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ON THE VECTOR-VALUED INDEX

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ABSTRACT. We give a definition of the vector-valued index for $\mathbb{Z}$-actions extending the numerical index in [9] and prove the extension theorem for $\mathbb{Z}$-actions for showing basic properties of the vector-valued index.

1. Introduction

In a dynamical system there can coexist various types of orbits. The complexity of orbits having some properties is measured by an index. Index theory for group actions plays an important role in variational analysis and differential equations, see [1], [2], [3], [7], [10].

The aim of this paper is to extend index theory for compact Lie groups to $\mathbb{Z}$-actions induced by a homeomorphism of a compact space. Using the rotation number of circle homeomorphisms and the generalized Borsuk-Ulam theorem, the numerical $\mathbb{Z}$-index was given in [9]. In this paper we extend it to the vector-valued $\mathbb{Z}$-index. It is unusual form but meaningful for the investigation of the possible behaviour of orbits.

In Section 2 we observe homeomorphisms of $(S^1)^n$ with a dense orbit under assumption of equicontinuity implying existence of rationally independent numbers as a generalization of the rotation number for circle homeomorphisms. We introduce the standard axioms for an index and then define a vector-valued index for $\mathbb{Z}$-actions. We give a $\mathbb{Z}$-version of the Borsuk-Ulam theorem and prove the extension of continuous equivariant maps for $\mathbb{Z}$-actions. Therefore, we can show in Section 3 that the vector-valued index has the desired properties as an

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index. It is noted that there is also an alternative way of introducing an index theory for $\mathbb{Z}$-actions based on group compactifications, for example, the Ellis group.

2. Rationally independent numbers

We consider a discrete dynamical system

$$\pi : \ \mathbb{Z} \times X \longrightarrow X, \qquad (m, x) \mapsto f^m(x)$$

induced by a homeomorphism $f : X \longrightarrow X$ of a compact space X. This $\mathbb{Z}$-space will be denoted by (X, f).

Let (X, f) and (Y, g) be discrete dynamical systems. A map $\Phi : X \longrightarrow Y$ is said to be *equivariant*, denoted by $\Phi : (X, f) \longrightarrow (Y, g)$, if $\Phi \circ f = g \circ \Phi$. A set A in X is said to be *invariant* under f if $f(A) = A$.

For $m \in \mathbb{Z}$ and $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{R}^n$, let $\widetilde{m\alpha} : S^{2k_1-1} \times \cdots \times S^{2k_n-1} \longrightarrow S^{2k_1-1} \times \cdots \times S^{2k_n-1}$ be defined by

$$\widetilde{m\alpha}(z) = \left(e^{2\pi i m\alpha_1} z_1, \cdots, e^{2\pi i m\alpha_n} z_n \right)$$

for every $z = (z_1, \cdots, z_n) \in S^{2k_1-1} \times \cdots \times S^{2k_n-1}$.

DEFINITION 1. The n real numbers $\alpha_1, \cdots, \alpha_n$ will be called *rationally independent* if any relation of the form $c_1\alpha_1 + \cdots + c_n\alpha_n = 0$ with rational numbers $c_1, \cdots, c_n$ implies that $c_1 = \cdots = c_n = 0$.

The following result which is deduced from the well-known Borsuk-Ulam theorem is fundamental in an index theory for $\mathbb{Z}$-actions. See [8], Theorem 6.

THEOREM 2. *For $j = 1, \cdots, n$, let k_j, l_j be natural numbers such that $k_{j_0} > l_{j_0}$ for some $j_0 \in \{1, \cdots, n\}$. If $1, \alpha_1, \cdots, \alpha_n$ are rationally independent, then there is no continuous map*

$$h : \ S^{2k_1-1} \times \cdots \times S^{2k_n-1} \longrightarrow S^{2l_1-1} \times \cdots \times S^{2l_n-1}$$

such that h is equivariant with respect to $\tilde{\alpha}$.

The following new result which is a generalization of the rotation number for circle homeomorphisms is based on group compactifications.

