\eta,$$

$$C_3 = \frac{\mu}{4}\int_0^t\int_{D_0}(f_\alpha f_\alpha + f_3^2)dAd\eta + \frac{\nu}{4}\int_0^t\int_{D_0} f_3^2 dAd\eta.$$

Among all such functions of the form (4.3), we make the choice of k which minimizes $F(w_i,\phi)$, i.e.,

$$k = (C_1/C_3)^{1/2}. \tag{4.4}$$

With this choice we obtain

$$F(w_i,\phi) = 2(C_1C_3)^{1/2} + C_2. \tag{4.5}$$

The insertion of (4.5) into (4.1) and the result back into (3.21) yields a bound for $E(0,t)$ in terms of data and the given geometry.

5. Continuous dependence on the parameter ζ

In this section we investigate the effect of a small perturbation of the parameter ζ on the decay of solution. It would of course be possible to investigate the effects of the perturbation of other parameters, the arguments being similar to those which we shall employ. We denote by $v_i(x,t)$ and $S(x,t)$ the solutions of (1.1), (1.3), (2.4)-(2.8) with ζ replaced by the constant ζ_1. We could allow u_i and v_i to satisfy different conditions on $x_3 = 0$, but since the problems are linear we may decompose and treat the two effects separately. The problem resulting from perturbation of the data is precisely the problem treated in Section 3 so we restrict our attention to the case in which $u_i = v_i$ and $T = S$ on $x_3 = 0$. If we now set

$$\varpi_i = u_i - v_i, \quad \theta = T - S, \tag{5.1}$$

we note that ϖ_i and θ satisfy

$$\tau\varpi_{i,t} = -\varpi_i - \kappa\theta_{,i} + \mu\Delta\varpi_i + \nu\varpi_{j,ji} \quad \text{in} \quad R \times (0,\infty), \tag{5.2}$$

$$c\theta_{,t} = -\varpi_{i,i} + \zeta\Delta\theta + \overline{\zeta}\Delta S \quad \text{in} \quad R \times (0,\infty), \tag{5.3}$$

where $\overline{\zeta} = \zeta - \zeta_1$,

$$\varpi_i(x_1, x_2, x_3, t) = 0, \quad \text{on} \quad \partial D, \quad 0 \le t, \ x_3 < \infty, \tag{5.4}$$

$$\theta(x_1, x_2, x_3, t) = 0, \quad \text{on} \quad \partial D, \quad 0 \le t, \ x_3 < \infty, \tag{5.5}$$

$$\varpi_i(x_1, x_2, 0, t) = 0, \quad \text{in} \quad \overline{D}_0 \times (0,\infty), \tag{5.6}$$

$$\theta(x_1, x_2, 0, t) = 0, \quad \text{in} \quad \overline{D}_0 \times (0,\infty), \tag{5.7}$$

$$\varpi_i(x_1, x_2, x_3, 0),\ S(x_1, x_2, x_3, 0) = 0 \quad \text{in} \quad R \times \{0\}. \tag{5.8}$$

We further assume that (u_i, T) and (v_i, S) satisfy (3.21). We now introduce a function $\Phi(z,t)$ for $z,\ t \ge 0$, given by

$$\begin{aligned} \Phi(z,t) = &\int_0^t \int_{R_z} \left(\mu\varpi_{i,j}\varpi_{i,j} + \nu\varpi_{j,j}^2 + \varpi_i\varpi_i + \kappa\zeta\theta_{,i}\theta_{,i}\right) dx d\eta \\ &+ \frac{1}{2}\int_{R_z} \left(\tau\varpi_i\varpi_i + \kappa c\theta^2\right) dx. \end{aligned} \tag{5.9}$$

Clearly

$$\begin{aligned} \frac{\partial\Phi}{\partial z} = -\Bigg\{ &\int_0^t \int_{D_z} \left(\mu\varpi_{i,j}\varpi_{i,j} + \nu\varpi_{j,j}^2 + \varpi_i\varpi_i + \kappa\zeta\theta_{,i}\theta_{,i}\right) dA d\eta \\ &+ \frac{1}{2}\int_{D_z} \left(\tau\varpi_i\varpi_i + \kappa c\theta^2\right) dA \Bigg\} \end{aligned} \tag{5.10}$$

On the other hand an integration of (5.9) by parts yields

$$\begin{aligned} \Phi(z,t) = &-\int_0^t \int_{D_z} \left(\mu\varpi_i\varpi_{i,3} + \nu\varpi_3\varpi_{j,j} - \kappa\varpi_3\theta + \kappa\zeta\theta\theta_{,3}\right) dA d\eta \\ &+ \kappa\overline{\zeta}\int_0^t \int_{R_z} \theta\Delta S dx d\eta. \end{aligned} \tag{5.11}$$

Using (3.17) we obtain

$$
\begin{aligned}
\Phi(z,t) &\leq -\frac{1}{\gamma}\frac{\partial \Phi}{\partial z} + \kappa\bar{\zeta}\int_0^t\int_{R_z}\theta S_{,33}\,dx\,d\eta - \kappa\bar{\zeta}\int_0^t\int_{R_z}\theta_{,\alpha}S_{,\alpha}\,dx\,d\eta \\
&= -\frac{1}{\gamma}\frac{\partial \Phi}{\partial z} + \kappa\bar{\zeta}\,(K_1 - K_2).
\end{aligned} \tag{5.12}
$$

