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# NOTES ON VANISHING THEOREMS ON RIEMANNIAN MANIFOLDS WITH BOUNDARY

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Dedicated to Professor T. Takahashi on his 60th Birthday

ABSTRACT. We shall discuss on some vanishing theorems of harmonic sections of a Riemannian vector bundle over a compact Riemannian manifold with boundary. In relating the results of H. Donnelly - P. Li ([4]), for special case of harmonic forms satisfying absolute or relative boundary problem, our results improve the vanishing results of T. Takahashi ([9]).

Since about 1950s, P. E. Conner ([3]), G. D. Duff - D. C. Spencer ([5]), T. Nakae ([7]), T. Takahashi ([9]) and others have studied harmonic forms on a compact Riemannian manifold with boundary. In this paper, we shall discuss some vanishing theorems of harmonic sections of a Riemannian vector bundle over a compact Riemannian manifold with boundary by the methods by P. H. Berard ([1]), H. Donnelly - P. Li ([4]), H. Kitahara - H. K. Pak ([6]). For special case of harmonic forms satisfying absolute or relative boundary problem, our results improve the vanishing results of T. Takahashi ([9]). We shall be in $C^\infty$-category. Manifolds are supposed to be paracompact, Hausdorff spaces.

Let $M$ be a compact connected (orientable) Riemannian manifold with boundary $\partial M$ of dimension $m$. We may consider $M$ as a closure of an open submanifold of a connected (orientable) Riemannian manifold $\mathcal{M}$ of dimension $m$. At each point $x$ in $\partial M$, there is a coordinate patch $(U; (x_i, x_m))$ $(1 \leq i \leq m-1)$ of $x$ in $\mathcal{M}$ such that $U \cap M$ is represented by $x_m \geq 0$. In particular, $U \cap M$ is represented by $x_m = 0$

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and $(x_i)$ is the induced coordinate system of $\partial M$. We call such a patch $(U;(x_i,x_m))$ a boundary coordinate patch.

Let $(V;(v_i,v_m))$ be an another boundary coordinate patch such that $U\cap V\neq\emptyset$. Then we have

$$\frac{\partial v_m}{\partial x_m}>0 \quad\text{and}\quad \frac{\partial v_m}{\partial x_i}=0,\quad 1\leq i\leq m-1.$$

Since the Jacobian of the coordinate transformation is positive, the Jacobian of the induced coordinate transformation restricted to $\partial M$ is positive.

There are two unit normal vector fields to $\partial M$. We always choose the inward pointing unit normal vector field $N$ along it, by which we also denote its dual 1-form.

A vector bundle $E\longrightarrow M$ may be also considered as the restriction of a vector bundle $\mathcal{E}\longrightarrow\mathcal{M}$.

**1.** Let $(M,g=\langle\ ,\ \rangle)$ be an oriented compact connected Riemannian manifold with boundary $\partial M$ of dimension $m$. A Riemannian vector bundle $E\longrightarrow M$ is a smooth vector bundle with a metric $(\ ,\ )$ along the fiber and a covariant differentiation $\nabla$ such that

$$X(s_1,s_2)=(\nabla_X s_1,s_2)+(s_1,\nabla_X s_2)\quad X\in\Gamma(TM),\ s_1,s_2\in\Gamma(E).$$

The Bochner Laplacian of $E$ is an invariantly defined second order differential operator $D:\Gamma(E)\longrightarrow\Gamma(E)$, defined by $D:=tr(\nabla\circ\nabla)$. Here $\Gamma(E)$ is denoted by the space of smooth sections of $E$. More explicitly, $D$ is given by the composition

$$\Gamma(E)\xrightarrow{\nabla}\Gamma(E\otimes T^*M)\xrightarrow{\nabla}\Gamma(E\otimes T^*M\otimes T^*M)\longrightarrow\Gamma(E),$$

where the last map is contraction.

