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QUATERNIONICALLY PROJECTIVE CORRESPONDENCE ON AN ALMOST QUATERNIONIC STRUCTURE

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ABSTRACT. In the present paper, we introduce the notions of quaternionically planar curves and quaternionically projective transformations to the case of almost quaternionic manifold with symmetric affine connection. Also, we obtain an invariant tensor field under the quaternionically projective transformation, and show that a quaternionic Kählerian manifold with such a vanishing tensor field is of constant Q-sectional curvature.

1. Introduction

In an Hermitian manifold, Otsuki and Tashiro ([4]) has studied the holomorphically projective change of the Riemannian connection, i.e. a change which preserves the system of holomorphically planar curves, and has obtained interesting results. In an almost complex manifold, Tashiro ([5]) has also studied such a change of a symmetric affine connection with respect to which the almost complex structure is covariant constant. He introduced the holomorphically projective curvature tensor which is invariant under holomorphically projective changes of the connection and has characterized the holomorphically projective flatness of the connection by the vanishing of its holomorphically projective correspondences of Kählerian manifolds.

In the present paper, generalizing those situations, we shall introduce the notions of quaternionically planar curves and quaternionically

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projective correspondence to the case of almost quaternionic manifold with symmetric affine connection. Next, a tensor invariant such a correspondence will be obtained. Finally, in a metric case we shall obtain the tensor of constant Q-sectional curvature.

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2. Almost quaternionic manifolds

Let M be a differentiable manifold of dimension n and assume that there is a subbundle V of the tensor bundle of type (1,1) over M such that V satisfies the following condition:

(a) In any coordinate neighborhood U of M, there is a local basis $\{F, G, H\}$ of the bundle V, where F, G and H are tensor fields of type (1,1) in U, and satisfy

$$(2.1)\quad \begin{aligned} &F^2 = -I, && G^2 = -I, && H^2 = -I, \\ &GH = -HG = F, && HF = -FH = G, && FG = -GF = H, \end{aligned}$$

I being the identity tensor field of type (1,1) in M. Such a local basis $\{F, G, H\}$ of the bundle V is said to be *canonical* in U. Thus the bundle V is 3-dimensional as a vector bundle. Such a bundle V is called an *almost quaternionic structure* and the pair (M, V) an *almost quaternionic manifold.* An almost quaternionic manifold is orientable and of dimension $n = 4m (m \geq 1)$ (See [2], for example).

For an almost quaternionic manifold (M, V), let $\{F, G, H\}$ and $\{F',$ $G', H'\}$ be canonical local basis of V in U and in another coordinate neighborhood U' of M, respectively. Then we have in $U \cap U'$

$$(2.2)\qquad \begin{aligned} F' &= s_{11}F + s_{12}G + s_{13}H, \\ G' &= s_{21}F + s_{22}G + s_{23}H, \\ H' &= s_{31}F + s_{32}G + s_{33}H, \end{aligned}$$

where $S_{U,U'} = (s_{\gamma\beta}) \in SO(3), (\beta, \gamma = 1, 2, 3)$, because $\{F, G, H\}$ and $\{F', G', H'\}$ satisfy (2.1). Thus, if we put in U

$$(2.3)\qquad \Lambda = F \otimes F + G \otimes G + H \otimes H,$$

then, using (2.2), we easily see that Λ determines in M a global tensor field of type (2,2), which will be also denoted by Λ.

Next, let there be given an almost quaternionic structure V in a Riemannian manifold (M, g) and assume that, for any canonical local basis $\{F, G, H\}$ of V, all of F, G, and H are almost Hermitian with respect to g. Moreover, we suppose that the set (M, g, V) satisfies the following condition:

(b) If ϕ is a cross-section of the bundle V, the $\nabla_X \phi$ is also a cross-section of V for any vector field X in M, where ∇ denotes the Riemannian connection of the Riemannian manifold (M, g).

