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A NEW MINIMUM THEOREM AND ITS APPLICATIONS

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ABSTRACT. In this paper we first prove a new minimum theorem using the upper semicontinuity of minimizing functions, which is comparable to Berge's theorem. Next, as applications, we shall prove the existence of equilibrium in generalized games and the existence theorem of zeros.

1. Introduction

In 1959, Berge [2] first proved the following maximum theorem which gives conditions under which a "maximizing correspondence" will be closed:

Let E and Y be topological spaces and let $u : E \times Y \to \mathbb{R}$ be a continuous real-valued function; let $F : E \to 2^Y$ be a continuous and compact valued correspondence; and, for each $x \in E$, let $M(x) := \{y \in F(x) : u(x,y) \geq u(x,z)$ for all $z \in F(x)\}$. Then the correspondence M is upper semicontinuous and non-empty compact valued.

Since then, this theorem, called Berge's maximum theorem, has become one of the most useful and powerful theorems in economics, optimization theory, and game theory. And there have been many generalizations and applications of Berge's theorem, e.g. Walker [12], Leininger [9], Tian-Zhou [11]. In their generalizations, continuity assumptions on u and F have been relaxed; but the properties of continuity assumptions of u and F are still needed in the different forms,

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e.g. graph-continuity in [9] and transfer-continuity in [11]. In this paper, by using the upper semicontinuity of minimizing functions, we can relax the continuity assumption of u by the continuity of the second variable only when E and Y are topological vector spaces and u is a bilinear functional. Till now there have been many kinds of maximum theorems generalized and related to Berge's theorem, so in this paper we shall prove a minimum theorem, which is actually equivalent to the maximum theorem.

The purpose of this paper is two-fold. First, we prove a new minimum theorem using the upper semicontinuity of minimizing functions, which is comparable to Berge's theorem. Next, as applications, we shall prove the existence of equilibrium in generalized games and the existence theorem of zeros.

2. Preliminaries

Let A be a subset of a topological space X. We shall denote by 2^A the family of all subsets of A, and by $int_X A$ the interior of A in X. If A is a subset of a vector space, we shall denote by $co\, A$ the convex hull of A. Let X, Y be topological spaces and $T : X \to 2^Y$ be a correspondence. The correspondence T is said to be *closed* or have *closed graph* if the graph of T ($Graph(T) = \{(x,y) \in X \times Y : x \in X, y \in T(x)\}$) is closed in $X \times Y$. A correspondence $T : X \to 2^Y$ is said to be (1) *upper semicontinuous* if for each $x \in X$ and each open set V in Y with $T(x) \subset V$, there exists an open neighborhood U of x in X such that $T(y) \subset V$ for each $y \in U$ and (2) *lower semicontinuous* if for each $x \in X$ and each open set V in Y with $T(x) \cap V \neq \emptyset$, there exists an open neighborhood U of x in X such that $T(y) \cap V \neq \emptyset$ for each $y \in U$, and (3) *continuous* if T is both upper semicontinuous and lower semicontinuous.

Let Φ denote either the real field or the complex field. Let E and F be vector spaces over Φ, $\langle\, , \rangle : F \times E \to \Phi$ be a bilinear functional. For each non-empty subset B of E and $\epsilon > 0$, let

$$U(B;\epsilon) = \{f \in F : \sup_{x \in B} |\langle f, x\rangle| < \epsilon\}.$$

If E is a topological vector space, we let $\eta(F,E)$ be the topology on F

generated by the family

$$\{U(B;\epsilon) : B \text{ is a non-empty bounded subset of } E \text{ and } \epsilon > 0\}$$

as a basis for the neighborhood system at 0. If F possesses the $\eta(F,E)$-topology, then F becomes a topological vector space.

We shall need the following lemmas.

LEMMA 1 [7]. *Let X be a non-empty convex subset of a locally convex Hausdorff topological vector space and D be a non-empty compact subset of X. Let $T : X \to 2^D$ be an upper semicontinuous correspondence such that for each $x \in X$, $T(x)$ is a non-empty closed convex subset of D. Then there exists a point $\bar{x} \in D$ such that $\bar{x} \in T(\bar{x})$.*

In a recent paper [4], Chang-Zhang proved the following very general theorem on lower semicontinuity of the minimum function, which is a slight generalization of Lemma 2 in [8]:

LEMMA 2. *Let E be a topological vector space over Φ and F be a vector space over Φ with the $\eta(F,E)$-topology. Let X be a non-empty bounded subset of E and $T : X \to 2^F$ be an upper semicontinuous multimap such that each $T(x)$ is non-empty compact, and $\langle , \rangle : F \times E \to \Phi$ a bilinear functional such that for each $f \in F$, $x \to \langle f, x\rangle$ is continuous.*

Then for each $y \in E$, the real-valued function $g_y : X \to \mathbb{R}$, defined by

$$g_y(x) = \inf_{w\in T(x)} Re\,\langle w, x-y\rangle, \quad \text{for each } x \in X,$$

is lower semicontinuous.