THEOREM 3. *For $n \in \mathbb{N}$, let $g : (S^1)^n \to (S^1)^n$ be a homeomorphism with a dense orbit such that $\{g^m : m \in \mathbb{Z}\}$ is equicontinuous. Then there are rationally independent numbers $1, \alpha_1, \cdots, \alpha_n \in [0,1]$ and a homeomorphism $\Psi : (S^1)^n \to (S^1)^n$ such that $\Psi(g(x)) = \tilde{\alpha}(\Psi(x))$ for every $x \in (S^1)^n$.*

Proof. It follows from the equicontinuity of $\{g^m : m \in \mathbb{Z}\}$ that for every $x \in (S^1)^n$, $\{g^m(x) : m \in \mathbb{Z}\}$ is dense in $(S^1)^n$. Hence, $(S^1)^n$ has the structure of a compact topological group such that $g(x) = a \cdot x$ for all $x \in (S^1)^n$ and some $a \in (S^1)^n$ and $\{a^m : m \in \mathbb{Z}\}$ is dense in $(S^1)^n$ (cf. [12], IV(3.42)). Therefore there exists a homeomorphism $\Psi : (S^1)^n \to (S^1)^n$ that is a morphism of groups because $(S^1)^n$ is a connected compact Lie group (cf. [4]). Thus there are $\alpha_1, \cdots, \alpha_n \in [0,1]$ such that $\Psi(a) = (e^{2\pi i\alpha_1}, \cdots, e^{2\pi i\alpha_n})$. For every $x \in (S^1)^n$, we have

$$\Psi(g(x)) = \Psi(a \cdot x) = \Psi(a)\Psi(x) = \tilde{\alpha}(\Psi(x)).$$

Since $\{a^m : m \in \mathbb{Z}\}$ is dense in $(S^1)^n$ and Ψ is a homeomorphism, $1, \alpha_1, \cdots, \alpha_n$ are rationally independent (cf. [12], III (1.14)). □

REMARK. Note that for the case $n = 1$ the condition of equicontinuity is unnecessary. See [12], IV(6.6).

We introduce the standard axioms for an index in more general situation and define the vector-valued index for $\mathbb{Z}$-actions in the next section.

Given a continuous action $\pi : G \times X \to X$ of a topological group G on a topological space X, we denote

$$\Sigma(X, G) := \{A \subset X : A \text{ is } G\text{-invariant}\}.$$

A *G-index* is a mapping

$$i : \Sigma(X, G) \to \mathbb{N} \cup \{0, \infty\}$$

which has the following properties

(1) $i(A) = 0$ if and only if $A = \emptyset$.
(2) If $A, B \in \Sigma(X, G)$ and $\Phi : A \to B$ is a continuous equivariant map, then $i(A) \le i(B)$.

(3) If $A \in \Sigma(X,G)$ is a closed set, then there exists an open neighborhood $U \in \Sigma(X,G)$ of A such that $i(A) = i(U)$.

(4) If $A, B \in \Sigma(X,G)$ are closed sets, then $i(A \cup B) \le i(A) + i(B)$.

For indexes the concept of join of topological spaces plays a vital role. See Theorem 6 below.

Let $\Delta^{n-1} := \{(t_1, \cdots, t_n) \in [0,1]^n : \sum_{i=1}^n t_i = 1\}$ be the standard $(n-1)$-simplex. For any topological spaces $Y_1, \cdots, Y_n$ we define the join

$$Y_1 * \cdots * Y_n := \Big(\Delta^{n-1} \times \prod_{i=1}^n Y_i\Big)/\sim$$

with the following equivalence relation:

$$(t_1, \cdots, t_n, y_1, \cdots, y_n) \sim (s_1, \cdots, s_n, y_1', \cdots, y_n')$$
$$\text{if } t_i = s_i \text{ and } (y_i = y_i' \text{ when } t_i \neq 0) \text{ for } i = 1, \cdots, n.$$

In the case of $Y_1, \cdots, Y_n$ compact, the quotient topology on $Y_1 * \cdots * Y_n$ coincides with the initial topology for which the map

$$q : Y_1 * \cdots * Y_n \to \Delta^{n-1}, \quad \big[(t_1, \cdots, t_n, y_1, \cdots, y_n)\big] \mapsto (t_1, \cdots, t_n)$$

and partial functions

$$p_j : Y_1 * \cdots * Y_n \rightarrowtail Y_j, \quad \big[(t_1, \cdots, t_n, y_1, \cdots, y_n)\big] \mapsto y_j$$

are continuous.

