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A UNIFIED FIXED POINT THEORY OF MULTIMAPS ON TOPOLOGICAL VECTOR SPACES

SEHIE PARK

ABSTRACT. We give general fixed point theorems for compact multimaps in the "better" admissible class $\mathfrak{B}^\kappa$ defined on admissible convex subsets (in the sense of Klee) of a topological vector space not necessarily locally convex. Those theorems are used to obtain results for Φ-condensing maps. Our new theorems subsume more than seventy known or possible particular forms, and generalize them in terms of the involving spaces and the multimaps as well. Further topics closely related to our new theorems are discussed and some related problems are given in the last section.

1. Introduction

Schauder [1930] showed that a continuous compact map $f : H \to H$ defined on a closed convex subset H of a Banach space has a fixed point. This theorem has an enormous influence on fixed point theory, differential equations, variational inequalities, equilibrium problems, and many other fields in mathematics. Moreover, there have appeared a large number of generalizations and their applications for various classes of multimaps defined on convex subsets of topological vector spaces more general than Banach spaces. Therefore, it would be desirable to present the most general result which can subsume many of generalizations of the Schauder theorem.

In this paper, we give general Schauder type fixed point theorems for compact multimaps in the class $\mathfrak{B}^\kappa$ of "better" admissible maps

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(see Park [1997b,d]) defined on admissible convex subsets (in the sense of Klee [1960]) of a topological vector space which is not necessarily locally convex. Moreover, we also obtain new fixed point theorems for condensing multimaps in "better" admissible classes. Our new theorems subsume more than seventy known or possible particular forms, and generalize them in terms of the involving spaces and the multimaps as well.

In Section 2, we introduce the concept of admissible subsets of a topological vector space in the sense of Klee [1960] and "better" admissible classes $\mathfrak{B}$, $\mathfrak{B}^\sigma$, and $\mathfrak{B}^\kappa$ of multimaps.

Section 3 deals with our new fixed point theorems for compact maps in these classes and their history. In fact, we list more than sixty papers in chronological order, from which we can deduce particular forms of our theorems.

In Section 4, we obtain new fixed point theorems for condensing maps in "better" admissible classes, and show that our new results subsume many of known theorems.

Section 5 deals with some further comments and remarks related to our new results and the Schauder conjecture. Some related problems are raised.

A part of this paper has appeared in the announcement of Park [1997d].

2. Better admissible classes of multimaps

A *multimap* or *map* $T : X \multimap Y$ is a function from X into the power set of Y with nonempty values, and $x \in T^-(y)$ if and only if $y \in T(x)$.

For topological spaces X and Y, a map $T : X \multimap Y$ is said to be *closed* if its graph $\mathrm{Gr}(T) = \{(x, y) : x \in X,\ y \in T(x)\}$ is closed in $X \times Y$, and *compact* if the closure $\overline{T(X)}$ of its range $T(X)$ is compact in Y.

A map $T : X \multimap Y$ is said to be *upper semicontinuous* (u.s.c.) if for each closed set $B \subset Y$, the set $T^-(B) = \{x \in X : T(x) \cap B \neq \emptyset\}$ is a closed subset of X; *lower semicontinuous* (l.s.c.) if for each open set $B \subset Y$, the set $T^-(B)$ is open; and *continuous* if it is u.s.c. and l.s.c. Note that every u.s.c. map $T : X \multimap Y$ with closed values is closed and that its converse is true whenever Y is compact.

Recall that a nonempty topological space is *acyclic* if all of its reduced Čech homology groups over rationals vanish. Note that any convex or star-shaped subset of a topological vector space is contractible, and that any contractible space is acyclic. A map $T : X \multimap Y$ is said to be *acyclic* if it is u.s.c. with compact acyclic values.

Throughout this paper, t.v.s. means Hausdorff topological vector spaces, and co denotes the convex hull.

A nonempty subset X of a t.v.s. E is said to be *admissible* (in the sense of Klee [1960]) provided that, for every compact subset K of X and every neighborhood V of the origin 0 of E, there exists a continuous map $h : K \to X$ such that $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a finite dimensional subspace L of E.

Note that every nonempty convex subset of a locally convex t.v.s. is admissible; see Nagumo [1950]. Other examples of admissible t.v.s. are l^p, L^p, the Hardy spaces H^p for $0 < p < 1$, the space $S(0,1)$ of equivalence classes of measurable functions on $[0,1]$, certain Orlicz spaces, ultrabarrelled t.v.s. admitting Schauder basis, and others. Moreover, any locally convex subset of an F-normable t.v.s. is admissible and that every compact convex locally convex subset of a t.v.s. is admissible. Note that an example of a nonadmissible nonconvex compact subset of the Hilbert space l^2 is known. For details, see Hadžić [1984], Weber [1992a,b], and references therein.