Such a set (M, g, V) is called a *quaternionic Kählerian manifold* and the set (g, V) a *quaternionic Kählerian structure* in M. The condition (b) is equivalent to the following condition:

(b') For a canonical local basis $\{F, G, H\}$ of V in U,

$$\begin{aligned} \nabla_X F &= \qquad\qquad\quad r(X)G - q(X)H, \\ \nabla_X G &= -r(X)F \qquad\qquad\quad + p(X)H, \\ \nabla_X H &= \quad q(X)F - p(X)G \end{aligned} \tag{2.4}$$

for any vector field X in M, where p, q and r are certain local 1-forms in U. Thus, using (2.4), we easily find

$$\nabla \Lambda = 0. \tag{2.5}$$

Here, we can easily verify that the condition (2.5) is equivalent to the condition (b'). Using the Ricci formula, from (2.4) we have

$$\begin{aligned} K_{kjt}{}^h F_i{}^t - K_{kji}{}^s F_s{}^h &= \qquad\qquad C_{kj} G_i{}^h - B_{kj} H_i{}^H, \\ K_{kjt}{}^h G_i{}^t - K_{kji}{}^s G_s{}^h &= -C_{kj} F_i{}^h \qquad\qquad + A_{kj} H_i{}^h, \\ K_{kjt}{}^h H_i{}^t - K_{kji}{}^s H_s{}^h &= -B_{kj} F_i{}^h - A_{kj} G_i{}^h, \end{aligned} \tag{2.6}$$

$K_{kji}{}^h$ being the components of curvature tensor of the quaternionic Kählerian manifold, and A, B, C being defined by

$$A = dp + q \wedge r, \;\; B = dq + r \wedge p, \;\; C = dr + q \wedge p, \tag{2.7}$$

where A_{ji}, B_{ji}, and C_{ji} are all skew- symmetric,

$$(2.8)\quad A=\frac{1}{2}A_{ji}dx^j\wedge dx^i,\ B=\frac{1}{2}B_{ji}dx^j\wedge dx^i,\ C=\frac{1}{2}C_{ji}dx^j\wedge dx^i.$$

Thus A, B, and C are local 2-forms defined in U. Transvecting the three equations of (2.6) respectively with $F_{hu}=F_h{}^tg_{tu}$, $G_{hu}=G_h{}^tg_{tu}$ and $H_{hu}=H_h{}^tG_{tu}$ and changing indices, we find respectively

$$(2.9)\quad \begin{aligned} -K_{kjts}F_i{}^tF_h{}^s+K_{kjih}&=C_{kj}H_{ih}+B_{kj}G_{ih},\\ -K_{kjts}G_i{}^tG_h{}^s+K_{kjih}&=A_{kj}F_{ih}+C_{kj}H_{ih},\\ -K_{kjts}H_i{}^tH_hs+K_{kjih}&=B_{kj}G_{ih}+A_{kj}F_{ih}, \end{aligned}$$

where $K_{kjih}=K_{kji}{}^sg_{sh}$. Transvecting the second equation of (2.9) with $F^{ih}=g^{is}F_s{}^h$, we have

$$-K_{khts}G_i{}^tG_h{}^sF^{ih}+K_{kjih}F^{ih}=4mA_{kj},$$

from which it follows that

$$A_{kj}=\frac{1}{2m}K_{kjih}F^{ih},$$

$dimM=4m$. Similarly, we obtain

$$(2.10)\quad A_{kj}=\frac{1}{2m}K_{kjih}F^{ih},\ B_{kj}=\frac{1}{2m}K_{kjih}G^{ih},\ C_{kj}=\frac{1}{2m}K_{kjih}H^{ih}.$$

Next, using (2.10) we have

$$\begin{aligned} K_{ktsh}F^{ts}&=\frac{1}{2}(K_{ktsh}-K_{ksth})F^{ts}=\frac{1}{2}(K_{ktsh}+K_{skth})F^{ts}\\ &=-\frac{1}{2}K_{khts}F^{ts}=-m\ A_{kh}, \end{aligned}$$

where we have used the identity $K_{kjih}+K_{jikh}+K_{ikjh}=0$.

