J. Korean Math. Soc. **35** (1998), No. 3, pp. 741–753

ON ALMOST SURE REPRESENTATIONS FOR LONG MEMORY SEQUENCES

HWAI-CHUNG HO

ABSTRACT. Let $G(\cdot)$ be a Borel function applied to a stationary long memory sequence $\{X_i\}$ of standard Gaussian random variables. Focusing on the process $\{G(X_i)\}$, the present paper establishes the almost sure representation for the empirical quantile process, that is, Bahadur's representation, and for the empirical process with respect to sample mean. Statistical applications of the representations are also addressed.

1. Introduction

Let $\{X_i, i \in Z\}$ be a stationary sequence of standard Gaussian random variables with covariance function satisfying

$$r(n) = EX_1X_{1+n} = |n|^{-\alpha}L(n),\ 0 < \alpha < 1, \tag{1.1}$$

where the function $L(\cdot)$ varies slowly at infinity. Stationary sequences with covariance function as specified in (1.1) is usually said to exhibit long memory or long-range dependence to reflect the fact $\sum_n r(n) = \infty$. As a result of (1.1), $\mathrm{var}(\sum_{i=1}^n X_i) = O(n^{2-2\alpha}L(n))$, which is the main feature often used to distinguish the long-memory processes from the traditional short-memory processes such as ARMA and Markov processes. A detailed account of the development of long-memory processes, both in theory and application, can be found in Künsch (1986), Robinson (1994) and Beran (1994). To allow more flexibility on distribution, we focus on the sequence modeled by $Y_i = G(X_i), i \in Z$ for some Borel function $G(\cdot)$. Denoted by $F(\cdot)$ and $f(\cdot)$, respectively, the cumulative distribution function and the density function of Y_i, and let $Q(y) = F^{-1}(y)$ be

Received February 24, 1998.

1991 Mathematics Subject Classification: primary 60F15; secondary 60G10, 60G15.

Key words and phrases: long-range dependence, empirical process, Bahadur's representation, trimmed mean, signed test.

the quantile function. Suppose the data set is $\{Y_i, \cdots, Y_n\}$. Denote by $Y_{n:i}$ the i-th order statistics of $\{Y_i, \cdots, Y_n\}$. Define for all $x \in \mathcal{R}$ and $0 < y \leq 1$

$$\begin{aligned} F_n(x) &= n^{-1}\sum_{i=1}^{n} I\{Y_i \leq x\}, \\ Q_n(y) &= \inf\{x : F_n(x) \geq y\} \\ &= Y_{n:k}, \text{ if } (k-1)/n < y \leq k/n, \ k = 1, \cdots, n. \end{aligned}$$

Being regarded as the building block in the modern theory of empirical quantile process, the Bahadur representations [Bahadur, 1966] that describe the almost sure relationship between the empirical quantile distribution and the empirical distribution has been extensively studied for decades [see, Csörgő, 1983, for more references] and extended from iid case to the setting of various types of weakly dependent stationary processes, for example, ϕ-mixing sequences [Sen, 1972], and linear processes [Hesse, 1990]. Another important type of representation which is analogous to Bahadur's representation arises from test for symmetry. It is an almost sure expansion for the empirical distribution with respect to sample mean. This paper aims to establish both representations for the stationary long-memory process$\{Y_i\}$ which is known not satisfying the strongly mixing property. The two representations for $\{Y_i\}$ are given in Theorems 1 and 2 of Section 2 respectively. Corollaries to the theorems address the statistical applications of the representations. Proofs of Theorems 1 and 2 are given in Section 3.

2. Main results

Let X be a standard normal random variable and let $H_j(x)$ be the j-th Hermite polynomial with leading coefficient one.In the sequel, we assume the Borel function $G(\cdot)$ is square-integrable, i.e., $EG^2(X) < \infty$. We let $EG(X) = \mu$ and make the following definition. (see, e.g., Taqqu, 1979, and Dehling and Taqqu, 1989a)

DEFINITION. $a_j = E[G(X) - \mu]H_j(X)$ and $\tilde{k} = \inf\{j \geq 1, a_j \neq 0\}$. $k^* = \inf\{j : a_j(y) \neq 0$ for some $y \in \mathcal{R}\}$ where $a_j(y) = EI\{G(X) \leq y\}H_j(X)$.