In many applications on the stability of minimization problems, we shall need the continuity on the minimum or maximum functions, so we do need the upper semicontinuity of g_y on X in Lemma 2.

By replacing the upper semicontinuity of T with the lower semicontinuity in Lemma 2, we first prove the upper semicontinuity of g_y on X as follows:

LEMMA 3. *Let E be a topological vector space over Φ and F be a vector space over Φ with the $\eta(F,E)$-topology. Let X be a non-empty bounded subset of E and $T : X \to 2^F$ be a lower semicontinuous multimap such that each $T(x)$ is non-empty compact, and $\langle , \rangle : F \times E \to \Phi$ a bilinear functional such that for each $f \in F$, $x \to \langle f, x\rangle$ is continuous. Then for each $y \in E$, the real-valued function $g_y : X \to \mathbb{R}$, defined by*

$$g_y(x) = \inf_{w \in T(x)} Re\,\langle w, x-y\rangle, \quad \text{for each } x \in X,$$

is upper semicontinuous.

Proof. Let $x_0 \in X$ and $\epsilon > 0$ be arbitrarily given. Then we shall show that there exists an open neighborhood $N(x_0)$ of x_0 such that

$$g_y(x) \leq g_y(x_0) + \epsilon \quad \text{for each } x \in N(x_0).$$

We first note that the function $w \to \langle w, x-y\rangle$ is a continuous function of w. In fact, for any given w' and $\epsilon > 0$, the set $V := \{w \in F : \sup_{x \in X} |\langle w - w', x-y\rangle| < \epsilon\}$ is an open neighborhood of w' in F, and for every $w \in V$, we have

$$|\langle w, x-y\rangle - \langle w', x-y\rangle| \leq |\langle w-w', x-y\rangle| < \epsilon,$$

and hence $w \to \langle w, x-y\rangle$ is continuous. Since $T(x_0)$ is compact, we can find $w_0 \in T(x_0)$ such that

$$Re\,\langle w_0, x_0-y\rangle \leq \inf_{w \in T(x_0)} Re\,\langle w, x_0-y\rangle + \frac{\epsilon}{2}.$$

For each $w \in T(x_0)$, we let

$$N_w := \left\{ w' \in F : \sup_{t \in \Omega} |\langle w' - w, t\rangle| < \frac{\epsilon}{10} \right\},$$

where $\Omega = \{z - y : z \in X\}$ is a bounded subset of E. Then N_w is an $\eta(F,E)$-open neighborhood of w in F. Since $T(x_0)$ is compact and $T(x_0) \subset \bigcup_{w \in T(x_0)} N_w$, there exists $\{w_1, \cdots, w_n\} \subset T(x_0)$ such that $T(x_0) \subset \bigcup_{i=1}^{n} N_{w_i}$, and there exists some N_{w_j} with $1 \leq j \leq n$ such that $w_0 \in N_{w_j}$. Now we simply denote N_{w_j} by $N(w_0)$ as an open neighborhood of w_0. Then we have $\sup_{t\in\Omega} |\langle w - w_j, t\rangle| < \frac{\epsilon}{10}$ for every $w \in N(w_0)$, and in particular, $|\langle w_0 - w_j, x_0 - y\rangle| < \frac{\epsilon}{10}$. For each $i = 1, \cdots, n$, since $x \to \langle w_i, x\rangle$ is continuous, there exists an open neighborhood N_i of x_0 in X such that $|\langle w_i, x - x_0\rangle| < \frac{\epsilon}{10}$ for each $x \in N_i$. Now let $N_1(x_0) := \bigcap_{i=1}^{n} N_i$; then $N_1(x_0)$ is an open neighborhood of x_0 in X. For every $x \in N_1(x_0)$ and $w \in N(w_0)$, we have

$$
\begin{aligned}
|\langle w, x - x_0\rangle| &\leq |\langle w - w_j, x - y\rangle| + |\langle w - w_j, y - x_0\rangle| + |\langle w_j, x - x_0\rangle| \\
(*) \qquad &< \frac{\epsilon}{10} + \frac{\epsilon}{10} + \frac{\epsilon}{10} = \frac{3\epsilon}{10}.
\end{aligned}
$$

