J. Korean Math. Soc. **35** (1998), No. 3, pp. 765–782

# CONVERGENCE THEOREMS FOR SET VALUED AND FUZZY VALUED MARTINGALES AND SMARTINGALES

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ABSTRACT. The purpose of this paper is to give convergence theorems both for closed convex set valued and relative fuzzy valued martingales, and sub- and super- martingales. These kinds of martingales, sub- and super-martingales are the extension of classical real valued martingales, sub- and super-martingales. Here we compare two kinds of convergences, in the Hausdorff metric and in the Kuratowski-Mosco sense. We also introduce a new convergence for the fuzzy valued case in the graph sense and obtain convergence theorems.

## 1. Introduction

In practice, we are often faced with random experiments whose outcomes are not numbers or vectors but are sets or are expressed in inexact linguistic terms. The former case is set valued random variables (also called random set). In the past 50 years, the study of set valued random variables has been developed extensively, with many applications to economics and optimal control problems, statistical problems and so on, see for examples, Arrow and Debreu [1], Arstein and Vitale [2], Aumman [3, 4], Debreu [8], Hiai and Umegaki [10, 11], Kendall [12], Klein and Thompson [14], Kudo [16], Papageorgiou [27-28], etc.. The concept of set valued random variable has been formalized as an extension of random variables and random vectors (cf. Matheron [23], Molchanov [24]). The latter case is fuzzy set random variables. As a simple example, consider a group of individuals chosen at random who are questioned about the

Received February 24, 1998.
1991 Mathematics Subject Classification: 60G42, 28B20, 60G48, 60D05.
Key words and phrases: set valued martingale, fuzzy valued martingale, sub-, super-martingale, the Kuratowski-Mosco convergence.
Research supported by Grant-in-Aid for Scientific Research No. 09440085.

weather of tomorrow in a particular city on a particular day. Some possible answers would be "a little raining day", "a heavy raining day", "an extremely heavy raining day" and so on. Whether it will rain tomorrow or not is a random phenomenon but each outcome is an inexact linguistic terms. A possible way of handling these inexact linguistic terms is by using the concepts of fuzzy sets (cf. Zadeh [34]). This kind of fuzzy valued random variable was introduced by Puri and Ralescu in [29], and was followed by many works such as strong law of large numbers and central limit theorems (cf. [5, 6, 13, 30] etc.).

Concerning set valued martingales, Hiai and Umegaki [10] have given the definitions and obtained convergence theorems. This theory is the basic foundation of the study of set valued martingale theory and its applications. Their results were used by many authors such as Ban [5, 6], Luu [22], Papageorgiou [27, 28] etc. The well cultivated results in above theory are, however, mainly on compact set valued random variables. This comes from the fact that the most typical method in the theory of set valued random variables is to embed the family of all of compact sets into a Banach space by using H. Rådström's embedding theorem [31], which is only available for the family of all of compact sets. For example, Hiai and Umegaki only got the convergence theorem for the compact set valued martingales by using embedding theorem. Although they obtained a regularity theorem, they did not get the convergence theorem for the closed set valued martingales. Since regular property does not imply convergence in Hausdorff metric for the set valued martingales, we can see that Hausdorff topology is a too strong topology for the study of set valued random variables.

In 1991, Puri and Ralescu [30] built a concept of fuzzy valued martingale and got a convergence theorem for fuzzy valued martingales. The basic route of this kind of study is to exploit the theory of set valued random variables, because fuzzy valued random variables are considered as a family of set valued random variables satisfying some additional conditions, and there are very rich mathematical properties in the theory of set valued random variables as that is mentioned above. Most of important results in this theory are, however, mainly on fuzzy valued random variables on the case where the level sets are compact. They still used the embedding method. We have to notice that many important results in that theory, such as central limit theorems and convergence theorem for fuzzy valued martingales, were obtained under Lipschitz condition, which is a very strong condition. For example, Lipschitz condition is not satisfied even for simple fuzzy valued random variables. This extra condition is only for keeping the embedding theorem to be available.

Here, we mainly develop the theory of closed convex set valued random variables, get convergence theorems for the closed convex set valued martingales and sub-, super-martingales, and then use them to study fuzzy valued random variables whose level sets are not compact in the underlying Banach space $\mathfrak{X}$.

For these purposes, we must first make up a theory on non-compact set valued random variables $F$, and then apply them to that on fuzzy valued ones. In more words, we firstly discuss the properties of Aumann integrals, conditional expectations for closed convex set valued random variables and give some sufficient conditions for Aumann integrals and relative conditional expectations to be closed, which are very necessary results to discuss sub- and super-martingales. Secondly we prove a representation theorem and then obtain convergence theorems for closed convex set valued martingales, sub- and super-martingales in the Kuratowski-Mosco sense under weaker conditions than that in former works. Then we find the relations between the theory of set valued random variables and fuzzy valued random variables so that we can extend the results to fuzzy cases.

Since the embedding theorem can not be used for the closed set valued random variables and relative fuzzy valued random variables, we must find a new method to solve the problems above mentioned. The main method here is the systematic use of the associated the set of measurable selections $S_F$ of closed set valued random variable $F$, which lies in $L^1[\Omega, \mathfrak{X}]$, the space of $\mathfrak{X}$-valued integrable random variable. This makes the objects more tractable, since $L^1[\Omega, \mathfrak{X}]$ is already a Banach space. To get convergence theorems for closed set valued martingales, sub- and super-martingales, we exploit the martingale selections. For the results of fuzzy valued random variables, we first use the cut sets of the fuzzy valued random variables and get the system of set valued random variables. Using the selection technique, we got relative results on the system of cut sets and then bring them back to the fuzzy valued cases. This is the first advantage.

Our second advantage is to make use of the Kuratowski-Mosco topology in place of Hausdorff topology, which was a main tool in most of former works. There is a rich history of Kuratowski-Mosco topology after the celebrated paper [25] (see also [26], [32, 33] e.g.). Actually, Both notions of convergence for set valued in a metric space, the Hausdorff metric convergence and the Kuratowski-Mosco convergence, are eminently useful in several areas of mathematics and applications such as optization and control, stochastic and integral geometry, mathematical economics. However, in an normed space, especially for infinite dimensional cases, the Kuratowski-Mosco convergence is more tractable than the Hausdorff one.