Similarly, we find

$$(2.11)\quad \begin{aligned} &K_{ktsh}F^{ts}=-m\ A_{kh},\ K_{ktsh}G^{ts}=-m\ B_{kh},\\ &K_{ktsh}H^{ts}=-m\ C_{kh}. \end{aligned}$$

On the other hand because of (2.11), transvecting (2.9) with g^{ji} gives

$$K_{kh} = -mA_{ks}F_h{}^s - B_{ks}G_h{}^s - C_{ks}H_h{}^s$$

where $K_{kh} = K_{kjih}g^{ji}$ are the components of the Ricci tensor K of (M, g, V). Similarly we obtain

$$\begin{aligned} K_{kh} &= -mA_{ks}F_h{}^s - B_{ks}G_h{}^s - C_{ks}H_h{}^s, \\ K_{kh} &= -A_{ks}F_h{}^s - mB_{ks}G_h{}^s - C_{ks}H_h{}^s, \\ K_{kh} &= -A_{ks}F_h{}^s - B_{ks}G_h{}^s - mC_{ks}H_h{}^s, \end{aligned}$$

from which, it follows that for $m > 1$,

(2.12)
$$\begin{aligned} &K_{kh} = -(m+2)A_{ks}F_h{}^s, \ K_{kh} = -(m+2)B_{ks}G_h{}^s, \\ &K_{kh} = -(m+2)C_{ks}H_h{}^s, \end{aligned}$$

and for $m = 1$,

(2.13)
$$K_{kh} = -A_{ks}F_h{}^s - B_{ks}G_h{}^s - C_{ks}H_h{}^s.$$

We find from (2.12) that if $m > 1$, then

(2.14)
$$\begin{aligned} &A_{kh} = \frac{1}{m+2}K_{ks}F_h{}^s, \ B_{kh} = \frac{1}{m+2}K_{ks}G_h{}^s, \\ &C_{kh} = \frac{1}{m+2}K_{ks}H_h{}^s. \end{aligned}$$

Substituting (2.14) in (2.9) we have for $m > 1$,

(2.15)
$$\begin{aligned} -K_{kjts}F_i{}^tF_h{}^s + K_{kjih} &= \frac{1}{m+2}K_{kt}(G_j{}^tG_{ih} + H_j{}^tH_{ih}), \\ -K_{kjts}G_i{}^tG_h{}^s + K_{kjih} &= \frac{1}{m+2}K_{kt}(H_j{}^tH_{ih} + F_j{}^tF_{ih}), \\ -K_{kjts}H_i{}^tH_h{}^s + K_{kjih} &= \frac{1}{m+2}K_{kt}(F_j{}^tF_{ih} + G_J{}^tG_{ih}). \end{aligned}$$

Since A_{kj}, B_{kj} and C_{kj} are all skew - symmetric, using (2.14) we find, for $m > 1$,

$$K_{ts}F_k{}^tF_j{}^s = K_{ts}G_k{}^tG_j{}^s = K_{ts}H_k{}^tH_j{}^s = K_{kj}. \tag{2.16}$$

It is well known that any quaternionic Kählerian manifold (M, g) is an Einstein space when $dimM \geq 8$, i.e., that the Ricci tensor K_{ji} of (M, g) has the form

$$K_{ji} = \frac{k}{4m} g_{ji}, \tag{2.17}$$

k being the scalar curvature of (M, g), which is a constant if M is connected, when $dimM = 4m$ (See [1], [2]).

3. Q-projective transformations

Let M be a differentiable manifold of dimension n with an almost quaternionic structure. Let M be endowed with a symmetric affine connection Γ^h_{ji}. We now introduce the curves $x^i = x^i(t)$ satisfying the differential equations

(3.1)
$$\frac{d^2x^h}{dt^2} + \Gamma^h_{ji}\frac{dx^j}{dt}\frac{dx^i}{dt} = \alpha(t)\frac{dx^h}{dt} + \beta(t)F_i{}^h\frac{dx^i}{dt} + \gamma(t)G_i{}^h\frac{dx^i}{dt} + \tau(t)H_i{}^h\frac{dx^i}{dt}.$$

