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INNOVATION OF SOME RANDOM FIELDS

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ABSTRACT. We apply the generalization of Lévy's infinitesimal equation

$$\delta X(t) = \phi(X(s), s \le t, Y_t, t, dt), \ \ t \in R^1,$$

for a random field $X(C)$ indexed by a contour C or by a more general set. Assume that the $X(C)$ is homogeneous in x, say of degree n, then we can appeal to the classical theory of variational calculus and to the modern theory of white noise analysis in order to discuss the innovation for the $X(C.)$

1. Introduction

As is well known, P. Lévy has proposed the so-called stochastic infinitesimal equation,

$$\delta X(t) = \phi(X(s), s \le t, Y_t, t, dt), \tag{1.1}$$

from which the structure of a stochastic process can be determined. Although it has only formal significance, it has profound suggestion in the investigation of a stochastic process $X(t), t \in R$. In the expression (1.1) the Y_t is the innovation for $X(t)$; namely $\{Y_t\}$ is an independent system such that each Y_t contains the same information as that is gained by the $X(t)$ during the time interval $[t, t + dt)$.

We are interested in a random field $X(C)$ indexed by a manifold C and wish to discuss its intricate probabilistic structure by observing the variations $\delta X(C)$ when C varies a little within a certain class $\mathbf{C}$. To this end we generalize the above method of the innovation approach to $X(t)$ with one dimensional parameter to a random field $X(C)$ with a parameter C.

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The basic idea of our investigation of a random field $X(C)$ is to introduce the stochastic infinitesimal equation for the field in order to characterize the $X(C)$. Our approach is of course in line with white noise analysis. We form the innovation process for the given field and express it as a functional of the obtained innovation.

T. Hida has proposed a counter part of (1.1) for a random field $X(C)$ depending on a contour (or a loop) C in the form

$$\delta X(C) = \Phi\left(X(C'), C' < C, Y(s), s \in C, C, \delta C\right), \tag{1.2}$$

where $C' < C$ means that C' is inside of C, that is, the domain (C') enclosed by a contour C' is a subset of (C), and where Φ is, as before, a nonrandom function and the system $Y = \{Y(s), s \in \mathbf{C}; C\}$ is the *innovation.* We note that the parameter set $\mathbf{C} = \{C\}$ is taken to be a class in which the variation can make enough contribution to our approach, and it will be specified later.

The first step to our problem is to establish a way of constructing an innovation and then we come to a problem to form the given field as a functional of the constructed innovation.

The discussion will be first done , particularly, on the innovation which is taken to be a (Gaussian) white noise, then come to a slight generalization. In any case such an innovation may be called a system of *idealized elementary random variables* (see [3]), because the system of those random variables is most elementary and atomic.

Important concept and tools from analysis are applied for our purpose; namely generalized white noise functionals which are like infinite dimensional Schwartz's distributions and the differential operator that will be prescribed later.

We have so far discussed only some particular cases, however it is our hope that the present technique can be applied to more general class of random fields formed from white noise.

2. Background

We first prepare some background, the theory of white noise, along which the innovation approach to random field is discussed.

Let (E^*, μ) be a white noise space, where E^* is a space of generalized functions on R^d ; it is the dual space of some nuclear space $E \subset L^2(R^d)$, and where μ is a Gaussian measure on E^* such that its characteristic functional $C(\xi)$, $\xi \in E$, is given by

$$\begin{aligned} C(\xi) &= \int_{E^*} \exp[i < x, \xi >] \, d\mu(x) \\ &= \exp\left[-\tfrac{1}{2}\|\xi\|^2\right], \quad \xi \in E. \end{aligned} \tag{2.1}$$

Then the complex Hilbert space $(L^2) = L^2(E^*, \mu)$ can be build in a usual manner. A member of (L^2) is denoted by $\varphi(x), \psi(x)$ and so forth.

We can now construct the Gel'fand triple

$$(S) \subset (L^2) \subset (S)^*, \tag{2.2}$$

where (S) and $(S)^*$ be the space of test functionals and that of generalized (white noise) functionals, respectively. For more details we refer to [1].

Since we will deal with some random fields with parameter which is taken to be lower dimensional manifolds, we need the following assertion to be proved.