We should notice that Papageorgiou got the convergence theorems for closed convex set valued martingales in Kuratowski-Mosco sense in his papers. However the assumption there that the conjugate functions were uniformly lower equi-continuous is not easy to check, especially when we apply them to fuzzy valued martingales. In this paper, we succeed in getting rid of this assumption by exploiting martingale selections. This is our third advantage.

We obtain the convergence theorems for both closed convex set valued sub- and super-martingales in Kuratowski-Mosco sense, since we successfully give the sufficient conditions for the Aumann integral and relative conditional expectation to be closed. Then we extend them to the fuzzy case. There were not any results on this topic before.

Finally, we introduce a new convergence called convergence in graph sense for fuzzy valued random variables and obtain convergence theorems, since it makes the topology clearly for the space of fuzzy valued random variables.

We organize this paper as follows: in section 2, we firstly give the basic definitions and results for set valued random variables, and then we state our main theorems for set valued martingales, sub- and supermartingales. In section 3, we discuss convergences for fuzzy valued cases. In section 4, we introduce convergence in graph for fuzzy random variables, give the equivalent definitions and then obtain some convergence theorems for a sequence of fuzzy valued random variables and martingales.

The proofs of theorems in sections 2 and 3 were given in our papers [17] - [21]. The proofs of theorems in section 4 were given in our preprint paper "Convergence in graph for fuzzy valued random variables and martingales".

## 2. Convergence theorems for set valued martingales

Throughout this paper, $(\Omega, \mathcal{A}, \mu)$ is a complete, probability space. Denote by $(\mathfrak{X}, \|\cdot\|_{\mathfrak{X}})$ a real separable Banach space, and by $\mathbf{K}(\mathfrak{X})$ the family of all nonempty, closed subsets of $\mathfrak{X}$, by $\mathbf{K}_c(\mathfrak{X})$ the family of all closed, convex subsets of $\mathfrak{X}$, and $\mathbf{K}_{cc}(\mathfrak{X})$ the family of all compact, convex subsets of $\mathfrak{X}$.

Define two operations in $\mathbf{K}(\mathfrak{X})$ as follows:

$$A + B = cl\{a + b; a \in A, b \in B\}, \tag{2.1}$$

$$\lambda A = \{\lambda a; a \in A\}, \tag{2.2}$$

where $\lambda$ is a real number.

REMARK 2.1. We do not define the Minkowski addition at (2.1) in the ordinary way, because $\{a + b; a \in A, b \in B\}$ is not a closed set in general, even if $A$ and $B$ are bounded closed sets (cf. Li and Ogura [17]).

The *Hausdorff metric* on $\mathbf{K}(\mathfrak{X})$ is defined as follows:

$$d_H(A, B) = \max\{\sup_{a\in A} \inf_{b\in B} \|a - b\|_{\mathfrak{X}}, \sup_{b\in B} \inf_{a\in A} \|a - b\|_{\mathfrak{X}}\} \tag{2.3}$$

for $A, B \in \mathbf{K}(\mathfrak{X})$. But if $A$ or $B$ are unbounded, $d_H(A, B)$ may be infinite. It is well-known (cf. Kuratowski [16, p.214, p.407]) that the family of all bounded elements in $\mathbf{K}(\mathfrak{X})$ is a complete space with respect to the Hausdorff metric $d_H$, and the family of all bounded elements in $\mathbf{K}_c(\mathfrak{X})$, $\mathbf{K}_{cc}(\mathfrak{X})$ are closed subsets of this complete space. For $A \in \mathbf{K}(\mathfrak{X})$, $\|A\|_{\mathbf{K}}$ denotes the norm of $A$ defined as $\|A\|_{\mathbf{K}} = \sup_{a\in A} \|a\|_{\mathfrak{X}}$.

A *set valued random variable* $F : \Omega \to \mathbf{K}(\mathfrak{X})$ is a measurable mapping, that is, for every $B \in \mathbf{K}(\mathfrak{X})$, $F^{(-1)}(B) := \{\omega \in \Omega; F(\omega) \cap B \neq \emptyset\} \in \mathcal{A}$ (cf. Hiai and Umegaki [10], for a few equivalent definitions).

A measurable mapping $f : \Omega \to \mathfrak{X}$ is called *a measurable selection* of $F$ if $f(\omega) \in F(\omega)$ for all $\omega \in \Omega$. Denote by $L^1[\Omega, \mathfrak{X}]$ the Banach space of all measurable mappings $g : \Omega \to \mathfrak{X}$ such that the norm $\|g\|_L = \int_\Omega \|g(\omega)\|_{\mathfrak{X}} d\mu$ is finite. For a measurable set valued random variable $F$, define the set

$$S_F = \{f \in L^1[\Omega, \mathfrak{X}] : f(\omega) \in F(\omega) \quad a.e.(\mu)\}.$$

For a sub-$\sigma$-field $\mathcal{A}_0$, denote by $S_F(\mathcal{A}_0)$ the set of all $\mathcal{A}_0$-measurable mappings in $S_F$.

A set random variable $F : \Omega \to \mathbf{K}(\mathfrak{X})$ is called *integrably bounded* iff the real valued random variable $\|F(\omega)\|_{\mathbf{K}}$ is integrable. Let $L^1[\Omega, \mathcal{A}, \mu; \mathbf{K}(\mathfrak{X})]$ denote the space of all integrably bounded set random variables where two set random variables $F_1, F_2 \in L^1[\Omega, \mathcal{A}, \mu; \mathbf{K}(\mathfrak{X})]$ are considered to be identical if $F_1(\omega) = F_2(\omega)$, a.e.$(\mu)$.

The following Theorem is about the relation between a set valued random variable $F \in L^1[\Omega, \mathcal{A}, \mu; \mathbf{K}(\mathfrak{X})]$ and $S_F$ (cf. Hiai and Umegaki [10]).