Let X be a nonempty convex subset of a t.v.s. E and Y a topological space. A *polytope* P in X is any convex hull of a nonempty finite subset of X; or a nonempty compact convex subset of X contained in a finite dimensional subspace of E.

An *admissible* class $\mathfrak{A}_c^\kappa(X,Y)$ of maps $T : X \multimap Y$ is a class such that, for each T and each compact subset K of X, there exists a map $\Gamma \in \mathfrak{A}_c(K,Y)$ satisfying $\Gamma(x) \subset T(x)$ for all $x \in K$; where $\mathfrak{A}_c$ consists of finite compositions of maps in $\mathfrak{A}$, and $\mathfrak{A}$ is a class of maps satisfying the following properties:

(i) $\mathfrak{A}$ contains the class $\mathbb{C}$ of (single-valued) continuous functions;
(ii) each $F \in \mathfrak{A}_c$ is u.s.c. and compact-valued; and
(iii) for any polytope P, each $F \in \mathfrak{A}_c(P,P)$ has a fixed point, where the intermediate spaces are suitably chosen.

We define the *"better" admissible class* $\mathfrak{B}$ of multimaps defined on X as follows:

$F \in \mathfrak{B}(X,Y) \iff F : X \multimap Y$ is a map such that for any polytope P in X and any continuous function $f : F(P) \to P$, the composition $f(F|_P) : P \multimap P$ has a fixed point.

Subclasses of $\mathfrak{B}$ are classes of continuous functions $\mathbb{C}$, the Kakutani maps $\mathbb{K}$ (with convex values and codomains are convex spaces), the Aronszajn maps $\mathbb{M}$ (with R_δ values), the acyclic maps $\mathbb{V}$ (with acyclic values), the Powers maps $\mathbb{V}_c$ (finite compositions of acyclic maps), the O'Neill maps $\mathbb{N}$ (continuous with values of one or m acyclic components, where m is fixed), the approachable maps $\mathbb{A}$ (whose domains and codomains are subsets of t.v.s.), admissible maps of Górniewicz [1976], σ-selectionable maps of Haddad and Lasry [1983], permissible maps of Dzedzej [1985], the class $\mathbb{K}_c^+$ of Lassonde [1991], the class $\mathbb{V}_c^+$ of Park *et al.* [1994], and approximable maps of Ben-El-Mechaiekh and Idzik [1994], and many others. Those subclasses are all examples of the admissible class $\mathfrak{A}_c^\kappa$; see Park [1993a,b], Park and Kim [1993]. Some examples of maps in $\mathfrak{B}$ not belonging to $\mathfrak{A}_c^\kappa$ were given recently in Park [1997c].

We define another classes related to $\mathfrak{B}(X,Y)$ as follows:

$F \in \mathfrak{B}^\sigma(X,Y) \iff F : X \multimap Y$ is a map such that for any σ-compact convex subset K of X, there is a closed map $\Gamma \in \mathfrak{B}(K,Y)$ such that $\Gamma(x) \subset F(x)$ for each $x \in K$.

$F \in \mathfrak{B}^\kappa(X,Y) \iff F : X \multimap Y$ is a map such that for any compact convex subset K of X, there is a closed map $\Gamma \in \mathfrak{B}(K,Y)$ such that $\Gamma(x) \subset F(x)$ for each $x \in K$.

Note that the classes $\mathbb{K}_c^\sigma$, $\mathbb{V}_c^\sigma$, and $\mathfrak{A}_c^\sigma$ are examples of $\mathfrak{B}^\sigma$ and that $\mathfrak{A}_c^\kappa \subset \mathfrak{B}^\kappa$; see Park [1993a,b, 1997b,d].

It is clear that $\mathfrak{B}^\sigma \subset \mathfrak{B}^\kappa$ and any subclass of $\mathfrak{B}$ or $\mathfrak{B}^\kappa$ will be called a "better" admissible class.