PROPOSITION. *Restrictions of the parameter of white noise to lower dimensional C^∞−manifolds (or equivalently, definitions of marginal distributions of the white noise measure μ) are defined in terms of generalized white noise functionals.*

Outline of the proof. The integral, in the formula of the characteristic functional $C(\xi)$, given by (2.1), can be restricted on a smooth manifold M. Then a Gel'fand triple

$$E(M) \subset L^2(M) \subset E^*(M) \tag{2.3}$$

is obtained and we are given a Gaussian measure μ_M on $E^*(M)$ is uniquely determined by $C_M(\xi)$. Note that the construction of the above Gel'fand triple heavily depends on the differential structure of the manifold M. □

REMARK. The measure μ_M is viewed as a marginal distribution of μ. It is easy to see that $(S(M))$ and $(S(M))^*$ can be defined as in the case of (S) and $(S)^*$. This fact is tacitly used in section 4, where M is specified to be a contour C.

The S-*transform* of a generalized functional $\varphi(x) \in (S)^*$ is defined by

$$(S\varphi)(\xi) = \int \varphi(x+\xi)d\mu(x), \quad \varphi \in (S)^*. \tag{2.4}$$

By using this transform, we can define a differential operator ∂_t acting on (S),

$$\partial_t = S^{-1}\left(\frac{\delta}{\delta\xi(t)}(S\varphi)(\xi)\right), \tag{2.5}$$

where $\frac{\delta}{\delta\xi(t)}$ denotes the Fréchet derivative.

REMARK. For the one dimensional parameter case, a concretized expression of white noise is $\dot{B}(t)$, the time derivative of a Brownian motion $B(t)$, and there μ-almost all $x \in E^*$ are viewed as sample paths of $\dot{B}(t)$. We may also consider the operator ∂_t as the partial differential operator $\frac{\partial}{\partial \dot{B}(t)}$, which can be defined rigorously.

The adjoint operator ∂_t^* for the operator ∂_t is defined on $(S)^*$ such that

$$\langle \partial_t\varphi, \psi\rangle = \langle \varphi, \partial_t^*\psi\rangle, \quad \varphi \in (S),\ \psi \in (S)^*. \tag{2.6}$$

For more details of the white noise analysis we refer to [1] and [4].

We note that the actions by the operators ∂_t and $\partial_t{}^*$ can be restricted to $(S(M))$ and $(S(M))^*$, respectively, for any C^∞-manifold M .

3. Innovations of Gaussian processes

We shall first deal with a visualized case in order to illustrate the idea behind our approach to random fields. Let $\{X(t)\}$ be an ordinary Gaussian process with one dimensional parameter $t \in T \subset R^1$. Assume, in particular, that the $X(t)$ has a representation in terms of a white noise $\dot{B}(t)$. More precisely, we assume that $X(t)$ has unit multiplicity (see [2]) and that it is expressed as a Wiener integral of the form

$$X(t) = \int^t F(t,u)\dot{B}(u)du, \quad t \in T, \tag{3.1}$$

where the kernel $F(t,u)$ is assumed to be smooth enough in both variables. Then, its variation over an infinitesimal time interval $[t, t+dt)$ is given by

$$(3.2)\quad \delta X(t) = F(t,t)\dot{B}(t)dt + dt\int^t F_t(t,u)\dot{B}(u)du + o(dt),$$

where $F_t(t,u) = \frac{\partial}{\partial t}F(t,u)$.

As is well known, the representation of the form (3.1) is not unique for a given $X(t)$. Let us take the canonical representation, which can give some advantageous for our innovation approach. Thus it satisfies the condition

$$(3.3)\quad E(X(t)/\mathbf{B}_s(X)) = \int^s F(t,u)\dot{B}(u)du, \quad \text{for any } s < t,$$

where $\mathbf{B}_s(X)$ is the smallest $\sigma-$field with respect to which all the $X(u), u \leq s$, are measurable.

PROPOSITION. *If a Gaussian process has a representation of the form* (3.1), *the function $F(t,t)^2$ is uniquely determined regardless the representation is canonical or not.*

Outline of the proof. Take an interval $I = [a,b]$ arbitrarily and let $\{\Delta_j\}$ be a partition of I. Consider $E(\sum_j(\Delta_j X)^2)$ which will converge to $\int_I F(t,t)^2 dt$ as $\Delta = max|\Delta_j| \to 0$ by the continuity of $F(t,u)$. Since I is arbitrary, the assertion is proved. □

We have a freedom to choose the sign of $F(t,t)$, but we do not care the sign, since $\dot{B}(t)$, which is to be associated to dt, has symmetric probability distribution. On the other hand $F(t,t)^2$ is determined by $\delta X(t)$, which means that the $F(t,t)^2$ is independent of the way of representation.