THEOREM 2.1. (1) *Let $F \in L^1[\Omega, \mathcal{A}, \mu; \mathbf{K}(\mathfrak{X})]$, then*

*(a) $S_F$ is a nonempty, closed and bounded subset in $L^1[\Omega, \mathfrak{X}]$,*

*(b) there exists a sequence $\{f_n\}$ contained in $S_F$ such that $F(\omega) = cl\{f_n(\omega)\}$ for all $\omega \in \Omega$,*

*(c) $S_F$ is convex if and only if $F(\omega)$ is convex for almost every $\omega \in \Omega$.*

(2) *Let $M$ be a nonempty closed subset of $L^1[\Omega; \mathfrak{X}]$. Then there exists a set random variable $F \in L^1[\Omega, \mathcal{A}, \mu; \mathbf{K}(\mathfrak{X})]$ such that $M = S_F$ if and only if $M$ is nonempty, closed, bounded and decomposable, i.e. $h = I_A f + I_{\Omega\setminus A} g$ belongs to $M$ for all $A \in \mathcal{A}$ and $f, g \in M$.*

Define the subset of $L^1[\Omega, \mathcal{A}, \mu; \mathbf{K}(\mathfrak{X})]$ as follows:

$$L^1[\Omega, \mathcal{A}, \mu; \mathbf{B}] = \{F \in L^1[\Omega, \mathcal{A}, \mu; \mathbf{K}(\mathfrak{X})] : F(\omega) \in \mathbf{B},\ a.e.(\mu)\},$$

and if $\mathcal{A}_0$ is a sub-$\sigma$-field of $\mathcal{A}$, we denote

$$L^1[\Omega, \mathcal{A}_0, \mu; \mathbf{B}] = \{F \in L^1[\Omega, \mathcal{A}, \mu; \mathbf{B}] : F \text{ is } \mathcal{A}_0 - \text{measurable}\},$$

where $\mathbf{B}$ is a subset of $\mathbf{K}(\mathfrak{X})$.

For each $F \in L^1[\Omega, \mathcal{A}, \mu; \mathbf{K}(\mathfrak{X})]$, Aumann integral (cf. Aumann [4]) of $F$ is

$$\int_\Omega F d\mu = \left\{ \int_\Omega f d\mu : f \in S_F \right\}, \tag{2.4}$$

where $\int_\Omega f d\mu$ is the usual Bochner integral. Define $\int_A F d\mu = \{\int_A f d\mu : f \in S_F\}$, for $A \in \mathcal{A}$.

If $F$ is a compact convex valued random variable, the Aumann integral and Debreu integral are equivalent (cf. Klein and Thompson [15]).

Let $\mathcal{A}_0$ be a sub-$\sigma$-field of $\mathcal{A}$. The *conditional expectation* $E[F|\mathcal{A}_0]$ of an $F \in L^1[\Omega, \mathcal{A}, \mu;\ \mathbf{K}_c(\mathfrak{X})]$ is determined as a $\mathcal{A}_0$-measurable element of

$L^1[\Omega, \mathcal{A}, \mu; \mathbf{K}_c(\mathfrak{X})]$ by

$$S_{E[F|\mathcal{A}_0]}(\mathcal{A}_0) = cl\{E(f|\mathcal{A}_0) : f \in S_F\}, \tag{2.5}$$

where the closure is taken in the $L^1[\Omega, \mathfrak{X}]$ (cf.Hiai and Umegaki [10]). If $\mathfrak{X}^*$ is separable, this is equivalent to the formula

$$cl \int_A F d\mu = cl \int_A E[F|\mathcal{A}_0] d\mu \quad \text{for} \quad A \in \mathcal{A}_0. \tag{2.6}$$

REMARK 2.2. The integral $\int_\Omega F d\mu$ of $F$ and the set $\{E(f|\mathcal{A}_0) : f \in S_F\}$ are not necessary closed in general (cf. counterexample in Li and Ogura [21]).

In the following, we will give some sufficient conditions for the closedness of Aumann integral of compact convex set value random variable and of closed convex set valued random variable $F$, and the set $\{E(f|\mathcal{A}_0) : f \in S_F\}$ concerning the conditional expectations $E[F|\mathcal{A}_0]$ of $F$ (cf. Li and Ogura [21]). They are important for our proof of the set valued submartingale convergence theorem.

A Banach space $\mathfrak{X}$ is said to have the *Radon-Nikodym property* (*RNP*) with respect to a finite measure space $(\Omega, \mathcal{A}, \mu)$ if for each $\mu$-continuous $\mathfrak{X}$-valued measure $m : \mathcal{A} \to \mathfrak{X}$ of bounded variation, there exists an integrable mapping $f : \Omega \to \mathfrak{X}$ such that $m(A) = \int\limits_A f d\mu$ for all $A \in \mathcal{A}$. It is known that every separable dual space and every reflexive space has the RNP (cf. Chatterji [7]).

THEOREM 2.2. (1) *If $\mathfrak{X}$ has the RNP and $F \in L^1[\Omega, \mathcal{A}, \mu; K_{cc}(\mathfrak{X})]$, then the set*

$$\int_\Omega F d\mu = \left\{ \int_\Omega f d\mu : f \in S_F \right\}$$

*is closed in $\mathfrak{X}$.*

(2) *If $\mathfrak{X}$ has the RNP, $F \in L^1[\Omega, \mathcal{A}, \mu; K_{cc}(\mathfrak{X})]$ and $\mathcal{A}_0$ is countably generated, that is $\mathcal{A}_0 = \sigma(\mathfrak{A})$ for a countable sub-class $\mathfrak{A}$ of $\mathcal{A}$, then the set*

$$\{E(f|\mathcal{A}_0) : f \in S_F\}$$

*is closed in $L^1[\Omega, \mathfrak{X}]$.*

THEOREM 2.3. (1) *If $\mathfrak{X}$is a reflexive Banach space, $F \in L^1[\Omega, \mathcal{A}, \mu; K_c(\mathfrak{X})]$. Then the set*

$$\int_\Omega F d\mu = \left\{ \int_\Omega f d\mu : f \in S_F \right\}$$

*is closed.*

(2) *If $\mathfrak{X}$ is a reflexive Banach space, $F \in L^1[\Omega, \mathcal{A}, \mu; K_c(\mathfrak{X})]$ and $\mathcal{A}_0 = \sigma(\mathfrak{A})$ is countably generated. Then the set*

$$\{E(f|\mathcal{A}_0) : f \in S_F\}$$

*is closed in $L^1[\Omega, \mathfrak{X}]$.*

In the following we discuss set valued martingales and smartingales.