3. New fixed point theorems for compact multimaps

The following is our new fixed point theorem for compact better admissible maps:

THEOREM 1. *Let E be a t.v.s. and X an admissible convex subset of E. Then any compact map $F \in \mathfrak{B}^\kappa(X,X)$ has a fixed point.*

Proof. Let $\mathcal{V}$ be a fundamental system of neighborhoods of the origin 0 of E and $V \in \mathcal{V}$. Since $\overline{F(X)}$ is a compact subset of the admissible subset X, there exist a continuous map $h : \overline{F(X)} \to X$ and a finite dimensional subspace L of E such that $x - h(x) \in V$ for all $x \in \overline{F(X)}$ and $h(\overline{F(X)}) \subset L \cap X$. Let $M := h(\overline{F(X)})$. Then M is a compact subset of L and hence $K := \operatorname{co} M$ is a compact convex subset of $L \cap X$. Note that $h : \overline{F(X)} \to K$ and $F|_K : K \to \overline{F(X)}$. Since $F \in \mathfrak{B}^\kappa(X, X)$ and K is a compact convex subset of X, there exists a closed map $\Gamma \in \mathfrak{B}(K, \overline{F(X)})$ such that $\Gamma(x) \subset F(x)$ for all $x \in K$. Then the composition $h\Gamma : K \multimap K$ has a fixed point $x_V \in (h\Gamma)(x_V)$ since K is a polytope. Let $x_V = h(y_V)$ for some $y_V \in \Gamma(x_V) \subset \overline{F(X)}$. We have $y_V - h(y_V) = y_V - x_V \in V$. Since $\overline{F(X)}$ is compact, we may assume that y_V converges to some $\hat{x}$. Then x_V also converges to $\hat{x}$ and hence $\hat{x} \in K$. Since the graph of Γ is closed in $K \times \overline{\Gamma(K)}$, we have $\hat{x} \in \Gamma(\hat{x}) \subset F(\hat{x})$. This completes our proof. □

It should be noted that, in Theorem 1, the admissibility of X can be replaced by that of $\overline{F(X)}$ without affecting its conclusion.

We have the following immediately:

COROLLARY 1.1. *Let E be a t.v.s. and X an admissible convex subset of E. Then any closed compact map $F \in \mathfrak{B}(X, X)$ has a fixed point.*

Proof. Any closed map $F \in \mathfrak{B}(X, X)$ belongs to $\mathfrak{B}^\kappa(X, X)$. □

COROLLARY 1.2. *Let E be a t.v.s. and X an admissible convex subset of E. Then any compact map $F \in \mathfrak{B}^\sigma(X, X)$ has a fixed point.*

In the remainder of this section, we list more than sixty papers in chronological order, from which we can deduce particular forms of Theorem 1. Some results are direct consequences of Theorem 1; and for some cases we can obviously obtain particular forms of Theorem 1 for a subclass of $\mathfrak{B}^\kappa$ and a special type of t.v.s.

Poincaré [1883, 1884]: An n-dimensional version of the intermediate value theorem equivalent to Brouwer's theorem [1912].

Bohl [1904]: An equivalent form of Brouwer's theorem [1912].

Brouwer [1912]: Corollary 1.1 for an n-simplex X and the subclass $\mathbb{C}$ of all continuous functions instead of $\mathfrak{B}$.

Alexander [1922]: Corollary 1.1 for the closed $(n-1)$-ball X and $\mathfrak{B} = \mathbb{C}$. (This means that the author obtained his result for the subclass $\mathbb{C}$ of $\mathfrak{B}$.)

Birkhoff and Kellogg [1922]: Fixed point theorems and invariant direction theorems for finite dimensional or function spaces $\mathbb{C}([a,b])$ and $L^2(a,b)$. Applied to the classical existence theorems for differential and integral equations, linear and nonlinear.

Schauder [1927]: Corollary 1.1 for a compact convex subset X of a Banach space and $\mathfrak{B} = \mathbb{C}$. Applied to solutions of second order partial differential equations of hyperbolic type.

Schauder [1930]: Corollary 1.1 for a closed convex subset X of a Banach space and $\mathfrak{B} = \mathbb{C}$, and two more theorems. Note that in view of Corollary 1.1, X can be a mere convex subset of a normed vector space.

Tychonoff [1935]: Corollary 1.1 for a compact convex subset X of a locally convex t.v.s. and $\mathfrak{B} = \mathbb{C}$. Applied to solutions of systems of differential equations.

von Neumann [1937]: An equivalent form of Corollary 1.1 for a compact convex subset X of $\mathbf{R}^m$ and $\mathfrak{B} = \mathbb{K}_c$. Applied to various problems of mathematical economics.

Krein and Šmulian [1940]: Two extensions of Schauder's theorem for separable Banach spaces.

Miranda [1940]: An n-dimensional version of the intermediate value theorem equivalent to Brouwer's theorem [1912].