If we assume

$$(3.4)\quad \delta X(t) \text{ is of order } \sqrt{dt},$$

then $F(t,t)$ may be taken to be positive and continuous. With this assumption and noting that $X(t)$ has unit multiplicity, we know $F(t,t)$ and can prove the following theorem.

THEOREM. *The limit*

$$(3.5)\quad \lim_{dt\to 0+} \frac{\delta X(t) - E[\delta X(t)/\mathbf{B}_t(X)]}{F(t,t)}$$

gives the innovation.

NOTE. The innovation obtained above will be denoted by the same symbol $\dot{B}(t)dt$ as was in (3.1). However, it may be different from the original one, if the representation (3.1) is not a canonical representation. The system $\{\dot{B}(t)dt\}$ well defines a Brownian motion $B(t)$ for the canonical representation.

Once the $\dot{B}(t)$ is given for every t, we can define the differential operator

$$\partial_u = \frac{\partial}{\partial \dot{B}(u)}, \quad u \leq t. \tag{3.6}$$

Apply ∂_u to $X(t)$ to have $F(t,u)$: $\partial_u X(t) = F(t,u), u \leq t$. It is the canonical kernel that we are looking for. Noting that $\dot{B}(t)$ is the innovation, we can establish the following proposition.

PROPOSITION. *The exact value of the canonical kernel $F(t,u)$ is obtained by applying the operator ∂_u, $u \leq t$, to the $X(t)$.*

Thus we can see that the expression (3.1) for the canonical representation can be completely determined and hence the structure of $X(t)$ can be known.

REMARK. As for the idea of the canonical representation of a Gaussian process we refer to [2].

4. Random fields and their variations

Let $X(C)$ be a random field with parameter C which is taken to be a smooth manifold running through the parameter space of the white noise $x(u), u \in R^d, x \in E^*$. Here, E^* is the space of generalized functions on R^d and the white noise measure μ is introduced on E^*. We are interested in the variation $\delta X(C)$, of $X(C)$, in which the innovation would be obtained.

To fix the idea and to avoid non-essential complex assumptions, we restrict our attention to the case where the parameter C is in $\mathbf{C}$ containing smooth contours (i.e. loops) in the plane. As a similar formula to the variation for a stochastic process $X(t)$, we propose a stochastic variation equation for $X(C)$ as in (1.2).

The innovation is understood more precisely in the following sense. The system $\{Y_s, s \in C\}$ is independent of every $X(C')$ with $C' < C$, and the equation (1.2) tells that the new information that the random field gains between C and $C+\delta C$ should be the same as that gained by Y_s's when s runs through the same region between C and $C+\delta C$.

It is claimed that the equation (1.2), if it exists, can determine the probabilistic structure of the given random field $X(C)$ completely, although (1.2) has only a formal significance.

In what follows we assume that

(4.1) $X(C)$ *is causal* in terms of white noise.

This means that $X(C)$ is a function only of the $x(u)$, $u \in (C)$, (C) being the domain enclosed by C, $x \in E^*$. We are now ready to discuss a random field $X(C)$ satisfying the condition (4.1) and

(4.2) $X(C) = X(C,x)$ is in $(S)^*$ and homogeneous in x.

Here homogeneity means that the S-transform $U(C,\xi)$ is a homogeneous polynomial in ξ of degree n in the sense of P. Lévy. In addition we assume that

(4.3) $X(C,x)$ is a regular function of x.

PROPOSITION. *Under these assumptions* (4.1), (4.2) *and* (4.3) *there is a positive integer n such that $X(C)$ can be expressed in the form*

$$X(C) = \int_{(C)^n} F(C; u_1, u_2, \dots u_n) : x(u_1)x(u_2)\dots x(u_n) : du^n, \tag{4.4}$$

where $F(C, u_1, u_2, \dots u_n)$ is symmetric in $u_1, u_2, \dots, u_n$ and where $:\ :$ is the Wick product. Chapter 4.

The formula (4.4) is simply denoted by

$$\int_{(C)^n} F(C; u) : x^{n\otimes}(u) : du. \tag{4.5}$$

Our final assumption is that the kernel

$$F(C; u) \ \ and \ \ F'_n(C, u; s) = \tfrac{\delta F(C,u)}{\delta n}(s) \tag{4.6}$$

are continuous in u and in (u,s), respectively.

DEFINITION. The representation (4.4) is a canonical representation if

$$\begin{aligned}(4.7)\qquad &\hat{E}(X(C)/X(C'), C' < C_1)\\ &= \int_{(C_1)^n} F(C; u_1, u_2, ..u_n) : x(u_1)..x(u_n) : du_1...du_n,\end{aligned}$$

for every $C_1 < C$.