Let $\{\mathcal{A}_n : n \in \mathbb{N}\}$ be a family of complete sub-$\sigma$-fields of $\mathcal{A}$ such that $\mathcal{A}_n \subset \mathcal{A}_{n+1}$ if $n \in \mathbb{N}$, and $\mathcal{A}_\infty$ the $\sigma$-field generated by $\bigcup_{n=1}^{\infty} \mathcal{A}_n$.

A system $\{F_n, \mathcal{A}_n : n \in \mathbb{N}\}$ is called a *set valued martingale* iff
(1) $F_n \in L^1[\Omega, \mathcal{A}_n, \mu : \mathbf{K}_c(\mathfrak{X})], \quad n \in \mathbb{N}$,
(2) $F_n = E[F_{n+1}|\mathcal{A}_n], \quad n \in \mathbb{N},\ a.e.(\mu)$.

A sequence of set random variables $\{F_n, n \geq 1\}$ is called *uniformly integrable* iff

$$\lim_{\lambda \uparrow \infty} \sup_{n \in \mathbb{N}} \int_{\{\|F_n(\omega)\|_{\mathbf{K}} > \lambda\}} \|F_n(\omega)\|_{\mathbf{K}} d\mu = 0.$$

By using embedding theorem, Hiai and Umegaki [10] obtained the following results.

(1) Let $\{F_n, \mathcal{A}_n; n \geq 1\}$ be a compact convex valued martingale such that $F_n = E[F|\mathcal{A}_n], n \geq 1$, where $F \in L^1[\Omega, \mathcal{A}, \mu; \mathbf{K}_{cc}(\mathfrak{X})]$ (or $F \in \mathbf{L}^1[\Omega, \mathcal{A}, \mu; \mathbf{K}_c(\mathfrak{X})]$ and $\mathfrak{X}$ is reflexive). Then

$$d_H(F_n, F_\infty) \to 0,$$

where $F_\infty = E[F|\mathcal{A}_\infty]$.

(2) Let $\{F_n, \mathcal{A}_n; n \leq -1\}$ be a set valued martingale such that $F_n = E[F|\mathcal{A}_n], n \leq -1$, where $F_{-1} \in L^1[\Omega, \mathcal{A}, \mu; \mathbf{K}_{cc}(\mathfrak{X})] (or F_{-1} \in \mathbf{L}^1[\Omega, \mathcal{A}, \mu; \mathbf{K}_c(\mathfrak{X})]$ and $\mathfrak{X}$ is reflexive). Then

$$d_H(F_n, F_{-\infty}) \to 0$$

as $n \to -\infty$, where $F_{-\infty} = E[F|\mathcal{A}_{-\infty}]$, and $\mathbf{L}^1[\Omega, \mathcal{A}, \mu; \mathbf{K}_c(\mathfrak{X})]$ denotes the closure of the set of all simple functions in $L^1[\Omega, \mathcal{A}, \mu; \mathbf{K}_c(\mathfrak{X})]$ (cf. Hiai and Umegaki [10]).

REMARK 2.3. The assertions above are not valid for $F \in L^1[\Omega, \mathcal{A}, \mu; \mathbf{K}_c(\mathfrak{X})]$, in general.

A set valued martingale $\{F_n, \mathcal{A}_n; n \in \mathbb{N}\}$ is called regular if there exists an $F \in L^1[\Omega, \mathcal{A}, \mu; \mathbf{K}_c(\mathfrak{X})]$ such that $F_n = E[F|\mathcal{A}_n]$ for all $n$.

Now we give the representation theorem for closed convex set valued martingales by exploiting the method of martingale selections (cf. Li and Ogura [17]).

THEOREM 2.4. *Assume that $\mathfrak{X}$ has the RNP and $\{F_n, \mathcal{A}_n; n \geq 1\}$ is a set valued martingale in $L^1[\Omega, \mathcal{A}, \mu; \mathbf{K}_c(\mathfrak{X})]$. If $\{F_n\}$ is uniformly integrable, then there exists an $F_\infty \in L^1[\Omega, \mathcal{A}_\infty, \mu; \mathbf{K}_c(\mathfrak{X})]$ such that*

$$F_n = E[F_\infty|\mathcal{A}_n] \quad a.e.(\mu), \quad n \in \mathbb{N}.$$

REMARK 2.4. The $\mathfrak{X}$-valued martingales with regular property, as we know, imply convergence in almost everywhere with respect to $\mu$. But in the case of set valued martingales , regular property does not imply convergence in the Hausdorff metric (cf. Li and Ogura[20, Example 4.2]). We can see from this fact that the Hausdorff topology is too strong for the study of set valued random variables. Now we introduce the Kuratowski-Mosco convergence.

Let $\{B_n\}_{n\in\mathbb{N}}$ is a sequence of closed sets of $\mathfrak{X}$. We will say that $B_n$ converges to $B$ *in the Kuratowski-Mosco sense* [25, 32] (denoted by $B_n \xrightarrow{K-M} B$) iff

$$w\text{-}\limsup_{n\to\infty} B_n = B = s\text{-}\liminf_{n\to\infty} B_n,$$

where

$$w\text{-}\limsup_{n\to\infty} B_n = \{x = w\text{-}\lim x_m : x_m \in B_m, m \in M \subset \mathbb{N}\}$$

and

$$s\text{-}\liminf_{n\to\infty} B_n = \{x = s\text{-}\lim x_n : x_n \in B_n, n \in \mathbb{N}\}.$$

In [20], we proved the following theorem by using martingale selection method. Since the Kuratowski-Mosco convergence and the Hausdorff convergence are equivalent when $\mathfrak{X}$ is a finite dimensional space, this result also generalizes the result of Hiai and Umegaki above mentioned.