Given $\mathbb{Z}$-spaces $Y_1, \ldots, Y_n$, the $\mathbb{Z}$-action on $\prod_{i=1}^n Y_i$ carries over in the canonical wise to $Y_1 * \cdots * Y_n$. For further informations about joins we refer to [5].

3. The vector-valued index

We consider in the sequel a homeomorphism $f : X \to X$ on a compact topological space X such that $\{f^m : m \in \mathbb{Z}\}$ is equicontinuous. Let

$$\Sigma(X,f) := \{A \subset X : A \text{ is invariant and } \big(\overline{O(x)}, f\big) \simeq \big((S^1)^n, \tilde{\alpha}\big)$$
$$\text{for all } x \in A\}$$

where $\tilde{\alpha}$ is given by rationally independent numbers $1, \alpha_1, \cdots, \alpha_n \in [0,1]$ as in Section 2 and $\overline{O(x)}$ denotes the closure of $O(x) = \{f^m(x) : m \in \mathbb{Z}\}$.

In this framework we define the map $g : \Sigma(X, f) \to \mathbb{N}^n \cup \{0, \infty\}$ by

$$g(A, f) := \min\{(k_1, \cdots, k_n) \in \mathbb{N}^n : \text{there exist an } m \in \mathbb{N} \text{ and a continuous map } \varphi : (A, f) \to (\prod_{j=1}^{n} S^{2k_j - 1}, \widetilde{m\alpha})\}.$$

for $A \in \Sigma(X, f) \setminus \{\emptyset\}$, $g(A, f) := \infty$ if such a map φ does not exist, and $g(\emptyset, f) := 0$. Here minimum means componentwise minimum by projections φ_j.

Theorem 2 implies the dimension property for the map g that $g(\prod_{j=1}^{n} S^{2k_j-1}, \tilde{\alpha}) = (k_1, \cdots, k_n)$, where $\tilde{\alpha}$ is defined as in Section 2 such that the numbers $1, \alpha_1, \cdots, \alpha_n$ are rationally independent. See [10] for the relationship between a Borsuk-Ulam theorem and index theories.

We will show that this is an index in the analogous sense of the definition stated in Section 2. The main tool for the proof is the theorem about existence of a continuous equivariant extension. The extension theorem for compact Lie groups has been proved by using Bochner integrals, see [11]. In general this does not hold for $\mathbb{Z}$-actions, but the almost periodicity allows us to do that.

In the following we always assume that $X \in \Sigma(X, f)$ and $\tilde{\alpha}$ is given by rationally independent numbers $1, \alpha_1, \cdots, \alpha_n \in [0, 1]$.

THEOREM 4. *For every invariant closed set A in X and for every continuous map $\varphi : (A, f) \to (\mathbb{C}^{k_1} \times \cdots \times \mathbb{C}^{k_n}, \tilde{\alpha})$, φ has a continuous extension*

$$\Phi : (X, f) \longrightarrow (\mathbb{C}^{k_1} \times \cdots \times \mathbb{C}^{k_n}, \tilde{\alpha}).$$

Proof. Let X be a compact space, regarded as a uniform space with a uniformity $\mathcal{U}$ on X. Let $A \subset X$ be an invariant closed set and let $\varphi = (\varphi_1, \cdots, \varphi_n) : (A, f) \to (\mathbb{C}^{k_1} \times \cdots \times \mathbb{C}^{k_n}, \tilde{\alpha})$ be a continuous map. Since A is a closed subset of normal space X, there is, by the extension theorem of Tietze-Urysohn, a continuous map $\hat{\varphi} = (\hat{\varphi}_1, \cdots, \hat{\varphi}_n) : X \to \mathbb{C}^{k_1} \times \cdots \times \mathbb{C}^{k_n}$ such that $\hat{\varphi}|_A = \varphi$. Let us define $\Phi = (\Phi_1, \cdots, \Phi_n) : X \to \mathbb{C}^{k_1} \times \cdots \times \mathbb{C}^{k_n}$ by

$$\Phi(x) := \lim_{k\to\infty} \frac{1}{2k+1} \sum_{|m|\le k} \widetilde{-m\alpha}\,(\hat{\varphi}(f^m(x))) \qquad \text{for each } x \in X.$$