By using $(*)$, we can obtain that for every $x \in N_1(x_0)$ and $w \in N(w_0)$, $Re\langle w, x - y\rangle \leq Re\langle w_0, x_0 - y\rangle + \frac{\epsilon}{2}$. In fact, we have

$$
\begin{aligned}
Re\langle w, x - y\rangle &= Re\langle w, x - x_0\rangle + Re\langle w, x_0 - y\rangle \\
&= Re\langle w, x - x_0\rangle + Re\langle w - w_j, x_0 - y\rangle \\
&\quad + Re\langle w_j - w_0, x_0 - y\rangle + Re\langle w_0, x_0 - y\rangle \\
&< \frac{3\epsilon}{10} + \frac{\epsilon}{10} + \frac{\epsilon}{10} + Re\langle w_0, x_0 - y\rangle \\
&= \frac{\epsilon}{2} + Re\langle w_0, x_0 - y\rangle.
\end{aligned}
$$

Since T is lower semicontinuous at x_0 and $w_0 \in T(x_0) \cap N(w_0)$, there exists an open neighborhood $N_2(x_0)$ of x_0 such that $T(x) \cap N(w_0) \neq \emptyset$ for every $x \in N_2(x_0)$. Finally we let $N(x_0) := N_1(x_0) \cap N_2(x_0)$; then $N(x_0)$ is the desired open neighborhood of x_0. In fact, for every $x \in N(x_0)$, we have

$$
\begin{aligned}
g_y(x) &= \inf_{w\in T(x)} Re\,\langle w, x - y\rangle \\
&\leq Re\langle w, x - y\rangle \quad (\text{ for every } w \in T(x) \cap N(w_0)) \\
&\leq Re\langle w_0, x_0 - y\rangle + \frac{\epsilon}{2} \\
&\leq \inf_{w\in T(x_0)} Re\,\langle w, x_0 - y\rangle + \frac{\epsilon}{2} + \frac{\epsilon}{2} = g_y(x_0) + \epsilon,
\end{aligned}
$$

so that g_y is upper semicontinuous at x_0. □

REMARKS. (1) In Lemma 3, we do not need the strong continuity assumption on the bilinear functional $\langle\, ,\, \rangle : F \times E \to \Phi$, but we only need the continuity assumption on the second variable. Hence we can obtain a general stability theorem as Theorem 3 in [1, p. 70] under the weaker assumption of the continuity on f.
(2) Using Lemma 3, we can prove new generalized variational inequalities and quasi-variational inequalities by following the method of Chang-Zhang in [4].

3. A minimum theorem and its applications

Combining Lemmas 2 and 3, we now prove the following general minimum theorem:

THEOREM 1. *Let E be a topological vector space over Φ and F be a vector space over Φ with the $\eta(F,E)$-topology. Let X be a non-empty bounded subset of E and $T : X \to 2^F$ be a continuous multimap such that each $T(x)$ is non-empty compact, and $\langle , \rangle : F \times E \to \Phi$ a bilinear functional such that for each $f \in F$, $x \to \langle f, x\rangle$ is continuous.*

Then for each $y \in E$, the real-valued function $g_y : X \to \mathbb{R}$, defined by

$$g_y(x) = \inf_{w \in T(x)} Re\,\langle w, x - y\rangle, \quad \textit{for each } x \in X,$$

is continuous, and the minimum correspondence $M : X \to 2^F$ defined by, for each $x \in X$,

$$M(x) = \{w \in T(x) : Re\,\langle w, x - y\rangle = \inf_{w' \in T(x)} Re\,\langle w', x - y\rangle\}$$

is upper semicontinuous and non-empty compact valued.