We call such a curve a *quaternionically planar curve.* Given two symmetric affine connections Γ^h_{ji} and $\bar{\Gamma}^h_{ji}$, we suppose that they have all quaternionically planar curves in common. Then we say these two symmetric affine connections are *Q-projectively related* to each other. Let Γ^h_{ji} and $\bar{\Gamma}^h_{ji}$ are connections in coordinate neighborhood U and $\bar{U}$, and ∇ and $\bar{\nabla}$ are covariant differentiations with respect to Γ^h_{ji} and $\bar{\Gamma}^h_{ji}$, respectively. We suppose that $\bar{\Gamma}^h_{ji}$ are also satisfying (3.1) and $\bar{\nabla}\Lambda = 0$. We have put

$$t_{ji}{}^h = \bar{\Gamma}^h_{ji} - \Gamma^h_{ji}. \tag{3.2}$$

From (3.1), we obtain

$$t_{ji}{}^{h}y^{j}y^{i} = ay^{h} + bF_{j}{}^{h}y^{j} + cG_{j}{}^{h}y^{j} + dH_{j}{}^{h}y^{j} \tag{3.3}$$

for any vector $y^{i}, a, b, c,$ and d being functions of y^{j}. On the other hand, from (2.5) it follows

$$t_{ls}{}^{k}\Lambda_{ji}^{sh} + t_{ls}{}^{h}\Lambda_{ji}^{ks} = t_{lj}{}^{s}\Lambda_{si}^{kh} + t_{li}{}^{s}\Lambda_{js}^{kh}, \tag{3.4}$$

where $\Lambda_{ji}^{kh} = F_{j}{}^{k}F_{i}{}^{h} + G_{j}{}^{k}G_{i}{}^{h} + H_{j}{}^{k}H_{i}{}^{h}$. Using (3.3), we can easily obtain the relations

$$t_{ji}{}^{h}y_{h}y^{j}y^{i} = ay^{r}y_{r}, \qquad t_{ji}{}^{r}F_{r}{}^{t}y_{t}y^{j}y^{i} = -by^{r}y_{r},$$
$$t_{ji}{}^{r}G_{r}{}^{t}y_{t}y^{j}y^{i} = -cy^{r}y_{r}, \qquad t_{ji}{}^{r}H_{r}{}^{t}y_{t}y^{j}y^{i} = -dy^{r}y_{r}.$$

Making use of these relations, we can eliminate a, b, c and d in (3.3) and get

$$t_{jirp}{}^{h}y^{j}y^{i}y^{r}y^{p} = 0, \tag{3.5}$$

where

$$t_{jirp}{}^{h} := t_{ji}{}^{h}g_{rp} - t_{jir}\delta_{p}^{h} + t_{ji}{}^{s}F_{sr}F_{p}{}^{h} + t_{ji}{}^{s}G_{sr}G_{p}{}^{h} + t_{ji}{}^{s}H_{sr}H_{p}{}^{h},$$
$$t_{jir} := t_{ji}{}^{s}g_{sr}.$$

Since (3.5) holds for arbitrary y^{i}, it follows $t_{jirp}{}^{h} + t_{jipr}{}^{h} + t_{jrip}{}^{h} + t_{jpir}{}^{h} + t_{jrpi}{}^{h} + t_{jpri}{}^{h} + t_{irjp}{}^{h} + t_{ipjr}{}^{h} + t_{rpji}{}^{h} + t_{irpj}{}^{h} + t_{iprj}{}^{h} + t_{rpij}{}^{h} = 0$, Transvecting this with g^{rp} and taking account of (3.2) and (3.4), we can obtain

$$\begin{aligned} \bar{\Gamma}_{ji}^{h} = \Gamma_{ji}^{h} &+ A_{j}\delta_{i}^{h} + A_{i}\delta_{j}^{h} + B_{j}F_{i}{}^{h} + B_{i}F_{j}{}^{h} \\ &+ C_{j}G_{i}{}^{h} + C_{i}G_{j}{}^{h} + D_{j}H_{i}{}^{h} + D_{i}H_{j}{}^{h}, \end{aligned} \tag{3.6}$$

where

$$A_{j} = \frac{1}{n+4}t_{jr}{}^{r}, B_{j} = -\frac{1}{n}t_{jr}{}^{s}F_{s}{}^{r}, C_{j} = -\frac{1}{n}t_{jr}{}^{s}G_{s}{}^{r}, D_{j} = -\frac{1}{n}t_{jr}{}^{s}H_{s}{}^{r}.$$