Kakutani [1941]: Corollary 1.1 for the case X is a simplex or a compact convex subset of a Euclidean space and $\mathfrak{B} = \mathbb{K}$ the class of Kakutani maps; that is, u.s.c. maps with compact convex values. Applied to simple proofs of von Neumann's intersection theorem [1937] and minimax theorem. Later, this result has been applied to numerous problems in various fields; see Park [1995b].

Eilenberg and Montgomery [1946]: Kakutani's theorem is extended to $\mathfrak{B} = \mathbb{V}$ the class of acyclic maps; that is, u.s.c. maps with compact acyclic values.

Begle [1950]: Extended the above result of Eilenberg and Montgomery. A consequence is that $F \in \mathbb{V}(X, X)$ has a fixed point if X is a compact convex subset of a locally convex t.v.s.

Bohnenblust and Karlin [1950]: Corollary 1.1 for the case X is a closed convex subset of a Banach space E and $\mathfrak{B} = \mathbb{K}$. (This means that the authors obtained their result for the subclass $\mathbb{K}$ of $\mathfrak{B}$.)

Hukuhara [1950]: Corollary 1.1 for the case E is a locally convex t.v.s. and $\mathfrak{B} = \mathbb{C}$. Proved the admissibility of a convex subset of a locally convex t.v.s. and noted that it was shown in 1948 by himself.

Nagumo [1951]: Every locally convex t.v.s. is admissible.

Fan [1952]: Corollary 1.1 for the case X is a compact convex subset of a locally convex t.v.s. and $\mathfrak{B} = \mathbb{K}$. Applied to an intersection theorem and a minimax theorem.

Glicksberg [1952]: Same as above. Applied to the Nash equilibrium points.

O'Neill [1957]: Corollary 1.1 for the case X is a compact convex subset of a Euclidean space and $\mathfrak{B} = \mathbb{N}$ the class of O'Neill maps; that is, continuous maps with values of one or m acyclic components, where m is fixed. Gave more general forms.

Schaefer [1959]: Corollary 1.1 for the case X is a complete convex subset of a locally convex t.v.s. and $\mathfrak{B} = \mathbb{C}$. Applied to nonlinear maps defined on (a subset of) the positive cone in a partially ordered locally convex t.v.s.

Nikaido [1959]: Let M be a compact Hausdorff topological space, N a finite dimensional compact convex set and $\sigma, \tau : M \to N$ continuous functions such that τ is onto and $\tau^-(q)$ is acyclic for each $q \in N$. Then there exists a $p \in M$ such that $\sigma(p) = \tau(p)$.

Note that $\sigma\tau^- \in \mathbb{V}_c(N, N)$ has a fixed point $q \in N$ by Corollary 1.1. Choose any $p \in \tau^-(q)$. Then we have the conclusion.

Singbal [1962]: A proof of Hukuhara's theorem [1950], as suggested by Bonsall [1962], who did not know that it was already done.

Fan [1964]: Corollary 1.1 for the case X is a compact convex subset of a t.v.s. E on which E^* separates points and $\mathfrak{B} = \mathbb{C}$.

Now it is well-known that such X is affinely embeddable in a locally convex t.v.s.; see Weber [1992b].

Powers [1970]: X is a (open or closed) convex subset of a Banach space and $\mathfrak{B} = \mathbb{V}$ in Corollary 1.1.

Powers [1972]: X is a convex subset of a Banach space and $\mathfrak{B} = \mathbb{V}_c$, where the intermediate spaces are metric, in Corollary 1.1.

Himmelberg [1972]: Corollary 1.1 for the case E is a locally convex t.v.s. and $\mathfrak{B} = \mathbb{K}$. Applied to generalizations of the von Neumann intersection theorem and the minimax theorem.

Rhee [1972]: X is a closed bounded convex subset of a Banach space and $\mathfrak{B} = \mathbb{K}$ in Corollary 1.1.

Fournier and Granas [1973]: $X = E$ and $\mathfrak{B} = \mathbb{C}$ in Corollary 1.1. This also holds for closed shrinkable neighborhoods X of zero in an admissible t.v.s.

Sehgal and Morrison [1973]: A variant of the fixed point theorem of Himmelberg [1972] is obtained.

Hahn and Pötter [1974]: Corollary 1.1 for the subclass $\mathbb{C}$ of $\mathfrak{B}$ under the additional assumption that X is closed.

Górniewicz [1976]: E is a normed space and $\mathfrak{B}$ is the subclass of admissible compact maps in Corollary 1.1.

In the category of metric spaces, for a map $\varphi : X \multimap Y$, a pair (p, q) of continuous functions of the form $X \xleftarrow{p} Z \xrightarrow{q} Y$ is called a *selected pair* of φ if the following two conditions are satisfied:

(i) p is a Vietoris map (that is, p is proper, $p^-(X) = Z$, and $p^-(x)$ is acyclic for all $x \in X$); and

(ii) $q(p^-(x)) \subset \varphi(x)$ for each $x \in X$.