Proof. The first assertion follows immediately from Lemma 2 and Lemma 3. To prove the upper semicontinuity of M, we write $M(x) = T(x) \cap S(x)$, for each $x \in X$, where $S(x) = \{w \in F : Re\,\langle w, x - y\rangle = \inf_{w' \in T(x)} Re\,\langle w', x - y\rangle\}$. Since the function $w \to \langle w, x - y\rangle$ is continuous and $T(x)$ is a non-empty compact subset of F, each $S(x)$ is non-empty closed and so $M(x)$ is non-empty compact. We first show that the graph of S is closed in $X \times F$. Let $(x_\alpha, w_\alpha)_{\alpha \in \Gamma} \subset Graph(S)$ be a net converging to $(\bar{x}, \bar{w})$ in $X \times F$. Let $\epsilon > 0$ be arbitrarily given. Since

$g_y : X \longrightarrow \mathbb{R}$ is continuous, there exists $\lambda_1 \in \Gamma$ such that $g_y(x_\alpha) \leq g_y(\bar{x}) + \frac{\epsilon}{3}$ for every $\alpha \geq \lambda_1$, i.e.,

$$\inf_{w \in T(x_\alpha)} Re\,\langle w, x_\alpha - y\rangle \leq \inf_{w \in T(\bar{x})} Re\,\langle w, \bar{x} - y\rangle + \frac{\epsilon}{3}.$$

Since F is equipped with $\eta(F, E)$-topology, for an open neighborhood $U(X, \frac{\epsilon}{3}) := \{f \in F : \sup_{x \in X} |\langle f, x - y\rangle| < \frac{\epsilon}{3}\}$ of 0 in F, $\bar{w} + U(X, \frac{\epsilon}{3})$ is an open neighborhood of $\bar{w}$ in F; and we simply denote it by $N(\bar{w})$ as an open neighborhood of $\bar{w}$ in F. Since $(w_\alpha)_{\alpha \in \Gamma}$ converges to $\bar{w}$ and $w \longrightarrow \langle w, x - y\rangle$ is a continuous function on w, for an open neighborhood $N(\bar{w})$ of $\bar{w}$, we can find $\lambda_2 \in \Gamma$ such that for each $\alpha \geq \lambda_2$, $w_\alpha \in N(\bar{w})$, i.e.,

$$\sup_{x \in X} |\langle w_\alpha - \bar{w}, x - y\rangle| < \frac{\epsilon}{3} \qquad \text{for each } \alpha \geq \lambda_2.$$

Also since $x \longrightarrow \langle f, x\rangle$ is a continuous function on x, we can find $\lambda_3 \in \Gamma$ such that for each $\alpha \geq \lambda_3$, $|\langle \bar{w}, x_\alpha - \bar{x}\rangle| < \frac{\epsilon}{3}$ for each $\alpha \geq \lambda_3$. Therefore we finally have $\lambda \in \Gamma$ such that $\lambda \geq \lambda_i$ for each $i = 1, 2, 3$. Then, for each $\alpha \geq \lambda$, we have

$$\begin{aligned}
0 &\geq Re\langle w_\alpha, x_\alpha - y\rangle - \inf_{w \in T(x_\alpha)} Re\,\langle w, x_\alpha - y\rangle \\
&= Re\langle w_\alpha - \bar{w}, x_\alpha - y\rangle + Re\langle \bar{w}, x_\alpha - y\rangle - \inf_{w \in T(x_\alpha)} Re\,\langle w, x_\alpha - y\rangle \\
&= Re\langle w_\alpha - \bar{w}, x_\alpha - y\rangle + Re\,\langle \bar{w}, x_\alpha - \bar{x}\rangle \\
&\quad + Re\langle \bar{w}, \bar{x} - y\rangle - \inf_{w \in T(x_\alpha)} Re\,\langle w, x_\alpha - y\rangle \\
&> -\frac{\epsilon}{3} - \frac{\epsilon}{3} + Re\langle \bar{w}, \bar{x} - y\rangle - \inf_{w \in T(\bar{x})} Re\,\langle w, \bar{x} - y\rangle - \frac{\epsilon}{3},
\end{aligned}$$

so that we have

$$Re\langle \bar{w}, \bar{x} - y\rangle - \inf_{w \in T(\bar{x})} Re\,\langle w, \bar{x} - y\rangle < \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we have $Re\langle \bar{w}, \bar{x} - y\rangle - \inf_{w \in T(\bar{x})} Re\,\langle w, \bar{x} - y\rangle \leq 0$, which implies $(\bar{x}, \bar{w}) \in Graph(S)$. Therefore the graph of S is closed in $X \times F$. Hence, by Proposition 2 in [1, p. 73], M is upper semicontinuous. □

REMARK 2. It should be noted that Theorem 1 is comparable to Berge's theorem in [2]. In fact, in Berge's theorem we must need the continuity of u on $F \times E$, however the choice sets E and F are general topological spaces and u may not be a bilinear functional.