Suppose that $\mathcal{R}$ is a selfadjoint endomorphism of $E$ and set $A:=-D+\mathcal{R}$. If $\partial M=\emptyset$, then $A$ defines a unique selfadjoint operator in $L^2(E)$ (:= the space of $L^2$-sections of $E$). Otherwise, we must impose suitable boundary conditions. It is most typical to use either Dirichlet boundary problem, i.e., $s(x)=0, x\in\partial M$, or Neumann boundary problem, i.e., $(\nabla_N s)(x) = 0, x \in \partial M$. Here $s \in \Gamma(E)$ and $N$ is the inward pointing unit normal vector field along $\partial M$. For $s_1, s_2 \in \Gamma(E)$,

$$
\begin{aligned}
(-Ds_1, s_2) &= -\sum_\mu (\nabla_\mu \nabla_\mu s_1, s_2) \\
&= -\sum_\mu \left(\frac{\partial}{\partial x_\mu}(\nabla_\mu s_1, s_2) - (\nabla_\mu s_1, \nabla_\mu s_2)\right) \\
&= \operatorname{div}(r) + (\nabla s_1, \nabla s_2),
\end{aligned} \tag{1.1}
$$

where $r$ is the vector field defined by the condition that $\langle r, X\rangle := (\nabla_X s_1, s_2)$ for all $X \in \Gamma(TM)$. In fact,

$$
\begin{aligned}
\operatorname{div}(r)|_x &= -\sum_\mu \langle \nabla_\mu r, \frac{\partial}{\partial x_\mu}\rangle|_x \\
&= -\sum_\mu \left\{ \frac{\partial}{\partial x_\mu}\langle r, \frac{\partial}{\partial x_\mu}\rangle - \langle r, \nabla_\mu \frac{\partial}{\partial x_\mu}\rangle \right\}|_x \\
&= -\sum_\mu \frac{\partial}{\partial x_\mu}\langle r, \frac{\partial}{\partial x_\mu}\rangle|_x \\
&= -\sum_\mu \frac{\partial}{\partial x_\mu}(\nabla_\mu s_1, s_2).
\end{aligned} \tag{1.2}
$$

Let $dvol_M$ (resp. $dvol_{\partial M}$) be the canonical volume form on $M$ (resp. $\partial M$) satisfying $dvol_M = N \wedge dvol_{\partial M}$. Then Stokes' formula implies

$$
\int_M \operatorname{div}(r) dvol_M = \int_{\partial M} \langle r, -N\rangle dvol_{\partial M}.
$$

Therefore we have, by definition, $\langle r, N\rangle = (\nabla_N s_1, s_2)$,

$$
\begin{aligned}
\int_M (As_1, s_2) dvol_M = &\int_M ((\nabla s_1, \nabla s_2) + (\mathcal{R}s_1, s_2)) dvol_M \\
&- \int_{\partial M} (\nabla_N s_1, s_2) dvol_{\partial M}.
\end{aligned} \tag{1.3}
$$

From

$$\left| \int_{\partial M} (\nabla_N s_1, s_2) dvol_{\partial M} \right| \leq ||\nabla_N s_1||_{\partial M} ||s_2||_{\partial M}, \tag{1.4}$$

it follows that if we assume the Dirichlet or Neumann boundary problem, then we conclude

$$\int_M (As_1, s_2) dvol_M = \int_M ((\nabla s_1, \nabla s_2) + (\mathcal{R}s_1, s_2)) dvol_M. \tag{1.5}$$

Let $X$ be a smooth vector field on $M$. If $|s|$ denote the (point-wise) norm of $s \in \Gamma(E)$, we can write $2(\nabla_X s, s) = X \cdot |s|^2 = 2(X \cdot |s|)|s|$ and $|\nabla_X s| \geq |X \cdot |s||$ with equality if and only if $s$ and $|\nabla_X s|$ are linearly dependent. Summing over a local orthonormal framing, we have the Kato inequality.

(1.6) For any $s \in \Gamma(E)$, $|d|s|| \leq |\nabla s|$, with equality if and only if for any $X \in \Gamma(TM)$ there is a function $\alpha_X$ such that $\nabla_X s = \alpha_X s$ (at least on the set $\{|s| \neq 0\}$).