An *admissible* map $\varphi : X \multimap Y$ (in the sense of Górniewicz) is one having such a selected pair. A composition of two admissible maps is admissible.

Krauthausen [1976]: $\mathfrak{B}$ is replaced by the subclass $\mathbb{C}$ in Corollary 1.1.

Skordev [1976]: General forms of the case $\mathfrak{B} = \mathbb{V}$ of Corollary 1.1 are obtained for an open acyclic subset X. A particular form for $\mathfrak{B} = \mathbb{C}$ was already given by Fournier and Granas [1973].

Idzik [1978]: The generalization of Himmelberg's theorem [1972] due to Sehgal and Morrison [1973] was shown to be equivalent to that theorem. Applications of Himmelberg's theorem to the existence of an equilibrium state in a special noncooperative game were also given.

Ha [1980]: Showed that $\mathbb{K}$ is a subclass of $\mathfrak{B}$.

Hadžić [1980]: It was shown that a class of subsets of type Φ is admissible in a t.v.s.

Górniewicz and Granas [1981]: Extended the concept of admissible maps of Górniewicz [1976] to morphisms in the category of Hausdorff topological spaces. Moreover, in this category, any finite composition of acyclic maps is admissible.

It is noted that the class of maps determined by some compact morphism is an example of $\mathfrak{B}$.

Ben-El-Mechaiekh, Deguire, et Granas [1982]: Theorem 1 for the case E is locally convex and $\mathfrak{B}^\kappa$ is restricted to the subclass of the Fan-Browder maps $A : X \multimap X$; that is, $A(x)$ is convex for each $x \in X$ and $X = \bigcup\{\operatorname{Int} A^-(y) : y \in X\}$.

Haddad and Lasry [1983]: A map $\Gamma : X \multimap Y$ is said to be *σ-selectionable* if there exists a sequence $\{\Gamma_n\}_{n\in\mathbf{N}}$ of u.s.c. maps $\Gamma_n : X \multimap Y$ having compact values and continuous selections such that

(a) $\Gamma_{n+1}(x) \subset \Gamma_n(x)$ for any $x \in X$ and any $n \in \mathbf{N}$;
(b) $\Gamma(x) = \bigcap_{n\in\mathbf{N}} \Gamma_n(x)$ for any $x \in X$.

Note that such Γ is u.s.c. with nonempty compact values. It is known that $\Gamma \in \mathfrak{B}(X, Y)$ whenever Γ is σ-selectionable; see Proposition A.I.3 and Theorem A.III.1 of Haddad and Lasry [1983].

Note that Corollary 1.1 is a far-reaching generalization of Theorem A.III.1 of that paper; see Park [1998a].

Arino, Gautier, and Penot [1984]: Corollary 1.1 for the case E is a metrizable locally convex t.v.s., X is a weakly compact convex subset of E, and $F = f : X \to X$ a weakly sequentially continuous map. Applied to ordinary differential equations.

Dzedzej [1985]: A *permissible* map $F : X \multimap Y$ is a compact-valued u.s.c. map such that there exists an $\tilde{F} \in \mathbb{M}_c(X, Y)$ satisfying $\tilde{F}(x) \subset F(x)$ for all $x \in X$, where $\mathbb{M}(X, Y)$ is given as follows:

Let $\mathbb{M}_0(X, Y) = \mathbb{V}(X, Y)$;

$F \in \mathbb{M}_m(X, Y) \Longleftrightarrow F$ is an O'Neill map; that is, F is continuous and has values consisting of one or m compact acyclic components; and

$F \in \mathbb{M}(X, Y) \Longleftrightarrow F \in \mathbb{M}_m(X, Y)$ for some integer $m \geq 0$.

The class of all permissible maps is an example of $\mathfrak{B}$.

Simons [1986]: For any polytope P, each $F = F_1 \cdots F_n \in \mathbb{K}_c(P, P)$ has a fixed point, where the range of F_i is contained in a convex set. This shows that $\mathbb{K}_c$ is an example of $\mathfrak{B}$.

McLinden [1989]: X is a compact convex subset of a locally convex t.v.s. and $\mathfrak{B} = \mathbb{V}$ in Corollary 1.1. This is a consequence of Begle [1950].

Ben-El-Mechaiekh [1990]: X is a compact convex subset of a locally convex t.v.s. and $\mathfrak{B} = \mathbb{K}_c$ in Corollary 1.1.