From (3.4) and (3.6), we can find

$$\begin{aligned} A_{i} &= B_{j}F_{i}{}^{j} = C_{j}G_{i}{}^{j} = D_{j}H_{i}{}^{j}, \\ B_{i} &= -A_{j}F_{i}{}^{j}, \quad C_{i} = -A_{j}G_{i}{}^{j}, \quad D_{i} = -A_{j}H_{i}{}^{j}. \end{aligned} \tag{3.7}$$

Consequently, we have

THEOREM 1. *Let Γ_{ji}^h and $\bar{\Gamma}_{ji}^h$ be two symmetric affine connections in an almost quaternionic manifold with $\nabla\Lambda = 0$ and $\bar{\nabla}\Lambda = 0$. Then these two connections are Q-projectively related to each other if and only if*

(3.8)
$$\begin{aligned}\bar{\Gamma}_{ji}^h = \Gamma_{ji}^h &+ A_j\delta_i^h + A_i\delta_j^h - A_kF_j{}^kF_i{}^h - A_kF_i{}^kF_j{}^h \\ &- A_kG_j{}^kG_i{}^h - A_kG_i{}^kG_j{}^h - A_kH_j{}^kH_i{}^h - A_kH_i{}^kH_j{}^h\end{aligned}$$

holds for a vector field A_i.

Let Γ_{ji}^h and $\bar{\Gamma}_{ji}^h$ be two symmetric affine connections satisfying (3.8) for a vector field A_i. Then this correspondence is called a *quaternionically projective one*, or shortly a *Q. P.* correspondence.

If we denote by $K_{kji}{}^h$ the curvature tensor with respect to Γ_{ji}^h, then, by a straightforward and rather complicated computation, we obtain

(3.9)
$$\begin{aligned}\bar{K}_{kji}{}^h = K_{kji}{}^h &+ (P_{kj} - P_{jk})\delta_i^h + P_{ki}\delta_j^h - P_{ji}\delta_k^h \\ &- P_{ks}\Lambda_{ji}^{sh} - P_{ks}\Lambda_{ij}^{sh} + P_{js}\Lambda_{ki}^{sh} + P_{js}\Lambda_{ik}^{sh},\end{aligned}$$

where

(3.10)
$$P_{ki} = \nabla_kA_i - A_iA_k + A_sA_t\Lambda_{ki}^{st}.$$

By contraction over h and k in (3.9), we have

(3.11)
$$\bar{K}_{ji} = K_{ji} + (P_{ji} + P_{ij}) - (n+4)P_{ji} - P_{ts}\Lambda_{ji}^{ts} - P_{ts}\Lambda_{ij}^{ts}$$

and, multiplying (3.11) by $F_b{}^jF_a{}^i, G_b{}^jG_a{}^i$ and $H_b{}^jH_a{}^i$, respectively. We have

(3.12)
$$\begin{aligned}\bar{K}_{ji} - \bar{K}_{ba}\Lambda_{ji}^{ba} = K_{ji} &- K_{ba}\Lambda_{ji}^{ba} + (n+4)P_{ba}\Lambda_{ji}^{ba} \\ &- (n+4)P_{ji} + 4(P_{ji} + P_{ij}).\end{aligned}$$