Let $\lambda_0$ be the infimum of the spectrum of the scalar Laplacian $\Delta$ acting on smooth functions on $M$. It is well-known that $\lambda_0 = 0$ in the case of either $\partial M = \emptyset$ or the Neumann boundary problem. Then we have

THEOREM 1. *Under the Dirichlet or Neumann boundary problem, if $\rho(x) \geq -\lambda_0$ for all $x \in M$ and $\rho(x_0) \geq -\lambda_0$ for some $x_0 \in M$, then $\mathcal{H}(E) := \{s \in \Gamma(E) : As = 0\} = \{0\}$.*

*Proof.* Let $s \in \Gamma(E)$ satisfy $As = 0$. Weitzenböck formula (1.5) and the Kato inequality (1.6) imply that

$$\begin{aligned} \int_M (-\rho)|s|^2 dvol_M \geq - \int_M (\mathcal{R}s, s) dvol_M = \int_M |\nabla s|^2 dvol_M \\ \geq \int_M |d|s||^2 dvol_M. \end{aligned}$$

It follows from the definition of $\lambda_0$ that

$$\int_M (\lambda_0 + \rho)|s|^2 dvol_M \leq 0.$$

Since $(\lambda_0 + \rho)(x) \geq 0$ for all $x \in M$ and $(\lambda_0 + \rho)(x_0) > 0$ for some $x_0 \in M$, we conclude that $s = 0$ on a neighborhood of $x_0$, hence $s = 0$ on $M$. $\square$

**2.** Let $M$ be an oriented compact connected Riemannian manifold with boundary $\partial M$ of dimension $m$ and $\nabla^M$ be the Levi-Civita connection on $M$. Let $F \longrightarrow M$ be a Riemannian vector bundle of fiber dimension $n$ with a metric along the fiber and a covariant differentiation $\nabla^F$ such that

$$X(s_1, s_2) = (\nabla^F_X s_1, s_2) + (s_1, \nabla^F_X s_2), \quad X \in \Gamma(TM),\ s_1, s_2 \in \Gamma(F).$$

We consider $E := \Lambda^* T^* M \otimes F$. In this case sections of $E$ are $F$-valued differential forms on $M$, which is denoted by $\Lambda^*(M, F)$. Let $\{e_i\}$ $(i, j = 1, \cdots, m)$ be an orthonormal framing with its dual framing $\{\omega^i\}$ and $\{f_\alpha\}$ $(\alpha, \beta = 1, \cdots, n)$ a framing of the fiber of $F$. Locally $s \in \Lambda^p(M, F)$ can be written as

$$s = \sum s^\alpha_{i_1 \cdots i_p} \omega^{i_1} \wedge \cdots \wedge \omega^{i_p} \otimes f_\alpha.$$

We define a differential operator $\partial : \Lambda^p(M, F) \longrightarrow \Lambda^{p+1}(M, F)$ by

$$\partial s := \sum \omega^j \wedge \{\nabla^M_{e_j}(s^\alpha_{i_1 \cdots i_p} \omega^{i_1} \wedge \cdots \wedge \omega^{i_p})\} \otimes f_\alpha + \sum \omega^j \wedge s^\alpha_{i_1 \cdots i_p} \omega^{i_1} \wedge \cdots \wedge \omega^{i_p} \otimes \nabla^F_{e_j} f_\alpha.$$

Let $\partial^* : \Lambda^{p+1}(M, F) \longrightarrow \Lambda^p(M, F)$ be its adjoint operator. The Laplacian acting on $\Lambda^p(M, F)$ is defined by

$$\square := \partial \partial^* + \partial^* \partial.$$

Let $D^E := tr(\nabla^E \circ \nabla^E)$ be the Bochner Laplacian, where $\nabla^E$ is the induced connection on $E$ from the connections $\nabla^M$ and $\nabla^F$. If either the boundary of $M$ is empty or $s \in \Lambda^p(M, F)$ satisfies Dirichlet or Neumann boundary problem, it follows from (1.5) that

$$\int_M (\square s, s) dvol_M = \int_M \{-(D^E s, s) + (\mathcal{R} s, s)\} dvol_M = \int_M \{(\nabla^E s, \nabla^E s) + (\mathcal{R} s, s)\} dvol_M,$$

where $\mathcal{R}$ is defined by the curvature operators of $\nabla^M$ and $\nabla^F$.