For $j = 1, \cdots, n$, Φ_j is well-defined because $F_j : \mathbb{Z} \to \mathbb{C}^{k_j}$, $m \mapsto e^{-2\pi i m\alpha_j}\hat{\varphi}_j(f^m(x))$ is an almost periodic map implying existence of the mean value of F_j (cf. [6]).

Since Φ_j is an equivariant map with $\Phi_j|_A = \varphi_j$, Φ is also an equivariant extension of φ.

To show that Φ is continuous, let $\epsilon > 0$ and $V_\epsilon := \{\, (y, y') \in \prod_{j=1}^{n} \mathbb{C}^{k_j} \times \prod_{j=1}^{n} \mathbb{C}^{k_j} : \| y - y' \| < \epsilon \,\}$. As $\hat{\varphi}$ is uniformly continuous, there exists an $M \in \mathcal{U}$ such that for all $z, z' \in X$

$$(z, z') \in M \ \text{ implies } \ (\,\hat{\varphi}(z),\ \hat{\varphi}(z')\,) \in V_{\frac{\epsilon}{2}}.$$

Since $\{f^n : n \in \mathbb{Z}\}$ is uniformly equicontinuous on X, there is an $N \in \mathcal{U}$ such that for all $x, y \in X$ and for all $m \in \mathbb{Z}$

$$(x, y) \in N \ \text{ implies } \ (\,f^m(x),\ f^m(y)\,) \in M,$$

hence we have

$$\Big(\hat{\varphi}(f^m(x)),\ \hat{\varphi}(f^m(y))\Big) \in V_{\frac{\epsilon}{2}}.$$

Consequently, we obtain that for all $x, y \in X$ with $(x, y) \in N$

$$\begin{aligned} &\|\Phi(x) - \Phi(y)\| \\ &= \Big\| \lim_{k\to\infty} \frac{1}{2k+1} \sum_{|m|\le k} \widetilde{-m\alpha}\Big(\hat{\varphi}(f^m(x)) - \hat{\varphi}(f^m(y))\Big) \Big\| \\ &\le \limsup_{k\to\infty} \frac{1}{2k+1} \sum_{|m|\le k} \Big\| \Big(\hat{\varphi}(f^m(x)) - \hat{\varphi}(f^m(y))\Big) \Big\| \\ &< \epsilon, \end{aligned}$$

that is, $(\Phi(x), \Phi(y)) \in V_\epsilon$. This completes the proof. □

From Theorem 4 and the fact that $S^{2k_1-1} \times \cdots \times S^{2k_n-1}$ is a neighborhood retract of $\mathbb{C}^{k_1} \times \cdots \times \mathbb{C}^{k_n}$, we can easily deduce the following theorem. See [11] for compact Lie groups.

THEOREM 5. *For every invariant closed set A in X and for every continuous map $\varphi : (A, f) \to (S^{2k_1-1} \times \cdots \times S^{2k_n-1}, \tilde{\alpha})$, there are an invariant open neighborhood $U \subset X$ of A and a continuous extension*

$$\Phi : \ (U, f) \longrightarrow (S^{2k_1-1} \times \cdots \times S^{2k_n-1}, \tilde{\alpha})$$

of φ.

Next, we obtain the following result by using the concept of join and its initial topology presented in Section 2.