As an application of Theorem 1, we can obtain a new equilibrium existence theorem. We first recall some terminologies of the equilibrium theory in mathematical economics. Let I be a finite or an infinite set of agents. Each agent i chooses a strategy x_i in the choice set X_i of a topological vector space. Denote by X the (Cartesian) product $\Pi_{j\in I}X_j$ and X_{-i} the product $\Pi_{j\in I\setminus\{i\}}X_j$. Denote by x_{-i} and $x = (x_{-i}, x_i)$ the generic elements of X_{-i} and X, respectively. Each agent i has a payoff (utility) function $u_i : X \to \mathbb{R}\cup\{-\infty\}$. Given x_{-i} (the strategies of others), the choice of the $i-$th agent is restricted to a non-empty set $F_i(x_{-i}) \subset X_i$, the feasible strategy set; the $i-$th agent chooses $x_i \in F_i(x_{-i})$ so as to minimize $u(x_{-i}, x_i)$ over $F_i(x_{-i})$.

DEFINITION. A *generalized game* $\Gamma = (X_i, F_i, u_i)_{i\in I}$ is defined as a family of ordered triples (X_i, F_i, u_i). An *equilibrium* for Γ is a point $\hat{x} = (\hat{x}_{-i}, \hat{x}_i) \in X$ such that for each $i \in I$,

(i) $\hat{x}_i \in F_i(\hat{x}_{-i})$, and

(ii) $u(\hat{x}_{-i}, \hat{x}_i) = \inf_{y_i\in F_i(\hat{x}_{-i})} u_i(\hat{x}_{-i}, y_i)$.

If $F_i(x_{-i}) = X_i$ for each $i \in I$, then the generalized game Γ reduces to the conventional game $\Gamma = (X_i, u_i)$ and the equilibrium is called a *Nash equilibrium.*

In 1952, Debreu [5] first proved an equilibrium existence of the generalized game, and since then, the classical Debreu theorem on contractible polyhedron choice set has been generalized in many directions, e.g., see [1,3].

Now we apply Theorem 1 to prove an existence theorem on equilibria in generalized games where the payoff functions u_i are neither continuous nor jointly continuous:

THEOREM 2. *Let I be a finite or an infinite set of agents. For each $i \in I$, X_i is a non-empty bounded convex subset in a locally convex Hausdorff topological vector space and D_i be a non-empty compact subset of X_i. Let $u_i : X_{-i} \times X_i \to \mathbb{R}\cup\{-\infty\}$ be a bilinear functional* *and equip* X_{-i} *with the* $\eta(X_{-i}, X_i)$*-topology generated by the bilinear functional* u_i. *Furthermore, we assume that*

(1) $F_i : X_{-i} \to 2^{D_i}$ *is a continuous correspondence such that for each* $x_{-i} \in X_{-i}$, $F_i(x_{-i})$ *is a non-empty compact convex subset of* D_i,

(2) *for each fixed* $z_{-i} \in X_{-i}$, $x_i \to u_i(z_{-i}, x_i)$ *is continuous.*

Then the generalized game $\Gamma = (X_i, F_i, u_i)_{i\in I}$ *has an equilibrium.*

Proof. For each $i \in I$, we define the minimizing correspondence $M_i : X_{-i} \to 2^{D_i}$ by,

$$M_i(x_{-i}) := \{y_i \in F_i(x_{-i}) \ : \ u_i(x_{-i}, y_i) = \inf_{z_i \in F_i(x_{-i})} u_i(x_{-i}, z_i)\}, \quad \text{for each } x_{-i} \in X_{-i}.$$