THEOREM 2.5. *Assume that $\mathfrak{X}$ is a Banach space satisfying the RNP with the separable dual $\mathfrak{X}^*$. Then, for every uniformly integrably set valued martingale $\{F_n, \mathcal{A}_n : n \in \mathbb{N}\}$, there exists a unique $F_\infty \in L^1[\Omega, \mathcal{A}, \mu; \mathbf{K}_c(\mathfrak{X})]$ such that $\{F_n, \mathcal{A}_n : n \in \mathbb{N} \cup \infty\}$ is a martingale and $F_n \xrightarrow{K-M} F_\infty$, a.e.$(\mu)$.*

A system $\{F_n, \mathcal{A}_n : n \in \mathbb{N}\}$ is called a *set valued submartingale* (cf. Li and Ogura [21]) iff it satisfies the following two conditions
(1) for each $n \in \mathbb{N}$, $F_n \in L^1[\Omega, \mathcal{A}_n, \mu; \mathbf{K}_c(\mathfrak{X})]$,
(2) for each $n \in \mathbb{N}$, $S_{F_n}(\mathcal{A}_n) \subset \{E[f|\mathcal{A}_n] : f \in S_{F_{n+1}}(\mathcal{A}_{n+1})\}$.

REMARK 2.5. 1) Condition (2) is equivalent to
(3) For each $n, m \in \mathbb{N}$ with $n \le m$, $S_{F_n}(\mathcal{A}_n) \subset \{E[f|\mathcal{A}_n] : f \in S_{F_m}(\mathcal{A}_m)\}$.
2) Condition (2) is stronger than the notion of submartingale in [10], where they use the condition

$$F_n(\omega) \subset E[F_{n+1}|\mathcal{A}_n](\omega), \quad a.e.(\mu),$$

that is,
(2′) $S_{F_n}(\mathcal{A}_n) \subset cl\{E(f|\mathcal{A}_n) : f \in S_{F_{n+1}}(\mathcal{A}_{n+1})\}$.
3) Condition (2) is equivalent to formula (2′) if

$$\hat{S} = \{E(f|\mathcal{A}_n) : f \in S_{F_{n+1}}(\mathcal{A}_{n+1})\}$$

is closed. Concerning its closedness, we have given the sufficient conditions in Theorems 2.2 and 2.3.

Now we give a convergence theorem for a set valued submartingale as follows.

THEOREM 2.6. *Assume that $\mathfrak{X}$ is a Banach space satisfying the RNP with the separable dual $\mathfrak{X}^*$. Then, for every uniformly integrably set valued submartingale $\{F_n, \mathcal{A}_n : n \in \mathbb{N}\}$, there exists a unique $F_\infty \in L^1[\Omega, \mathcal{A}_\infty, \mu; \mathbf{K}_c(\mathfrak{X})]$ such that $F_n \xrightarrow{K-M} F_\infty$, a.e.$(\mu)$.*

To prove this theorem, we first get a selection representation lemma for the closed convex set valued submartingale (cf. Li and Ogura [21], Lemma 5.1).

A system $\{F_n, \mathcal{A}_n : n \in \mathbb{N}\}$ is called a *set valued supermartingale* iff it satisfies the following two conditions

(1) for each $n \in \mathbb{N}$, $F_n \in L^1[\Omega, \mathcal{A}_n, \mu; \mathbf{K}_c(\mathfrak{X})]$,
(2) for each $n \in \mathbb{N}$, $E[F_{n+1}|\mathcal{A}_n](\omega) \subset F_n(\omega)$, a.e.$(\mu)$.

REMARK 2.6. Condition (2) is equivalent to $E[F_m|\mathcal{A}_n](\omega) \subset F_n(\omega)$, a.e.$(\mu)$ for each $n, m \in \mathbb{N}$ with $n \leq m$.

The following is the convergence theorem for set valued super-martingales in Kuratowski-Mosco sense.

THEOREM 2.7. *Assume that $\mathfrak{X}$ is a Banach space satisfying the RNP with the separable dual $\mathfrak{X}^*$, $\{F_n, \mathcal{A}_n : n \in \mathbb{N}\}$ is uniformly integrably set valued supermartingale and*

$$M = \bigcap_{n=1}^{\infty} \{f \in L^1[\Omega, \mathcal{A}_\infty, \mu; \mathfrak{X}] : E(f|\mathcal{A}_n) \in S_{F_n}(\mathcal{A}_n)\} \tag{2.5}$$

*is a nonempty set. Then there exists a unique $F_\infty \in L^1[\Omega, \mathcal{A}_\infty, \mu; \mathbf{K}_c(\mathfrak{X})]$ such that $F_n \xrightarrow{K-M} F_\infty$, a.e.$(\mu)$.*

## 3. Convergence theorems for fuzzy valued martingales

Let $\mathbf{F}(\mathfrak{X})$ denote the family of all fuzzy sets $\nu : \mathfrak{X} \to [0,1]$ such that its each $\alpha$-cut set $\nu_\alpha = \{x \in \mathfrak{X} : \nu(x) \geq \alpha\}$ is nonempty, closed subset of $\mathfrak{X}$, for every $0 < \alpha \leq 1$ with $\nu_0 = cl\{x \in \mathfrak{X} : \nu(x) > 0\}$ is bounded subset of $\mathfrak{X}$. For two fuzzy sets $\nu^1, \nu^2 \in \mathbf{F}(\mathfrak{X}), \nu^1 \preceq \nu^2$ iff $\nu^1_\alpha \subset \nu^2_\alpha$ for every $\alpha \in (0,1]$. Obviously, $(\mathbf{F}(\mathfrak{X}), \preceq)$ is a partial ordered set.

The Hausdorff metric in $\mathbf{K}(\mathfrak{X})$ can be extended to $\mathbf{F}(\mathfrak{X})$ by

$$d(\nu^1, \nu^2) = \sup_{0<\alpha\leq 1} d_H(\nu^1_\alpha, \nu^2_\alpha). \tag{3.1}$$

Then we can prove similarly as Puri and Ralescu in [29] that $(\mathbf{F}(\mathfrak{X}), d)$ is a complete metric space.