By the way, we have from (3.11)

$$\begin{aligned}\bar{K}_{ji} = K_{ji} - (n+2)P_{ji} &+ \frac{1}{n+4}\{\bar{K}_{ji} - \bar{K}_{ij} - K_{ji} + K_{ij}\} \\ - P_{ts}\Lambda_{ji}^{ts} &- P_{ts}\Lambda_{ij}^{ts},\end{aligned}$$

and

$$P_{ts}\Lambda_{ij}^{ts} = P_{ts}\Lambda_{ji}^{ts} + \frac{1}{n+4}\{\bar{K}_{ts} - \bar{K}_{st} - K_{ts} + K_{st}\}\Lambda_{ij}^{ts}.$$

Therefore

(3.13)

$$\begin{aligned}\bar{K}_{ji} = K_{ji} - (n+2)P_{ji} + \frac{1}{n+4}\{\bar{K}_{ji} - \bar{K}_{ij} - K_{ji} + K_{ij}\} \\ - 2P_{ts}\Lambda_{ji}^{ts} - \frac{1}{n+4}\{\bar{K}_{ts} - \bar{K}_{st} - K_{ts} + K_{st}\}\Lambda_{ij}^{ts}.\end{aligned}$$

From (3.12),

$$\begin{aligned}\bar{K}_{ji} - \bar{K}_{ba}\Lambda_{ji}^{ba} = K_{ji} - K_{ba}\Lambda_{ji}^{ba} - (n-4)P_{ji} + (n+4)P_{ba}\Lambda_{ji}^{ba} \\ + \frac{4}{n+4}\{\bar{K}_{ji} - \bar{K}_{ij} - K_{ji} + K_{ij}\}.\end{aligned}$$

We have

(3.14)

$$\begin{aligned}\frac{2}{n+4}\bar{K}_{ji} - \frac{2}{n+4}\bar{K}_{ba}\Lambda_{ji}^{ba} = \frac{2}{n+4}K_{ji} - \frac{2}{n+4}K_{ba}\Lambda_{ji}^{ba} - \frac{2(n-4)}{n+4}P_{ji} \\ + 2P_{ba}\Lambda_{ji}^{ba} + \frac{8}{(n+4)^2}\{\bar{K}_{ji} - \bar{K}_{ij} - K_{ji} + K_{ij}\}.\end{aligned}$$

On the other hand, using (3.13) and (3.14), we have

$$n(n+8)P_{ji} = M_{ji} - \bar{M}_{ji}, \qquad (3.15)$$

where we have put

$$M_{ji} = \frac{n^2+9n+12}{n+4}K_{ji} + \frac{n+12}{n+4}K_{ij} - 3K_{ba}\Lambda_{ji}^{ba} + K_{ba}\Lambda_{ij}^{ba}. \qquad (3.16)$$

Consequently, substituting (3.15) into (3.9), we have

THEOREM 2. *An invariant tensor field under the Q. P. correspondence is given by*

(3.17)
$$Q_{kji}{}^h = K_{kji}{}^h + \frac{1}{n(n+8)}\{(M_{kj} - M_{jk})\delta_i^h + M_{ki}\delta_j^h - M_{ji}\delta_k^h - M_{ks}\Lambda_{ji}^{sh} - M_{ks}\Lambda_{ij}^{sh} + M_{js}\Lambda_{ki}^{sh} + M_{js}\Lambda_{ik}^{sh}\}.$$

We call it the *Q. P. curvature tensor field.* We can verify that

$$Q_{kji}{}^k = 0. \tag{3.18}$$

An almost quaternionic manifold with a symmetric affine connection ∇ with $\nabla\Lambda = 0$ is said to be *Q. P. flat* if it can be related to a Euclidean space by a *Q. P.* correspondence (3.8). The necessary condition to be *Q. P.* flat is clearly $Q_{kji}{}^h = 0$. Conversely, if $Q_{kji}{}^h = 0$, then putting

$$P'_{ji} = \frac{1}{n(n+8)} M_{ji},$$

the curvature tensor $K_{kji}{}^h$ satisfies the equation

$$K_{kji}{}^h = -\{(P'_{kj} - P'_{jk})\delta_i^h + P'_{ki}\delta_j^h - P'_{ji}\delta_k^h - P'_{ks}\Lambda_{ji}^{sh} - P'_{ks}\Lambda_{ij}^{sh} + P'_{js}\Lambda_{ki}^{sh} + P'_{js}\Lambda_{ik}^{sh}\}.$$