Since the bilinear functional u_i satisfies the assumption of Theorem 1, so we know that M_i is upper semicontinuous and each $M_i(x_{-i})$ is non-empty compact. Furthermore, $M_i(x_{-i})$ is convex for each $x_{-i} \in X_{-i}$. In fact, for any y_i, $y_i' \in M_i(x_{-i})$ and $t \in [0, 1]$,

$$\begin{aligned} & u_i(x_{-i}, t y_i + (1-t)y_i') \\ &= t \cdot u_i(x_{-i}, y_i) + (1-t) \cdot u_i(x_{-i}, y_i') \\ &= t \cdot \inf_{z_i \in F_i(x_{-i})} u_i(x_{-i}, z_i) + (1-t) \cdot \inf_{z_i \in F_i(x_{-i})} u_i(x_{-i}, z_i) \\ &= \inf_{z_i \in F_i(x_{-i})} u_i(x_{-i}, z_i), \end{aligned}$$

so that $ty_i + (1-t)y_i' \in M_i(x_{-i})$. Hence $M_i(x_{-i})$ is convex. We now define the correspondence $M : X \to 2^D$ by

$$M(x) := \Pi_{i\in I} M_i(x_{-i}), \quad \text{for each } x = (x_{-i}, x_i) \in X,$$

where $D = \Pi_{i \in I} D_i$ is a non-empty compact subset of X. Then, by Lemma 3 in [6], we know that the correspondence M is also upper semicontinuous such that each $M(x)$ is non-empty compact convex. Therefore, by Lemma 1, there exists $\hat{x} \in M(\hat{x})$. Thus $\hat{x}$ is an equilibrium for the generalized game Γ, i.e., for each $i \in I$, $\hat{x}_i \in F_i(\hat{x}_{-i})$ and $u_i(x_{-i}, x_i) \leq u_i(x_{-i}, z_i)$ for all $z_i \in F_i(x_{-i})$. □

REMARK. Theorem 2 is different from Theorem 5 in [11] in the following aspects:

(1) X_i need not be compact but the feasible strategy set $F_i(x_i)$ is contained in a compact set D_i; however we do need the strong continuity assumption on F_i.

(2) The continuity assumption on $u_i(x_{-i}, x_i)$ is quite different. In fact, in Theorem 2, we do not need the continuity assumption of u_i on $X_{-i} \times X_i$, and even more do not need the continuity assumption on the first variable x_{-i}. We do need the weaker assumption of the continuity of the second variable x_i only; but in Theorem 5 in [8], they need some kind of jointly continuity assumption on u_i.

Now let $\Delta_n = \{x = (x_1, \cdots, x_{n+1}) \ : \ x_i \geq 0, \ i = 1, \cdots, n+1, \text{ and } \sum_{i=1}^{n+1} x_i = 1\}$ be a closed simplex in $\mathbb{R}^{n+1}$ and $S_n = \{x \in \Delta_n \ : \ x_i > 0, \ i = 1, \cdots, n+1\}$ be the standard n-simplex. Define the partial order in $\mathbb{R}^{n+1}$ as follows: when $x = (x_1, \cdots, x_{n+1})$, $y = (y_1, \cdots, y_{n+1}) \in \mathbb{R}^{n+1}$, we define $x \prec y$ if $x_i < y_i$ for every $i = 1, \cdots, n+1$; and $x \preceq y$ if $x_i \leq y_i$ for every $i = 1, \cdots, n+1$. And we denote the standard dot product of x, y in $\mathbb{R}^{n+1}$ by $x \cdot y$, and denote the origin $(0, \cdots, 0)$ in $\mathbb{R}^{n+1}$ by O.

Finally, we prove the following existence theorem of zeros:

THEOREM 3. *Let S be a non-empty open convex subset of S_n and $f : S \to \mathbb{R}^{n+1}$ be continuous such that*

(*) $p \cdot f(p) \leq 0$ *for every* $p \in S$, *and* $p \cdot f(p) = 0$ *whenever* $f(p) \preceq 0$.

Let $p \in S$ be arbitrarily fixed and $K_1 = \{x \in S : p \cdot f(x) \leq 0\}$. If we assume that $\inf_{z \in \Delta_n \setminus S} z \cdot f(x) \leq 0$ for every x with $\inf_{z \in K_1} z \cdot f(x) \leq 0$, then the set $\{p \in S : f(p) = O\}$ of zeros is non-empty compact.

Proof. Since f is continuous, the set K_1 is non-empty closed. Then we can find a non-empty compact convex set K such that $K_1 \subset int_{\Delta_n} K \subset K \subset S$.