PROPOSITION 2. *If* $\square s = 0$, *then* $\mathcal{R}$ *is a negative operator.*

By a similar argument of Theorem 1, we have

PROPOSITION 3. *Let* $M$ *be compact and assume that either the boundary of* $M$ *is empty or* $s \in \Lambda^p(M, F)$ *satisfies Dirichlet or Neumann boundary problem. If* $\rho(x) \geq -\lambda_0$ *for all* $x \in M$ *and* $\rho(x_0) \geq -\lambda_0$ *for some* $x_0 \in M$, *then* $\mathcal{H}^p(E) := \{s \in \Lambda^p(M, F) : \square s = 0\} = \{0\}$ *for all* $p$.

**3.** In this section we shall discuss on the case that $\Gamma(E) = \Lambda^* M$, the space of smooth differential forms on $M$. Let $M$ be an oriented compact connected Riemannian manifold of dimension $m$ with boundary $\partial M$. Let $A = \Delta := d\delta + \delta d$ be the Hodge Laplacian acting on smooth differential $p$-forms $\Lambda^p M$, $(1 \leq p \leq m-1)$. Here $d$ is the exterior derivative, $d : \Lambda^p M \longrightarrow \Lambda^{p+1} M$, and $\delta$ is its adjoint operator. The Laplacian $\Delta$ is positive semi-definite on $\Lambda^* M$.

The results of section 1 would apply to $\Delta$ if we imposed Dirichlet or Neumann boundary problem. However, it is more interesting to consider the Hodge Laplacian with absolute or relative boundary problem ([2], [4]). Let $N$ be an inward pointing normal vector field along $\partial M$, which is sometimes identified with its dual. If $a \in \Lambda^p M$, then along $\partial M$ we may decompose $a$ into its tangential and normal components, i.e.,

$$a = a_{tan} + N \wedge a_{nor}, \quad a_{tan} \in \Lambda^p \partial M, \; a_{nor} \in \Lambda^{p-1} \partial M.$$

The form $a$ is said to satisfy the relative boundary problem if $a_{tan} = (\delta a)_{tan} = 0$, and $a$ is said to satisfy the absolute boundary problem if $a_{nor} = (da)_{nor} = 0$. Clearly, the Hodge star operator $*$ maps forms satisfying the absolute boundary problem to those satisfying the relative boundary problem,

$$* : \Lambda^p M \longrightarrow \Lambda^{m-p} M.$$

Recall that $a$ is said to be harmonic if $\Delta a = 0$. The significance of the absolute and the relative boundary problem stems from well-known;

FACT 4 (SEE ([8])).

(1) *The singular cohomology group* $H^p M$ *is isomorphic to the space of harmonic* $p$*-forms satisfying the absolute boundary problem.*

(2) *The singular cohomology group $H^p(M, \partial M)$ is isomorphic to the space of harmonic p-forms satisfying the relative boundary problem.*

We take a decomposition of normal coordinates $x = (y, t) \in \partial M \times [0, t_0[$ along $\partial M$ with respect to the normal exponential map $\exp_N$. The volume form $dvol_M$ on $M$ can be written as

$$dvol_M = N \wedge dvol_{\partial M_t},$$

where $dvol_{\partial M_t}$ is the volume form of the submanifold $\exp_N(\partial M \times \{t\})$.

We begin with rewriting the formula (1.3) for $A := \Delta$ ;

$$\begin{aligned}\int_M (\Delta a, a) dvol_M = \int_M &((\nabla a, \nabla a) + (\mathcal{R}^p a, a)) dvol_M \\ &- \int_{\partial M} (\nabla_N a, a) dvol_{\partial M}.\end{aligned} \tag{3.1}$$

From now on we want to estimate $\int_{\partial M} (\nabla_N a, a) dvol_{\partial M}$ in terms of the eigenvalues of the second fundamental form of $\partial M$. For $y \in \partial M$, we choose an orthonormal coframing $\{\omega_1, \omega_2, \cdots, \omega_{m-1}, \omega_m\}$ so that the second fundamental form of $\partial M$ is diagonalized at $y$. Let $\{\gamma_1, \gamma_2, \cdots, \gamma_{m-1}\}$ be the eigenvalues of the second fundamental form of $\partial M$. We define

$$\sigma_p := \min_{y \in \partial M} \min_I (\gamma_{i_1} + \cdots + \gamma_{i_p}), \quad \tilde{\sigma}_p := \max_{y \in \partial M} \max_I (\gamma_{i_1} + \cdots + \gamma_{i_p}),$$

where $I := (i_1, \cdots, i_p)$ is a multi-index.