On the other hand, if the space is *Q. P.* flat under (3.8), then P_{ji} satisfies the equation (3.9) in which the left hand side vanishes. Hence P_{ji} should be equal to the above P'_{ji}. Therefore, in order to prove that the space with $Q_{kji}{}^h = 0$ is *Q. P.* flat, it is sufficient that there exists a vector field A_i such that

$$\nabla_j A_i = P_{ji} + A_j A_i - A_s A_t \Lambda_{ji}^{st}, \tag{3.19}$$

in the space having the curvature

(3.20)
$$K_{kji}{}^h = -\{(P_{kj} - P_{jk})\delta_i^h + P_{ki}\delta_j^h - P_{ji}\delta_k^h - P_{ks}\Lambda_{ji}^{sh} - P_{ks}\Lambda_{ij}^{sh} + P_{js}\Lambda_{ki}^{sh} + P_{js}\Lambda_{ik}^{sh}\}.$$

The integrability condition of (3.19) is

$$-K_{kji}{}^{h}A_h = \nabla_k P_{ji} - \nabla_j P_{ki} + (\nabla_k A_j - \nabla_j A_k)A_i + A_j\nabla_k A_i - A_k\nabla_j A_i - (\nabla_k A_s)A_t\Lambda_{ji}^{st} - A_s(\nabla_k A_t)\Lambda_{ji}^{st} + (\nabla_j A_s)A_t\Lambda_{ki}^{st} + A_s(\nabla_j A_t)\Lambda_{ki}^{st}$$

or, substituting (3.19) and (3.20),

$$\nabla_k P_{ji} - \nabla_j P_{ki} = 0. \tag{3.21}$$

Now, if the identity of Bianchi is applied to (3.20), we have

(3.22)

$$\begin{aligned}&\{\nabla_l(P_{kj} - P_{jk}) + \nabla_k(P_{jl} - P_{lj}) + \nabla_j(P_{lk} - P_{kl})\}\delta_i^h \\ &+ (\nabla_l P_{ki} - \nabla_k P_{li})\delta_j^h + (\nabla_j P_{li} - \nabla_l P_{ji})\delta_k^h + (\nabla_k P_{ji} - \nabla_j P_{ki})\delta_l^h \\ &- (\nabla_l P_{ks} - \nabla_k P_{ls})(\Lambda_{ji}^{sh} + \Lambda_{ij}^{sh}) - (\nabla_j P_{ls} - \nabla_l P_{js})(\Lambda_{ki}^{sh} + \Lambda_{ik}^{sh}) \\ &- (\nabla_k P_{js} - \nabla_j P_{ks})(\Lambda_{li}^{sh} + \Lambda_{il}^{sh}) = 0.\end{aligned}$$

By contraction over h and i, we have

$$\nabla_l(P_{kj} - P_{jk}) + \nabla_k(P_{jl} - P_{lj}) + \nabla_j(P_{lk} - P_{kl}) = 0 \tag{3.23}$$

and, by contraction over h and j,

(3.24)

$$\begin{aligned}&(n+2)(\nabla_l P_{ki} - \nabla_k P_{li}) + \nabla_i(P_{lk} - P_{kl}) - \nabla_l P_{ik} + \nabla_k P_{il} \\ &- (\nabla_t P_{ls} - \nabla_l P_{ts})(\Lambda_{ki}^{st} + \Lambda_{ik}^{st}) - (\nabla_k P_{ts} - \nabla_t P_{ks})(\Lambda_{li}^{st} + \Lambda_{il}^{st}) = 0.\end{aligned}$$

Alternating indices i, k, l in (3.24) and considering (3.23), we have

$$(\nabla_t P_{ls} - \nabla_l P_{ts})(\Lambda_{ki}^{st} + \Lambda_{ik}^{st}) + (\nabla_k P_{ts} - \nabla_t P_{ks})(\Lambda_{li}^{st} + \Lambda_{il}^{st}) = 0, \tag{3.25}$$

substituting (3.25) into (3.24),

$$(n+2)(\nabla_l P_{ki} - \nabla_k P_{li}) + \nabla_i(P_{lk} - P_{kl}) - \nabla_l P_{ik} + \nabla_k P_{il} = 0. \tag{3.26}$$