The notation $\hat{E}$ means the weak conditional expectation in the sense of Doob. It means the projection of $\delta X(C)$ on a closed linear manifold spanned by $X(C')$, $C' < C$.

We can prove that the representation (4.4) is a canonical representation if and only if

$$(4.8)\qquad \int_{(C)^n} F(C; u_1, u_2, ..u_n) f(u_1, .., u_n) du^n = 0,$$

for all $(C) < (C_0)$ implies $f = 0$ on (C_0).

Let us take the variation $\delta X(C)$ of(4.4). Then it is of the form

$$\begin{aligned}\delta X(C) &= n \int_C \int_{(C)^{n-1}} F(C, u_1'; s) : x^{(n-1)\otimes}(u_1') x(s) : du_1' \delta n(s) ds\\ (4.9)\qquad &+ \int_C \int_{(C)^n} F'_n(C, u; s) : x^{n\otimes}(u) : \delta n(s) du ds,\end{aligned}$$

where $u_1' = (u_2, u_3, \cdots, u_n)$ and F_n' denotes the functional derivative in C.

Take the weak conditional expectation.

$$\begin{aligned}&\hat{E}(\delta X(C)/X(C'), C' < C)\\ (4.10)\qquad &= \int_C \int_{(C)^n} F'_n(C, u; s) : x^{n\otimes}(u) : \delta n(s) du ds.\end{aligned}$$

Then we have

$$\begin{aligned}&\delta X(C) - \hat{E}(\delta X/X(C'), C' < C)\\ (4.11)\qquad &= n \int_C \int_{(C)^{n-1}} F(C, u_1'; s) : x^{(n-1)\otimes}(u_1') x(s) : du_1' \delta n(s) ds.\end{aligned}$$

Let δn vary in the class of C^∞–functions so that δC is taken outward and that the integrand over C is determined as a function of s and the R.H.S. will give

$$(4.12) \qquad x(s) \int_{(C)^{n-1}} F(C, u_1'; s) : x^{(n-1)\otimes}(u_1') du_1'.$$

Let us denote it by

$$(4.13) \qquad x(s)\varphi(s)$$

and use the same technique as in one dimensional parameter space. Thus we know the value

$$(4.14) \qquad \varphi(s)^2.$$

We may ignore its sign to determine $\varphi(s)$. Divide (4.13) by $\varphi(s)$ to obtain the generalized innovation $x(s)$. Since the representation is canonical, it can be regarded as the same as the original $x(s)$. It means that it is the real innovation (not in a generalized sense). Thus we can prove the following theorem.

THEOREM. *The innovation for the random field $X(C)$ given by* (4.4) *is obtained as*

$$(4.15) \qquad x(s) = \frac{1}{\varphi(s)} \left\{ \frac{\delta X(C) - \hat{E}(\delta X(C)/X(C'), C' < C)}{\delta n}(s) \right\}.$$

Note that if the representation is canonical (4.15) gives the original x in (4.4). However for the noncanonical case, we can see that

$$(4.16) \qquad \begin{aligned} &\delta X(C) - \hat{E}(\delta X/X(C'), C' < C) \neq \\ &n \int_C \int_{(C)^{n-1}} F(C, u_1'; s) : x^{(n-1)\otimes}(u_1') x(s) : du_1' \delta n(s) ds. \end{aligned}$$

Thus in this case we are given a generalized innovation which is different from the original x.

REMARK. Observe that the situation is somewhat different from the one dimensional parameter case, i.e. Gaussian case. That is

$$(4.17) \qquad \delta X(C) - \hat{E}(\delta X(C)/X(C'), C' < C)$$

is orthogonal to $X(C')$ where C' is inside of C, however may not be independent.

Let ∂_u be the differential operator defined as in (3.6), where $\dot{B}(u)$ is replaced by $x(u)$.

PROPOSITION. *The kernel function F in* (4.4) *is obtained by*

$$F(C; u_1, u_2, ..., u_n) = \frac{1}{n!}\partial_{u_1}\partial_{u_2}...\partial_{u_n}X(C), \tag{4.18}$$

in which $u_1, u_2, ..., u_n$ are different.

CONCLUDING REMARKS.
1. We hope to generalize this theorem to the case where $X(C)$ is a normal function of x (cf. L-functional in the sense of Saito [6]) to have a similar result.
2. Details of the proofs and generalizations will be reported in a separate paper.

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