THEOREM 6. *For $l = 1, 2$, let A_l be an invariant closed set in X and let $\varphi_l : (A_l, f) \to (\prod_{j=1}^{n} S^{2k_{lj}-1}, \tilde{\alpha})$ be a continuous map. Then there exists a continuous map*

$$\Phi : \ (A_1 \cup A_2, f) \longrightarrow (\prod_{j=1}^{n} S^{2k_{1j}-1} * S^{2k_{2j}-1}, \tilde{\alpha}).$$

Furthermore, there is a continuous map

$$\Psi : (A_1 \cup A_2, f) \longrightarrow (\prod_{j=1}^{n} S^{2(k_{1j}+k_{2j})-1}, \tilde{\alpha}).$$

Proof. Let $A_1, A_2 \subset X$ be invariant closed sets, and let φ_1, φ_2 be continuous maps. It is sufficient to prove the result for the case

$$\Phi_j : \ (A_1 \cup A_2, f) \longrightarrow (S^{2k_{1j}-1} * S^{2k_{2j}-1}, \tilde{\alpha}_j) \ \text{ for } j = 1, \cdots, n.$$

Let $j \in \{1, \cdots, n\}$ be fixed. For $l = 1, 2$, by Theorem 5, there are an invariant open neighborhood $U_{lj} \subset X$ of A_l and a continuous extension

$\Phi_{lj} : (U_{lj}, f) \to (S^{2k_{lj}-1}, \tilde{\alpha}_j)$ of φ_{lj}. Since X is normal, there exists a continuous function $\gamma'_{lj} : X \to [0,1]$ such that

$$\gamma'_{lj}(x) = \begin{cases} 1 & \text{for } x \in A_l \\ 0 & \text{for } x \in X \setminus U_{lj} \end{cases}.$$

Let $\gamma_{lj} : X \to \mathbb{R}$, $x \mapsto \lim\limits_{k\to\infty} \frac{1}{2k+1} \sum\limits_{|m|\le k} \gamma'_{lj}(f^m(x))$. Then γ_{lj} is continuous and invariant, that is, $\gamma_{lj} \circ f = \gamma_{lj}$. It follows that $\gamma_{lj}|_{A_l} = 1$, $\gamma_{lj}|_{X\setminus U_{lj}} = 0$ and $\gamma_{lj}(X) \subset [0,1]$.
Let $\tilde{\gamma}_j : A_1 \cup A_2 \to \Delta^1 = \{ (t_1, t_2) \in [0,1]^2 : t_1 + t_2 = 1 \}$,

$$\tilde{\gamma}_j(x) := (\tilde{\gamma}_{1j}(x), \tilde{\gamma}_{2j}(x)),\ \tilde{\gamma}_{lj}(x) := \frac{\gamma_{lj}(x)}{\gamma_{1j}(x) + \gamma_{2j}(x)} \quad \text{for } l = 1, 2.$$

Then $\tilde{\gamma}_j$ is continuous, invariant and $\tilde{\gamma}_{lj}|_{(A_1\cup A_2)\setminus U_{lj}} = 0$.
Define $\Phi_j : A_1 \cup A_2 \to S^{2k_{1j}-1} * S^{2k_{2j}-1}$ by

$$\Phi_j(x) := [\,(\tilde{\gamma}_{1j}(x), \tilde{\gamma}_{2j}(x), \tilde{\Phi}_{1j}(x), \tilde{\Phi}_{2j}(x))\,],$$

where $\tilde{\Phi}_{lj} : X \to S^{2k_{lj}-1}$ is an arbitrary extension of Φ_{lj}. Then Φ_j is well-defined, continuous and equivariant. Recall the initial topology and the group action on $S^{2k_1-1} * S^{2k_2-1}$.
Moreover, since $S^{2k_1-1} * S^{2k_2-1}$ is homeomorphic to $S^{2(k_1+k_2)-1}$, it is clear that there is a continuous map

$$\Psi_j : (A_1 \cup A_2, f) \longrightarrow (S^{2(k_{1j}+k_{2j})-1}, \tilde{\alpha}_j).$$

This completes the proof. □

REMARK. Theorem 4 - 6 generalizes the results in [9].

Now we prove that the map g has the basic properties for an index, that is, g is a vector-valued index for $\mathbb{Z}$-actions. Note that $(k_1, \cdots, k_n) \le (l_1, \cdots, l_n)$ means that $k_j \le l_j$ for $j = 1, \cdots, n$.