Lassonde [1990]: E is locally convex and $\mathfrak{B} = \mathbb{K}_c$ in Corollary 1.1.

Ben-El-Mechaiekh [1991]: E is locally convex and $\mathfrak{B} = \mathbb{A}$, the class of approachable maps, in Corollary 1.1.

Given two open neighborhoods U and V of the origins in t.v.s. E and F, respectively, and $X \subset E$ and $Y \subset F$, a (U, V)-*approximative continuous selection* of $T : X \multimap Y$ is a continuous function $s : X \to Y$ satisfying

$$s(x) \in (T[(x + U) \cap X] + V) \cap Y \quad \text{for each} \quad x \in X.$$

A map $T : X \multimap Y$ is said to be *approachable* if it admits a (U, V)-approximative continuous selection for every U and V as above.

Ben-El-Mechaiekh and Deguire [1991]: E is locally convex and $\mathfrak{B}$ is the subclass of approachable maps $\mathbb{A}$ or $\mathbb{A}_c$ in Corollary 1.1.

Lassonde [1991]: E is locally convex and $\mathfrak{B}^\sigma = \mathbb{K}_c^\sigma$ in Corollary 1.2.

Shioji [1991]: Noted that $\mathbb{V}$ is an example of $\mathfrak{B}$. This is already known by Eilenberg and Montgomery [1946] and Górniewicz [1976].

Park [1992a]: Corollary 1.1 for the case X is a compact convex subset of a locally convex t.v.s. E and $\mathfrak{B} = \mathbb{V}_c$ where intermediate spaces are all compact convex subsets of locally convex t.v.s. Some related results are also given.

Park [1992b]: E is locally convex and $\mathfrak{B} = \mathbb{V}$ in Corollary 1.1. Applications are followed by Kum [1994] to generalized quasi-variational inequalities and by Lee *et al.* [1996] to vector quasi-variational inequalities.

Ben-El-Mechaiekh [1993]: E is locally convex and $\mathfrak{B} = \mathbb{A}$, the class of approachable maps, in Corollary 1.1. Some applications to fixed point or coincidence theorems are obtained.

Park [1993a]: E is locally convex and $\mathfrak{B} = \mathfrak{A}_c$ in Corollary 1.1.

Park [1993b]: X is a compact convex subset of a t.v.s. E on which E^* separates points and $\mathfrak{B}^\kappa = \mathfrak{A}_c^\kappa$ in Theorem 1.

Park [1994]: E is locally convex and $\mathfrak{B}^\sigma = \mathfrak{A}_c^\sigma$ in Corollary 1.2.

Park, Singh, and Watson [1994]: E is locally convex and $\mathfrak{B}^\sigma = \mathbb{V}_c^\sigma$ in Corollary 1.2. Applied to best approximation and fixed point theorems.

Park [1995a]: X is a compact convex subset of a locally convex t.v.s. E and $\mathfrak{B} = \mathbb{V}$ in Corollary 1.1. Some results on best approximations and fixed points are also given.

Chang and Yen [1996]: E is locally convex and $\mathfrak{B} = \text{KKM}$ in Corollary 1.1.

For a convex space X, a topological space Y, and a map $T : X \multimap Y$, the authors defined

$T \in \text{KKM}(X,Y) \iff$ the family $\{S(x) : x \in X\}$ has a finite intersection property whenever $S : X \multimap Y$ has closed values and $T(\operatorname{co} N) \subset S(N)$ for each nonempty finite subset N of X.

The authors applied their result to some coincidence theorems and other problems.

Note that, in the class of closed compact maps, two subclasses $\mathfrak{B}$ and KKM coincide; see Park [1997b].

Chu [1996]: E is locally convex and $\mathfrak{B} = \mathbb{V}$ in Corollary 1.1. This is already appeared in Park [1992b].

Park [1996]: E is locally convex and $\mathfrak{B} = \mathbb{V}$ in Corollary 1.1. Other results for a t.v.s. E on which E^* separates points and their applications are given.

Park and H. Kim [1996]: Another proofs of Park's earlier results for locally convex t.v.s. and $\mathfrak{B}^\sigma = \mathfrak{A}_c^\sigma$ in Corollary 1.2.

Park [1997b]: E is locally convex in Corollary 1.1. Applied to condensing maps and other problems.

Park [1998b]: $\mathfrak{B} = \mathbb{V}$ in Corollary 1.1. Applied to existence of solutions of quasi-equilibrium problems.

4. New fixed point theorems for condensing multimaps

In this section, we deduce new theorems for condensing maps.