In coordinates with respect to this framing, let $a_{i_1 \cdots i_p}$ be the components of $a_{tan}$ and $a_{j_1 \cdots j_{p-1} m}$ the components of $a_{nor}$, where the indices $i, j$ run from 1 to $m-1$. The relative boundary problem reads $a_{i_1 \cdots i_p} = 0$ for $a_{tan} = 0$ and

$$\sum_k a_{j_1 \cdots j_{p-1} k, k} + a_{j_1 \cdots j_{p-1} m, m} = 0$$

for $(\delta a)_{tan} = 0$, where the index $k$ runs from 1 to $m - 1$. An index following a comma means covariant differentiation. Thus we have

$$\frac{1}{2} \nabla_N |a|^2 = - \sum_J \sum_{k \notin J} \gamma_k (a_{j_1 \cdots j_{p-1} m})^2, \tag{3.2}$$

where $J := (j_1, \cdots, j_{p-1})$ is summed over all increasing multi-indices. Then we have along $\partial M$

$$\frac{1}{2}\nabla_N|a|^2 \leq -\sigma_{m-p}\sum_J (a_{j_1\cdots j_{p-1}m})^2 = -\sigma_{m-p}|a|^2. \tag{3.3}$$

Next, for $a$ supported in $\exp_N(\partial M \times [0, t_0[)$,

$$\begin{aligned}\int_{\partial M} |a|^2 dvol_{\partial M} &= \int_M d(|a|^2 dvol_{\partial M_t}) \\ &= \int_M N(|a|^2) dvol_M + |a|^2 N \wedge \nabla_N(dvol_{\partial M_t}).\end{aligned} \tag{3.4}$$

But we note $(dvol_{\partial M_t}, \nabla_N(dvol_{\partial M_t})) \leq \tilde{\sigma}_{m-1}|dvol_{\partial M_t}|^2$. Moreover,

$$\int_M N(|a|^2) dvol_M = 2\int_M (\nabla_N a, a) dvol_M \leq ||\nabla a||^2 + ||a||^2,$$

where $||a|| := \{\int_M (a, a) dvol_M\}^{1/2}$.

Therefore we find an upper bound

$$\int_{\partial M} |a|^2 dvol_{\partial M} \leq ||\nabla a||^2 + (1 + \tilde{\sigma}_{m-1})||a||^2. \tag{3.5}$$

On the other hand, since $(dvol_{\partial M_t}, \nabla_N(dvol_{\partial M_t})) \geq \sigma_{m-1}|dvol_{\partial M_t}|^2$, a similar way gives rise to a lower bound

$$\int_{\partial M} |a|^2 dvol_{\partial M} \geq (\sigma_{m-1} - 1)||a||^2 - ||\nabla a||^2. \tag{3.6}$$

Considering $(3.3), (3.5)$ and $(3.6)$, the estimate of $\int_{\partial M}(\nabla_N a, a) dvol_{\partial M}$ may be divided into two cases, either $\sigma_{m-p} \leq 0$ or $\sigma_{m-p} > 0$.

(Case I) In case $\sigma_{m-p} \leq 0$, we find from (3.3) and (3.5) that

$$\int_{\partial M} (\nabla_N a, a) dvol_{\partial M} \leq -\sigma_{m-p}\{||\nabla a||^2 + (1 + \tilde{\sigma}_{m-1})||a||^2\}.$$

Hence (3.1) becomes

$$||\nabla a||^2 - \int_M (\Delta a, a) dvol_M \leq -\mathcal{R}^p_{min}||a||^2 - \sigma_{m-p}\{||\nabla a||^2 + (1+\tilde{\sigma}_{m-1})||a||^2\},$$

where $\mathcal{R}^p(x) := \inf\{(\mathcal{R}^p a, a)_x : a \in \Lambda^p_x M,\ |a|_x = 1\}$, and $\mathcal{R}^p_{min} := \inf\{\mathcal{R}^p(x) : x \in M\}$. If we assume that $\Delta a = 0$, then

$$(3.7) \qquad (1+\sigma_{m-p})||\nabla a||^2 \leq -\{\sigma_{m-p}(1+\tilde{\sigma}_{m-1}) + \mathcal{R}^p_{min}\}||a||^2.$$