A *fuzzy valued random variable*(cf. [17]) or *fuzzy random variable* is a function $X : \Omega \to \mathbf{F}(\mathfrak{X})$, such that $X_\alpha(\omega) = \{x \in \mathfrak{X} : X(\omega)(x) \geq \alpha\}$ is a set random variable for every $\alpha \in (0,1]$.

A fuzzy random variable $X$ is called *integrably bounded* if for every $\alpha \in (0,1]$, the real valued random variable $\|X_\alpha(\omega)\|_K$ is integrable. A fuzzy random variable $X$ is called *strongly integrably bounded* if there exists a $\mu$-integrable function $f : \Omega \to \mathbb{R}$ such that $\|X_\alpha(\omega)\|_K \leq f(\omega)$ for almost every $\omega \in \Omega$ and for all $\alpha \in (0,1]$.

Let $L^1[\Omega,\mathcal{A},\mu;\mathbf{F}(\mathfrak{X})]$ be the set of all integrably bounded fuzzy random variables and $L^1[\Omega,\mathcal{A},\mu;\mathbf{F}_c(\mathfrak{X})]$ denote the set of all fuzzy random variables $X\in L^1[\Omega,\mathcal{A},\mu;\mathbf{F}(\mathfrak{X})]$ such that $X_\alpha\in L^1[\Omega,\mathcal{A},\mu;\mathbf{K}_c(\mathfrak{X})]$ for all $\alpha\in(0,1]$. Similarly we have the notation $L^1[\Omega,\mathcal{A},\mu;\mathbf{F}_{cc}(\mathfrak{X})]$

For each $X,Y\in L^1[\Omega,\mathcal{A},\mu;\mathbf{F}(\mathfrak{X})]$, we can define the metric function $\mathbf{D}_\infty : L^1[\Omega,\mathcal{A},\mu;\mathbf{F}(\mathfrak{X})]\times L^1[\Omega,\mathcal{A},\mu;\mathbf{F}(\mathfrak{X})]\to\mathbb{R}$ by

$$\mathbf{D}_\infty(X,Y)=\sup_{0<\alpha\leq 1}\Delta(X_\alpha,Y_\alpha), \tag{3.2}$$

where $\Delta(X_\alpha,Y_\alpha)=\int_\Omega d_H(X_\alpha,Y_\alpha)d\mu$.

Two fuzzy random variables $X,Y\in L^1[\Omega,\mathcal{A},\mu;\mathbf{F}(\mathfrak{X})]$ are considered to be identical if $\mathbf{D}_\infty(X,Y)=0$. We also can define the metric function $\mathbf{D}_1 : L^1[\Omega,\mathcal{A},\mu;\mathbf{F}(\mathfrak{X})]\times L^1[\Omega,\mathcal{A},\mu;\mathbf{F}(\mathfrak{X})]\to\mathbb{R}$ by

$$\mathbf{D}_1(X,Y)=\int_0^1\Delta(X_\alpha,Y_\alpha)d\lambda(\alpha),$$

where $\lambda$ is the Lebegue measure. Then both of $(L^1[\Omega,\mathcal{A},\mu;\mathbf{F}(\mathfrak{X})],\mathbf{D}_\infty)$ and $(L^1[\Omega,\mathcal{A},\mu;\mathbf{F}(\mathfrak{X})],\mathbf{D}_1)$ are complete metric spaces, and $L^1[\Omega,\mathcal{A},\mu;\mathbf{F}_c(\mathfrak{X})]\supset L^1[\Omega,\mathcal{A},\mu;\mathbf{F}_{cc}(\mathfrak{X})]$ are the closed subsets of $L^1[\Omega,\mathcal{A},\mu;\mathbf{F}(\mathfrak{X})]$ (cf. Li and Ogura [17]).

In the study of fuzzy random variables, the typical method is to use set valued theory, since all of the cut sets of a fuzzy random variable is a family of set valued random variables satisfying additional properties. The following is the basic Lemma.

LEMMA 3.1. *Let $\{S_\alpha : \alpha\in[0,1]\}$ be a family of subsets of $L^1[\Omega,\mathfrak{X}]$, $S_\alpha$ be nonempty, closed, bounded and decomposable for every $\alpha>0$ and satisfy the following conditions:*
(1) $\alpha\leq\beta\Longrightarrow S_\beta\subset S_\alpha$,

(2) $\alpha_1\leq\alpha_2\leq\cdots\leq\alpha_n\leq\cdots,\ \lim_{n\to\infty}\alpha_n=\alpha\Longrightarrow S_\alpha=\bigcap_{n=1}^{\infty}S_{\alpha_n}$.

*Then, there exists a unique $Y\in L^1[\Omega,\mathcal{A},\mu;\mathbf{F}(\mathfrak{X})]$ such that for every $\alpha$,*

$$S_\alpha=\{f\in L^1[\Omega,\mathfrak{X}];f(\omega)\in Y_\alpha(\omega),\ \text{a.e.}(\mu)\}. \tag{3.3}$$

*If $\{S_\alpha : \alpha\in[0,1]\}$ have the above conditions* (1) - (2) *and*
(3) *for any $\alpha\in(0,1]$, $S_\alpha$ is convex.*

*Then, there exists a unique $Y\in L^1[\Omega,\mathcal{A},\mu;\mathbf{F}_c(\mathfrak{X})]$ which satisfies* (3.3)

The *expected value* of any fuzzy random variable $X$, denoted by $E[X]$, is a fuzzy set such that, for every $\alpha \in (0,1]$,

$$(E[X])_\alpha = cl \int_\Omega X_\alpha d\mu = cl\{E(f); f \in S_{X_\alpha}\}, \tag{3.4}$$

where the closure is taken in $\mathfrak{X}$. From the existence theorem, we can get an equivalent definition as follows,

$$E(X)(x) = \sup\{\alpha \in [0,1] : x \in clE(X_\alpha)\}.$$

From now on, we limit fuzzy random variable in $L^1[\Omega, \mathcal{A}_0, \mu; \mathbf{F}_c(\mathfrak{X})]$ only for simplicity of notations.