Let E be a t.v.s. and C a lattice with a least element, which is denoted by 0. A function $\Phi : 2^E \to C$ is called a *measure of noncompactness* on E provided that the following conditions hold for any $X, Y \in 2^E$:

(1) $\Phi(X) = 0$ if and only if X is relatively compact;
(2) $\Phi(\overline{\text{co}}\, X) = \Phi(X)$; and
(3) $\Phi(X \cup Y) = \max\{\Phi(X), \Phi(Y)\}$.

It follows that $X \subset Y$ implies $\Phi(X) \le \Phi(Y)$.

The above notion is a generalization of the set-measure γ and the ball-measure χ of noncompactness defined in terms of a family of seminorms or a norm.

For $X \subset E$, a map $T : X \multimap E$ is said to be Φ-*condensing* provided that if $A \subset X$ and $\Phi(A) \le \Phi(T(A))$, then A is relatively compact; that is, $\Phi(A) = 0$.

From now on, we assume that Φ is a measure of noncompactness on the given t.v.s. E if necessary.

Note that each map defined on a compact set is Φ-condensing. If E is locally convex, then a compact map $T : X \multimap E$ is γ- or χ-condensing whenever X is complete or E is quasi-complete.

The following is well-known; for example, see Mehta *et al.* [1997].

LEMMA. *Let X be a nonempty closed convex subset of a t.v.s. E and $T : X \multimap X$ a Φ-condensing map. Then there exists a nonempty compact convex subset K of X such that $T(K) \subset K$.*

From Theorem 1 and Lemma, we have the following:

THEOREM 2. *Let X be an admissible closed convex subset of a t.v.s. E. Then any Φ-condensing map $F \in \mathfrak{B}^{\kappa}(X, X)$ has a fixed point.*

Proof. By Lemma, there is a nonempty compact convex subset K of X such that $F(K) \subset K$. Since $F \in \mathfrak{B}^{\kappa}(X, X)$, there exists a closed map $\Gamma \in \mathfrak{B}(K, K)$ such that $\Gamma(x) \subset F(x)$ for all $x \in K$. Since Γ is compact, by Corollary 1.1, it has a fixed point $x_0 \in K$; that is, $x_0 \in \Gamma(x_0) \subset F(x_0)$. This completes our proof. □

COROLLARY 2.1. *Let X be an admissible closed convex subset of a t.v.s. E. Then any closed Φ-condensing map $F \in \mathfrak{B}(X, X)$ has a fixed point.*

COROLLARY 2.2. *Let X be an admissible closed convex subset of a t.v.s. E. Then any Φ-condensing map $F \in \mathfrak{B}^{\sigma}(X, X)$ has a fixed point.*

In the remainder of this section, we list more than ten papers in chronological order, from which we can deduce particular forms of Theorem 2.

Darbo [1955]: A form of Corollary 2.1 for a Banach space E and $F = f \in \mathbb{C}(X, X)$ was obtained.

Sadovskii [1967]: The same as above.

Lifšic and Sadovskii [1968]: The above result was extended to a locally convex t.v.s. E.

Himmelberg, Porter, and Van Vleck [1969]: A form of Corollary 2.1 for a locally convex t.v.s. and $F \in \mathbb{K}(X, X)$.

Daneš [1970], Furi and Vignoli [1970], and Nussbaum [1971] obtained particular forms of Corollary 2.1 for a Banach space E.

Reich [1971] extended Sadovskii's theorem to a locally convex t.v.s.

Reinermann [1971]: A form of Corollary 2.1 for a Banach space and $F = f \in \mathbb{C}(X, X)$.

This direction of study moved to the type $\mathfrak{B}(X, E)$ of maps with certain boundary conditions (for example, so-called Leray-Schauder condition, inwardness, and so on) and there have appeared a lot of results on such maps; see Park [1995b, 1997a]. However, we still have particular forms of Theorem 2 as follows:

Mehta, Tan, and Yuan [1997]: Particular forms of Theorem 1 and Corollary 2.1 for the Fan-Browder type maps and the Kakutani maps, respectively, were obtained for a locally convex t.v.s.

Park [1997b]: Theorem 2 and Corollary 2.1 for a locally convex t.v.s. E.

5. Related results and problems

In this section, we collect known results which are closely related to our new theorems but are not included in. Consequently, a number of natural questions occur.

Many of the results quoted in this paper are closely related to the following problem known as the Schauder conjecture posed by Schauder in the Scottish Book in around 1935:

PROBLEM 1. Does every compact convex subset X of a (metrizable) t.v.s. have the fixed point property? That is, does any continuous map $f : X \to X$ have a point $x_0 \in X$ such that $x_0 = f(x_0)$?