Now

$$
K_1^2 \leq \int_0^t\int_{R_z}\theta^2\,dx\,d\eta \int_0^t\int_{R_z} S_{,33}^2\,dx\,d\eta. \tag{5.13}
$$

To bound the second term in (5.13) we observe that under additional assumptions on the smoothness of f_i and ∂D together with appropriate compatibility assumptions on $\partial D \times \{0\}$ for $t \geq 0$, the system $(v_{i,3}, S_{,3})$ satisfies (1.1), (1.3), (2.4)-(2.8) and hence using the results of Section 3 we conclude that

$$
\int_0^t\int_{R_z} S_{,33}S_{,33}\,dx\,d\eta \leq \frac{1}{\kappa\zeta_1}E_1(0,t)e^{-\gamma(t)z}, \tag{5.14}
$$

where we define a higher-order energy as

$$
\begin{aligned}
E_1(0,t) &= \int_0^t\int_R \left(\mu v_{i,j3}^2 + \nu v_{j,j3}^2 + v_{i,3}v_{i,3} + \kappa\zeta_1 S_{,i3}S_{,i3}\right)dx\,d\eta \\
&\quad + \frac{1}{2}\int_R \left(\tau v_{i,3}v_{i,3} + \kappa c S_{,3}^2\right)dx.
\end{aligned} \tag{5.15}
$$

Thus

$$
\begin{aligned}
K_1^2 &\leq \frac{E_1(0,t)}{\kappa\zeta_1}e^{-\gamma z}\int_0^t\int_{R_z}\theta^2\,dx\,d\eta \\
&\leq \frac{E_1(0,t)}{\kappa\lambda\zeta_1}e^{-\gamma z}\int_0^t\int_{R_z}\theta_{,\alpha}\theta_{,\alpha}\,dx\,d\eta.
\end{aligned} \tag{5.16}
$$

Similarly, using (3.21) we obtain

$$
\begin{aligned}
K_2^2 &\leq \int_0^t\int_{R_z}\theta_{,\alpha}\theta_{,\alpha}\,dx\,d\eta \int_0^t\int_{R_z} S_{,\beta}S_{,\beta}\,dx\,d\eta \\
&\leq \frac{E(0,t)}{\kappa\zeta_1}e^{-\gamma z}\int_0^t\int_{R_z}\theta_{,j}\theta_{,j}\,dx\,d\eta.
\end{aligned} \tag{5.17}
$$

Using the triangle inequality we find

$$
\begin{aligned}
|K_1 - K_2| &\le |K_1| + |K_2| \\
&\le (\kappa\zeta_1)^{-1/2}[E^{1/2}(0,t) + \lambda^{-1/2}E_1^{1/2}(0,t)]e^{-\gamma z/2} \\
&\quad \times \left(\int_0^t \int_{R_z} \theta_{,j}\theta_{,j}\,dx d\eta\right)^{1/2}.
\end{aligned} \tag{5.18}
$$

Returning to (5.10) we have

$$
\Phi(z,t) \le -\frac{1}{\gamma}\frac{\partial \Phi(z,t)}{\partial z} + \overline{\zeta}\, C(t) e^{-\gamma z/2}\Phi^{1/2}(z,t), \tag{5.19}
$$

where

$$
C(t) = (\zeta\zeta_1)^{-1/2}\{E^{1/2}(0,t) + \lambda^{-1/2}\, E_1^{1/2}(0,t)\}. \tag{5.20}
$$

This inequality integrates to give

$$
\Phi^{1/2}(z,t) \le \{\Phi^{1/2}(0,t) + \overline{\zeta}\; z\, C(t)\}e^{-\gamma z/2}. \tag{5.21}
$$

From (5.11) we observe that

$$
\begin{aligned}
\Phi(0,t) &= \kappa\overline{\zeta}\int_0^t \int_R \theta \Delta S\, dx d\eta \\
&= -\kappa\overline{\zeta}\int_0^t \int_R \theta_{,j} S_{,j}\, dx d\eta.
\end{aligned} \tag{5.22}
$$

Thus

$$
\Phi(0,t) \le \overline{\zeta}(\zeta\zeta_1)^{-1/2}\{\Phi(0,t)E(0,t)\}^{1/2}, \tag{5.23}
$$

which is

$$
\Phi^{1/2}(0,t) \le \overline{\zeta}(\zeta\zeta_1)^{-1/2}\,\{E(0,t)\}^{1/2}. \tag{5.24}
$$

The insertion of (5.24) into (5.21) gives the desired bound,

$$
\Phi(z,t) \le \overline{\zeta}^2\, \{(\zeta\zeta_1)^{-1/2}E^{1/2}(0,t) + z\, C(t)\}^2 e^{-\gamma z}. \tag{5.25}
$$

This inequality exhibits not only exponential decay in z, but also shows that the amplitude term becomes small as $\zeta_1 \to \zeta$. To make (5.25) explicit one would need a bound for $E_1(0,t)$ in terms of data; however, we do not pursue this question in the present investigation.

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