Now we define a minimum correspondence $\phi : K \to 2^K$ by

$$\phi(x) := \{y \in K : \ y \cdot f(x) = \inf_{z \in K} \ z \cdot f(x)\}, \quad \text{for each } x \in K.$$

Then, by Theorem 1, ϕ is upper semicontinuous and it is easy to see that each $\phi(x)$ is non-empty compact convex. Therefore, by Kakutani's fixed point theorem, there exists a point $\hat{x} \in K$ such that $\hat{x} \in \phi(\hat{x})$, i.e., $\hat{x} \cdot f(\hat{x}) = \inf_{z \in K} z \cdot f(\hat{x})$. By the assumption $(*)$, we have $0 \geq \hat{x} \cdot f(\hat{x}) = \inf_{z \in K} z \cdot f(\hat{x})$. Since $K_1 \subset K$, $\inf_{z \in K_1} z \cdot f(\hat{x}) \leq 0$, and so by the assumption, $\inf_{z \in \Delta_n \setminus S} z \cdot f(\hat{x}) \leq 0$. For any given $z \in S \setminus K$, we can find $x_1 \in K$, $x_2 \in \Delta_n \setminus S$ and $\lambda \in (0,1)$ such that $z = \lambda x_1 + (1-\lambda)x_2$. Therefore we have

$$\begin{aligned} z \cdot f(\hat{x}) &= (\lambda x_1 + (1-\lambda)x_2) \cdot f(\hat{x}) \\ &= \lambda x_1 \cdot f(\hat{x}) + (1-\lambda)x_2 \cdot f(\hat{x}) \leq 0, \end{aligned}$$

so that $\inf_{z \in S \setminus K} z \cdot f(\hat{x}) \leq 0$, and hence $\inf_{z \in \Delta_n} z \cdot f(\hat{x}) \leq 0$. Therefore, by Proposition 2.14 in [3], we have $f(\hat{x}) \preceq 0$. By the assumption, we have $\hat{x} \cdot f(\hat{x}) = 0$. Since $\hat{x} \in S$, we must have $f(\hat{x}) = O$. □

REMARK. The assumption $(*)$ in Theorem 3 is weaker than strong Walras' law ($p \cdot f(p) = 0$ for every $p \in S$), and stronger than weak Walras' law ($p \cdot f(p) \leq 0$ for every $p \in S$).

The following example can be suitable for Theorem 3 and the strong Walras' law is not satisfied, so that the previous many theorems on the market equilibrium using the strong Walras' law, e.g., Theorems 8.16 and 18.13 in [3] due to Neuefeind and Grandmont, respectively, can not be applied:

EXAMPLE. Let Δ_1 be the compact convex simplex in R^2 and $S = \{(r,\theta) : r\cos(\theta - \frac{\pi}{4}) = \frac{1}{\sqrt{2}},\ \frac{\pi}{8} < \theta < \frac{3\pi}{8}\}$ be an open convex subset of Δ_1. Let $f : S \to R^2$ be a continuous excess demand mapping defined by

$$f(r,\theta) = \begin{cases} (|\theta - \frac{\pi}{4}|r,\ \theta + \frac{\pi}{2} + \frac{1}{2}|\theta - \frac{\pi}{4}|), & \text{if } \frac{\pi}{8} < \theta \leq \frac{\pi}{4}, \\ (|\theta - \frac{\pi}{4}|r,\ \theta - \frac{\pi}{2} - \frac{1}{2}|\theta - \frac{\pi}{4}|), & \text{if } \frac{\pi}{4} < \theta < \frac{3\pi}{8}. \end{cases}$$

Then it is easy to see that f is a continuous mapping such that $f(r,\theta) \to O$ as $\theta \to \frac{\pi}{4}$, and it is noted that for every $(r,\theta) \in S$, $(r,\theta) \cdot f(r,\theta) \leq 0$ since f is a mapping of a kind of rotation of more than $\frac{\pi}{2}$. For any fixed $p \in S$, $\inf_{z \in K_1} z \cdot f(x) \leq 0$ for every $x \in S$, and also we know that $\inf_{z \in \Delta_n \setminus S} z \cdot f(x) \leq 0$ for every $x \in S$. Therefore we can show that all the hypotheses of Theorem 3 are satisfied, so that we can find a zero $p^* = (\frac{1}{\sqrt{2}}, \frac{\pi}{4}) \in S$ of f, i.e., $f(p^*) = O$. Finally, it should be noted that $(\frac{1}{\sqrt{2}} \cos^{-1} \frac{\pi}{16}, \frac{3\pi}{16}) \cdot f(\frac{1}{\sqrt{2}} \cos^{-1} \frac{\pi}{16}, \frac{3\pi}{16}) < 0$, so that the strong Walras' law does not hold.

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