Note that Theorem 1 is one of the most general partial solution to (the generalized forms) of Problem 1. Note also that there are many open problems related to the Schauder conjecture. See Hadžić [1984], Idzik [1987, 1988], Weber [1992a,b], Nguyen [1996], and references therein.

Theorem 1 would be the generalized solution of the Schauder conjecture if the following long-standing problem raised by Klee [1960] had a negative solution:

PROBLEM 2. Is there a (compact) convex non-admissible subset of a t.v.s.?

The following is noteworthy:

Glebov [1969]: The fixed point theorem of Fan [1952] and Glicksberg [1952] is extended to partially closed maps.

Far-reaching generalizations of this result including a large number of historically well-known particular forms were due to Park [1992b,c], [1993b].

PROBLEM 3. Does (any particular form of) Theorem 1 hold for partially closed maps instead of closed maps?

There are some other extensions of the Schauder fixed point theorem:

Zima [1977]: The Schauder theorem was extended to paranormed spaces (not necessarily locally convex).

Rzepecki [1979]: Let X be a convex subset of a t.v.s. E, and $f : X \to X$ a continuous compact map such that $K = \overline{f(X)}$ is locally convex; that is, for every $x \in K$ and every neighborhood V of x, there exists a neighborhood U of x such that $\text{co}(U \cap K) \subset V$. Then f has a fixed point.
This theorem is applied to obtain fixed point theorems of the Sadovskii or Krasnoselskii types and results on theory of equations.

Girolo [1981]: Let X be a compact convex subset of a normed vector space. Then a function $f : X \to X$ is called a *connectivity* map whenever the graph of f over each connected subset of X is a connected set. It was shown that the class of connectivity maps is a subclass of $\mathfrak{B}$.

Hadžić [1981]: A generalization of Zima's theorem [1977] to the Kakutani maps and other results were obtained.

Hadžić [1982a]: A particular form of Rzepecki's theorem [1979] for metrizable case and its generalization to the Kakutani maps were obtained.

Hadžić [1982b]: Introduced the concept of a set of Z type and noted that every convex subset of Z type is locally convex. Collected theorems due to the author and others.

Idzik [1987]: Rzepecki's theorem [1979] is extended to the case $K = \overline{f(X)}$ is convexly totally bounded; that is, for every neighborhood V

of $0 \in E$ there exists a finite subset $\{x_i\}_{i \in I}$ of E and a finite family of convex subsets $\{C_i\}_{i \in I}$ of V such that $K \subset \bigcup_{i \in I}\{x_i + C_i\}$.

Idzik [1988]: The above and other results were extended to the Kakutani maps.

It is also noted that in a locally convex t.v.s. E, every subset is of Z type and is locally convex, and that any compact set (in a t.v.s.) which is locally convex or of Z type is convexly totally bounded.

Therefore Idzik's theorems generalize works of Zima [1977], Rzepecki [1979], Hadžić [1981, 1982a,b] and others related to sets of Z type.

Moreover, if the following is affirmative, then Idzik's theorem will follow from Theorem 1.

PROBLEM 4. Is any compact and convexly totally bounded set admissible?

References on the study on convexly totally bounded sets can be found in Weber [1992a,b].

Arandelović [1995]: Hadžić's extension [1981] of Zima's theorem [1977] was proved by using the KKM theorem due to Ky Fan.

Pasicki [1995]: Proved Himmelberg's theorem [1972] without assuming the Hausdorffness of the locally convex t.v.s.

Nguyen [1996]: In a metrizable t.v.s. E, the admissibility is extended to weak admissibility for compact convex subsets. Showed that such subset has the fixed point property. This gives a partial solution to the Schauder conjecture.

He also raised the following and other problems:

PROBLEM 5. Is every compact convex subset of E weakly admissible?

PROBLEM 6. Is every weakly admissible compact convex subset of E: (i) an AR? or (ii) admissible?

Chang [1998]: Pasicki's theorem [1995] holds for the KKM class of multimaps without assuming the Hausdorffness of the locally convex t.v.s.

PROBLEM 7. Does (any particular form of) Theorem 1 hold without assuming the Hausdorffness of the t.v.s.?

Finally we should mention the following important paper:

Górniewicz and Rozploch-Nowakowska [1996]: Contains a survey of various results concerning the Schauder theorem for metric spaces both in single-valued or multi-valued cases. These important results are not comparable to our Theorem 1.

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Department of Mathematics
Seoul National University
Seoul 151–742, Korea
E-mail: shpark@math.snu.ac.kr