THEOREM 7. *Let $f : X \to X$ be a homeomorphism on a compact topological space X such that $\{f^m : m \in \mathbb{Z}\}$ is equicontinuous, and let $X \in \Sigma(X, f)$. Then the following statements hold:*

(1) *If $A, B \subset X$ are invariant and there exists a continuous map $h : (A, f) \to (B, f)$, then $g(A, f) \leq g(B, f)$.*
(2) *If $A, B \subset X$ are invariant and $A \subset B$, then $g(A, f) \leq g(B, f)$.*
(3) *If A_1, A_2 are invariant closed sets in X, then $g(A_1 \cup A_2, f) \leq g(A_1, f) + g(A_2, f)$.*
(4) *If A is an invariant closed set in X, then there is an invariant open neighborhood $U \subset X$ of A such that $g(A, f) = g(U, f)$.*

Proof. Let $\tilde{\alpha}$ be a map given by $X \in \Sigma(X, f)$.

(1) If $g(B, f) = \infty$, the statement (1) holds. Let now $g(B, f) = (k_1, \cdots, k_n)$. By definition of g, there are an $m \in \mathbb{N}$ and a continuous map $\varphi : (B, f) \to (\prod_{j=1}^{n} S^{2k_j - 1}, \widetilde{m\alpha})$. Setting $\Phi := \varphi \circ h$, the map $\Phi : (A, f) \to (\prod_{j=1}^{n} S^{2k_j-1}, \widetilde{m\alpha})$ is continuous. It is obvious that $g(A, f) \leq (k_1, \cdots, k_n) = g(B, f)$.

(2) follows from (1), with the inclusion map $i : A \hookrightarrow B$.

(3) If $g(A_1, f) = \infty$ or $g(A_2, f) = \infty$, then the conclusion is clear by (2). Let $g(A_1, f) = (k_{11}, \cdots, k_{1n})$ and $g(A_2, f) = (k_{21}, \cdots, k_{2n})$. Then there are $m_1, m_2 \in \mathbb{N}$ and continuous maps

$$\varphi_1 : (A_1, f) \to (\prod_{j=1}^{n} S^{2k_{1j}-1}, \widetilde{m_1\alpha})$$

$$\varphi_2 : (A_2, f) \to (\prod_{j=1}^{n} S^{2k_{2j}-1}, \widetilde{m_2\alpha}).$$

Hence there are continuous maps

$$\psi_1 : (A_1, f) \to (\prod_{j=1}^{n} S^{2k_{1j}-1}, \widetilde{m_2 m_1\alpha})$$

$$\psi_2 : (A_2, f) \to (\prod_{j=1}^{n} S^{2k_{2j}-1}, \widetilde{m_1 m_2\alpha}).$$

By Theorem 6, there exists a continuous map

$$\Psi:\ (A_1 \cup A_2, f) \ \rightarrow\ \Big(\prod_{j=1}^{n} S^{2(k_{1j}+k_{2j})-1},\ \widetilde{m_1 m_2 \alpha}\Big).$$

Therefore, $g(A_1 \cup A_2, f) \leq\ (k_{11}+k_{21}, \cdots, k_{1n}+k_{2n})\ =\ g(A_1, f)\ +\ g(A_2, f)$.

(4) Let $g(A, f) = \infty$, and let $V \subset X$ be an arbitrary invariant open neighborhood of A. By (2), we have $g(A, f) \leq g(V, f)$, and hence $g(A, f) = g(V, f)$. Now let $g(A, f) = (k_1, \cdots, k_n)$. Then there are an $m \in \mathbb{N}$ and a continuous map $\varphi : (A, f) \rightarrow (\prod_{j=1}^{n} S^{2k_j-1}, \widetilde{m\alpha})$. By Theorem 5, there are an invariant open neighborhood $U \subset X$ of A and a continuous map

$$\Phi:\ (U, f)\ \rightarrow\ \Big(\prod_{j=1}^{n} S^{2k_j-1}, \widetilde{m\alpha}\Big).$$

Hence we have $g(U, f) \leq (k_1, \cdots, k_n) = g(A, f) \leq g(U, f)$, therefore $g(A, f) = g(U, f)$. This completes the proof. □

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