By an elementary computation, we see that if $\mathcal{R}^p_{min} > 1 + \tilde{\sigma}_{m-1}$ and $\sigma_{m-p} \geq -1$ (or $\mathcal{R}^p_{min} \geq 1 + \tilde{\sigma}_{m-1}$ and $\sigma_{m-p} > -1$), then there are no harmonic $p$-forms ($0 < p < m$) satisfying relative boundary problem other than zero.

(Case II) In case $\sigma_{m-p} > 0$, we find from (3.3) and (3.6) that

$$\int_{\partial M} (\nabla_N a, a) dvol_{\partial M} \leq -\sigma_{m-p}\{(\sigma_{m-1} - 1)||a||^2 - ||\nabla a||^2\}.$$

If we assume that $\Delta a = 0$, then (3.1) becomes

$$(1-\sigma_{m-p})||\nabla a||^2 \leq -\{\sigma_{m-p}(\sigma_{m-1} - 1) + \mathcal{R}^p_{min}\}||a||^2.$$

From this formula we see that if $\mathcal{R}^p_{min} > \sigma_{m-p}(\sigma_{m-1} - 1)$ and $\sigma_{m-p} \leq 1$ (or $\mathcal{R}^p_{min} \geq \sigma_{m-p}(\sigma_{m-1} - 1)$ and $\sigma_{m-p} < 1$), then there are no harmonic $p$-forms ($0 < p < m$) satisfying relative boundary problem other than zero.

Let $b^p(M) := \dim H^p M$ and $b^p(M, \partial M) := \dim H^p(M, \partial M)$. Summing up, we get the following vanishing results.

THEOREM 5. *Let $M$ be an oriented compact connected Riemannian manifold of dimension $m$ with boundary $\partial M$.*

(I) *If we assume one of two cases*

(1) $\mathcal{R}^p_{min} > 1 + \tilde{\sigma}_{m-1}$ *and* $0 \geq \sigma_{m-p} \geq -1$,

(2) $\mathcal{R}^p_{min} \geq 1 + \tilde{\sigma}_{m-1}$ *and* $0 \geq \sigma_{m-p} > -1$,

*then* $b^p(M, \partial M) = b^{m-p}(M) = 0$ *for all* $0 < p < m$.

(II) *If we assume one of two cases*

(1) $\mathcal{R}^p_{min} > \sigma_{m-p}(\sigma_{m-1} - 1)$ *and* $0 < \sigma_{m-p} \leq 1$,

(2) $\mathcal{R}^p_{min} \geq \sigma_{m-p}(\sigma_{m-1} - 1)$ *and* $0 < \sigma_{m-p} < 1$,

*then* $b^p(M, \partial M) = b^{m-p}(M) = 0$ *for all* $0 < p < m$.

REMARKS. (1) For the case of the absolute boundary problem, we find the following inequality

$$\frac{1}{2}\nabla_N|a|^2 \leq -\sigma_p \sum_I (a_{i_1\cdots i_p})^2 = -\sigma_p|a|^2.$$

In this case, we deduce the corresponding vanishing results replacing $\sigma_{m-p}$ by $\sigma_p$ in Theorem 5.

(2) Given a weaker condition imposed on $\tilde{\sigma}_p$ or $\sigma_p$, H. Donnelly-P. Li ([4]) obtained the following upper bounds for $b^p(M, \partial M)$ or $b^p(M)$.

FACT 6 ([4], COROLLARY 6.5). *If $\tilde{\sigma}_p \leq 0$, in particular if $\sigma_p \leq 0$, then*

$$b^p(M, \partial M) \leq \binom{m}{p} e^{-\mathcal{R}^p_{min}}(1 + C vol\, M),$$

*where $C$ is a constant depending on certain Sobolev constant.*

It should be noted that Theorem 5 shows the vanishing result of $b^p(M, \partial M)$ in terms of lower bounds for $\mathcal{R}^p_{min}$.

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