The *conditional expectation* of fuzzy valued random variable $X$, denoted by $E[X|\mathcal{A}_0]$, is a fuzzy random variable satisfying the following two conditions.

$$E[X|\mathcal{A}_0] \in L^1[\Omega, \mathcal{A}_0, \mu; \mathbf{F}_c(\mathfrak{X})], \tag{3.5}$$

$$\int_A E[X|\mathcal{A}_0] d\mu = \int_A X d\mu \quad \text{for every} \quad A \in \mathcal{A}_0. \tag{3.6}$$

From [17], the conditional expectation $E[X|\mathcal{A}_0]$ uniquely exists.

$\{X^n, \mathcal{A}_n; n \in \mathbb{N}\}$ is called a *fuzzy valued martingale* or *fuzzy martingale* iff

(1) $X^n \in L^1[\Omega, \mathcal{A}_n, \mu; \mathbf{F}_c(\mathfrak{X})]$, for all $n \in \mathbb{N}$,
(2) $X^n = E[X^{n+1}|\mathcal{A}_n]$, for all $n \in \mathbb{N}$.

A sequence of fuzzy random variables $\{X^n, n \in \mathbb{N}\}$ is called *uniformly integrable* if

$$\lim_{\lambda\to\infty} \sup_{n\in\mathbb{N}} \int_{\{|X^n(\omega)|>\lambda\}} |X^n(\omega)| d\mu = 0,$$

where $|X^n(\omega)| = d(X^n(\omega), I_{\{0\}})$, and $I_{\{0\}}$ is the indicator function.

THEOREM 3.2. *Let $\{X^n, \mathcal{A}_n; n \geq 1\}$ be a fuzzy valued martingale such that $X^n = E[X|\mathcal{A}_n], n \geq 1$, where $X \in L^1[\Omega, \mathcal{A}, \mu; \mathbf{F}_{cc}(\mathfrak{X})]$ (or $X \in \mathbf{L}^1[\Omega, \mathcal{A}, \mu; \mathbf{F}_c(\mathfrak{X})]$ and $\mathfrak{X}$ is reflexive) and $X$ is strongly integrably bounded. Then*

$$\mathbf{D}_1(X^n, X^\infty) \to 0, \tag{3.7}$$

*where $X^\infty = E[X|\mathcal{A}_\infty]$ and $\mathbf{L}^1[\Omega, \mathcal{A}, \mu; \mathbf{F}_c(\mathfrak{X})]$ is the closure of all simply fuzzy random variables* (cf. [17]).

THEOREM 3.3. *Let $\{X^n, \mathcal{A}_n; n \leq -1\}$ be a fuzzy valued martingale such that $X^n = E[X^{-1}|\mathcal{A}_n], n \leq -1$, where $X^{-1} \in L^1[\Omega, \mathcal{A}, \mu; \mathbf{F}_{cc}(\mathfrak{X})]$ (or $X^{-1} \in \mathbf{L}^1[\Omega, \mathcal{A}, \mu; \mathbf{F}_c(\mathfrak{X})]$ and $\mathfrak{X}$ is reflexive) and $X^{-1}$ is strongly integrably bounded. Then*

$$\mathbf{D}_1(X^n, X^{-\infty}) \to 0 \tag{3.8}$$

*as $n \to -\infty$, where $X^{-\infty} = E[X|\mathcal{A}_{-\infty}]$.*

THEOREM 3.4. *Let $\mathfrak{X}^*$ be separable, $\{X^n, \mathcal{A}_n; n \geq 1\}$ be a fuzzy valued martingale in $L^1[\Omega, \mathcal{A}, \mu; \mathbf{F}_c(\mathfrak{X})]$. If $\tau_1 \leq \tau_2 \leq ... \leq \tau_n \leq ...$ is a sequence of stopping times, and $\liminf_{N\to\infty} \int_{\{\tau_n \geq N\}} |X^n| d\mu = 0$ for every $n \in N$. Then $\{X^{\tau_n}, \mathcal{A}_{\tau_n}; n \geq 1\}$ forms a martingale. That is, for every $n \geq 1$,*

$$E[X^{\tau_{n+1}}|\mathcal{A}_{\tau_n}] = X^{\tau_n}.$$

*In the above $|X^{\tau_n}(\omega)| = d(X^{\tau_n}(\omega), I_{\{0\}})$, where $I_{\{0\}}$ is the indicator function.*

In [20], we defined the Kuratowski-Mosco convergence for the fuzzy valued random martingales by using every cut set of them to be convergent in the Kuratowski-Mosco sense as we defined in section 2. We proved the following theorem.

THEOREM 3.5. *Assume that $\mathfrak{X}$ is a Banach space with the separable dual $\mathfrak{X}^*$ satisfying RNP. Then, for every uniformly integrable fuzzy valued martingale $\{X^n, \mathcal{A}_n; n \in \mathbb{N}\}$, there exists a unique $X^\infty \in L^1[\Omega, \mathcal{A}_\infty, \mu; \mathbf{F}_c(\mathfrak{X})]$ such that $\{X^n, \mathcal{A}_n; n \in (\mathbb{N} \cup \infty)\}$ is a fuzzy valued martingale, a.e. for each $n \in \mathbb{N}$, $X^n = E[X^\infty|\mathcal{A}_n]$, and $X^n \xrightarrow{K-M} X^\infty$.*

A system $\{X^n, \mathcal{A}_n; n \in \mathbb{N}\}$ is called a *fuzzy valued submartingale* (resp. *fuzzy valued supermartingale* ) iff

(1) $X^n \in L^1[\Omega, \mathcal{A}_n, \mu; \mathbf{F}_c(\mathfrak{X})]$, for all $n \in \mathbb{N}$,

(2) $X^n \preceq E[X^{n+1}|\mathcal{A}_n]$, for $n \in \mathbb{N}$ (resp. $\succeq$).