Using (3.23), we have

$$\nabla_l P_{ki} - \nabla_k P_{li} = 0. \tag{3.27}$$

Hence, the integrability condition (3.21) of (3.19) is a consequence of (3.20). This proves

THEOREM 3. *An almost quaternionic manifold with a symmetric affine connection ∇ with $\nabla\Lambda = 0$ is Q. P. flat if and only if the Q. P. curvature tensor field $Q_{kji}{}^h$ vanishes.*

In a quaternionic Kählerian manifold case, we can find from (2.16),

$$M_{ji} = nK_{ji}. \tag{3.28}$$

Therefore, substituting (3.28) into (3.17), we have

(3.29)

$$\begin{aligned} Q_{kjih} &= Q_{kji}{}^t g_{th} \\ &= K_{kjih} + \frac{1}{n+8}\{K_{ki}g_{jh} - K_{ji}g_{kh} - 2K_{ks}F_j{}^sF_{ih} \\ &\quad - K_{ks}F_i{}^sF_{jh} + K_{js}F_i{}^sF_{kh} - 2K_{ks}G_j{}^sG_{ih} - K_{ks}G_i{}^sG_{jh} \\ &\quad + K_{js}G_i{}^sG_{kh} - 2K_{ks}H_j{}^sH_{ih} - K_{ks}H_i{}^sH_{jh} + K_{js}H_i{}^sH_{kh}\}. \end{aligned}$$

If the space is Q. P. flat, i.e., $Q_{kjih} = 0$, then contracting by g^{ji}, we have

$$K_{kh} = \frac{K}{n}g_{kh}. \tag{3.30}$$

Hence K is a constant, and we put

$$k = \frac{4K}{n(n+8)} \tag{3.31}$$

or

$$K_{kh} = \frac{n+8}{4}kg_{kh}. \tag{3.32}$$

Then we obtain

(3.33)

$$\begin{aligned} K_{kjih} = \frac{k}{4}\{&g_{kh}g_{ji} - g_{jh}g_{ki} + F_{kh}F_{ji} - F_{jh}F_{ki} - 2F_{kj}F_{ih} + G_{kh}G_{ji} \\ &- G_{jh}G_{ki} - 2G_{kj}G_{ih} + H_{kh}H_{ji} - H_{jh}H_{ki} - 2H_{kj}H_{ih}\}, \end{aligned}$$

which is the expression of constant Q-sectional curvature. Thus we have

THEOREM 4. *If a quaternionic Kählerian manifold is Q. P. flat, then it is of constant Q-sectional curvature.*

Remark On a quaternionic Kählerian manifold M of dimension $n(=4m \geq 8, m > 1)$, the square of the norm of Q_{kjih} is given by

$$\|Q_{kjih}\|^2 = \|H_{kjih}\|^2 - \frac{2(10n+8)}{(n+8)^2}\|Q_{ji}\|^2$$

with the use of (2.11) and (2.16), where we have put

$$\begin{aligned} H_{kjih} = K_{kjih} - \frac{K}{n(n+8)}\{&g_{kh}g_{ji} - g_{jh}g_{ki} \\ &+F_{kh}F_{ji} - F_{jh}F_{ki} - 2F_{kj}F_{ih} + G_{kh}G_{ji} \\ &-G_{jh}G_{ki} - 2G_{kj}G_{ih} + H_{kh}H_{ji} - H_{jh}H_{ki} - 2H_{kj}K_{ih}\}, \end{aligned}$$

and

$$Q_{ji} = K_{ji} - \frac{K}{n}g_{ji}.$$

By the way, since M is an Einstein space, Q_{ji}=0, and consequently $\|Q_{kjih}\|^2 = \|H_{kjih}\|^2$.

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