By using the same method as in Theorem 3.5, we can get convergence theorems for submartingales and supermartingales. However, it is a little difficult to understand the topology of the space of fuzzy valued random variables, induced by the family of set valued random variables in the Kuratowski-Mosco convergence.

## 4. Graph convergence for sequences of fuzzy valued random variables and martingales

In this section, we introduce convergence in graph for fuzzy random variables.

Let $\nu \in \mathbf{F}(\mathfrak{X})$, write

$$Gr\ \nu = \{(x, y) \in \mathfrak{X} \times [0,1], \nu(x) \geq y\}.$$

It is clear that $Gr\ \nu$ denotes the domain between the curve of $\nu$ and $\mathfrak{X}$-axis if $\mathfrak{X}$ is $\mathbb{R}$. We call it *the graph of* $\nu$. Since all the cut set of $\nu$ are nonempty closed set, the graph of $\nu$ is a closed set of space $\mathfrak{X} \times [0,1]$.

For $\nu_n$, $\nu \in \mathbf{F}(\mathfrak{X})$, $\nu_n$ is called to converge to $\nu$ *in graph* (denoted by $\nu_n \xrightarrow{Gr} \nu$) iff $Gr\ \nu_n$ converges to $Gr\ \nu$ in $\mathfrak{X} \times [0,1]$ in the Kuratowski-Mosco sense.

REMARK 4.1. If, for any $\alpha \in [0,1]$, $(\nu_n)_\alpha \xrightarrow{K-M} (\nu)_\alpha$, we can prove easily that $\nu_n \xrightarrow{Gr} \nu$. But the opposite is not correct. We can see it from the following example.

EXAMPLE 4.1. Let $\mathfrak{X} = \mathbb{R}^1$, $a < b < c$ and

$$\nu(x) = \left\{ \begin{array}{ll} 0, & x < a, x > c, \\ 1/2, & a \leq x < b, \\ 1, & b \leq x \leq c, \end{array} \right\}$$

and

$$\nu_n(x) = \left\{ \begin{array}{ll} 0, & x < a, x > c, \\ 1/2 - 1/2n, & a \leq x < b, \\ 1, & b \leq x \leq c. \end{array} \right\}.$$

Then $\nu_n \xrightarrow{Gr} \nu$, but $(\nu_n)_{1/2}$ does not converge to $\nu_{1/2}$ in the Kuratowski-Mosco sense.

THEOREM 4.2. *Let $\nu_n$, $\nu \in \mathbf{F}(\mathfrak{X})$, then $\nu_n \xrightarrow{Gr} \nu$ iff the following two conditions are satisfied,*

(1) *for any $x \in \mathfrak{X}$, there exists a sequence $\{x_n, n \in \mathbb{N}\}$ of $\mathfrak{X}$ converging to $x$ in strong topology of $X$ such that*

$$\liminf_{n \to \infty} \nu_n(x_n) \geq \nu(x),$$

(2) *for any given subsequence* $\{\nu_{n_k}\}$ *of* $\{\nu_n\}$ *and any sequence* $\{x_{n_k}\}$ *which converges to* $x$ *in the weakly topology of* $\mathfrak{X}$, *we have*

$$\limsup_{k\to\infty} \nu_{n_k}(x_{n_k}) \leq \nu(x).$$

In fact, we can prove that (1) is equivalent to

$$Gr\ \nu \subset s\text{-}\liminf_{n\to\infty} Gr\ \nu_n, \qquad \text{in } \mathfrak{X}\times[0,1],$$

and (2) is equivalent to

$$w\text{-}\limsup_{n\to\infty} Gr\ \nu_n \subset Gr\ \nu, \qquad \text{in } \mathfrak{X}\times[0,1].$$

By using Theorem 4.2, we can prove the following theorem.

A sequence of fuzzy random variables $\{X^n : n \in \mathbb{N}\}$ is called *uniformly integrably bounded* iff there exists an $\mu$-integrable function $f : \Omega \to \mathbb{R}$ such that $\|X^n_\alpha(\omega)\|_K \leq f(\omega)$ for almost every $\omega \in \Omega$, for all $\alpha \in (0,1]$ and all $n \in \mathbb{N}$.

THEOREM 4.3. *Assume that a sequence of fuzzy random variables* $\{X^n : n \in \mathbb{N}\}$ *is uniformly integrably bounded. Then, if* $\{X^n : n \in \mathbb{N}\}$ *converges to an integrably bounded fuzzy random variable* $X$ *in graph, we have* $E[X^n] \xrightarrow{Gr} E[X]$. *Furthermore, if* $\mathcal{A}_0$ *is a sub-*$\sigma$*-filed, then* $E[X^n|\mathcal{A}_0] \xrightarrow{Gr} E[X|\mathcal{A}_0]$.

Now we give the following convergence theorems for fuzzy valued martingale, sub- and super-martingales.

THEOREM 4.4. *Assume that* $\mathfrak{X}$ *is a reflexive Banach space. Then, for every uniformly integrably fuzzy valued martingales or submartingale* $\{X^n, \mathcal{A}_n; n \in \mathbb{N}\}$, *there exists a unique* $X^\infty \in L^1[\Omega, \mathcal{A}_\infty, \mu; \mathbf{F}_c(\mathfrak{X})]$ *such that* $X^n \xrightarrow{Gr} X^\infty$ *a.e.*$(\mu)$.

THEOREM 4.5. *Assume that* $\mathfrak{X}$ *is a reflexive Banach space,* $\{X^n, \mathcal{A}_n; n \in \mathbb{N}\}$ *is uniformly integrably fuzzy valued supermartingale and for each* $\alpha \in (0,1]$,

$$M_\alpha = \bigcap_{n=1}^{\infty} \{f \in L^1[\Omega, \mathcal{A}_\infty, \mu; \mathfrak{X}] : E(f|\mathcal{A}_n) \in S_{X^n_\alpha}(\mathcal{A}_n)\}$$

*is a nonempty set. Then there exists a unique* $X^\infty \in L^1[\Omega, \mathcal{A}_\infty, \mu; \mathbf{F}_c(\mathfrak{X})]$ *such that* $X^n \xrightarrow{Gr} X^\infty$ *a.e..*

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