Following the above definition, we write

$$G(X) - \mu = \sum_{j=\tilde{k}}^{\infty} \frac{a_j}{j!} H_j(X) \text{ and } I\{G(X) \le y\} - F(y) = \sum_{j=k^*}^{\infty} \frac{a_j(y)}{j!} H_j(X).$$

If $G(x) = x$ it can be easily calculated that

$$a_j(y) = (-1)^j \Phi^{(j)}(y), \tag{2.1}$$

where $\Phi(x)$ is the standard normal distribution function. The positive integers $\tilde{k}$ and k^* which play a critical role in deciding the normalization constants are usually called the Hermite rank of $G(X)$ and $I\{G(X) \le y\}$ respectively. Set $\sigma_n^2(k) = \mathrm{var}(\sum_{i=1}^n H_k(X_i))$. It is known that $\sigma_n^2(k) = O(L(n)^k n^{2-k\alpha})$ if $k\alpha < 1$ (cf. Dobrushin and Major, 1979, and Taqqu, 1979).

LEMMA 1. *Assume $k^*\alpha < 1$. Then with probability one* ($\log_2 n = \log\log n$)

$$\overline{\lim}_{n\to\infty} \sigma_n^{-1}(k^*) n (\log_2 n)^{-k^*/2} \sqrt{k^*!} \sup_{x\in\mathcal{R}} |F_n(x) - F(x)| \le 1 \tag{2.2}$$

and

$$\overline{\lim}_{n\to\infty} \sigma_n^{-1}(k^*) (\log_2 n)^{-k^*/2} (\sum_{i=1}^n H_{k^*}(X_i)) \le 1. \tag{2.3}$$

Proof. Set

$$\mathcal{L}_1 = \Big\{ l(x) = (k^*!)^{-1} a_{k^*}(x) \int_{\mathcal{R}^{k^*}} K(u_1, \cdots, u_{k^*}) g(u_1) \cdots g(u_{k^*}) du_1 \cdots du_{k^*} : \int_{\mathcal{R}} g^2(u) du \le 1 \Big\}$$

and

$$\mathcal{L}_2 = \Big\{ \int_{\mathcal{R}^{k^*}} K(u_1, \cdots, u_{k^*}) g(u_1) \cdots g(u_{k^*}) du_1 \cdots du_{k^*} : \int_{\mathcal{R}} g^2(u) du \le 1 \Big\}$$

where

$$K(u_1, \cdots, u_{k^*}) = C(k^*, \alpha) \cdot \frac{e^{i(u_1 + \cdots + u_{k^*})} - 1}{i(u_1 + \cdots + u_{k^*})} \prod_{j=1}^{k^*} u_j^{-(1-\alpha)/2}. \tag{2.4}$$

The constant $C(k^*,\alpha) = \{\frac{(1-\frac{k^*\alpha}{2})(1-k^*\alpha)}{k^*![2\Gamma(\alpha)\sin\frac{(1-\alpha)\pi}{2}]^m}\}^{1/2}$ is to guarantee

$$\int_{\mathcal{R}^{k^*}} |K(u_1,\cdots,u_{k^*})|^2 du_1\cdots du_{k^*} = 1$$

(cf. (1.7) of Dehling and Taqqu, 1989b). We have from Hölder's inequality

$$\int_{\mathcal{R}^{k^*}} |K(u_1,\cdots,u_{k^*})g(u_1)\cdots g(u_{k^*})|du_1\cdots du_{k^*} \leq 1$$

and

$$\begin{aligned}|a_{k^*}(x)| &\leq (\int I(G(u)\leq x)d\Phi(u))^{1/2}(\int H_{k^*}^2(u)d\Phi(u))^{1/2}\\ &= (F(x))^{1/2}(k^*!)^{1/2}\end{aligned}$$

yielding

$$\sqrt{k^*!}\sup_{l\in\mathcal{L}_1}\sup_{x\in\mathcal{R}}|l(x)| \leq 1 \text{ and } \sup\mathcal{L}_2 \leq 1. \tag{2.5}$$

By applying Corollary 5.3.5 of Stout (1979) to the FLIL obtained by Dehling and Taqqu (1989b), (2.2) and (2.3) are immediate from (2.5). □

The next lemma indicates that under the long-memory dependence the empirical distribution may have a rate of convergence slower than that of the sample mean, violating the commonly acknowledged $\sqrt{n}$-rule in the conventional short-memory cases.

LEMMA 2. *Under $EG^2(X) < \infty$, $k^* \leq \tilde{k}$.*

Proof. For any positive integer $m \geq 1$, define a finite signed measure $d\nu_m = H_m(x)d\Phi(x)$. Then for all $c > 0$

$$\begin{aligned}a_m(y) &= \int I\{G(x)\leq y\}d\nu_m(x)\\ &= (\int_{\{G(x)<-c\}\cap\{G(x)\leq y\}} + \int_{\{G(x)\in[-c,c]\}\cap\{G(x)\leq y\}}\\ &\qquad + \int_{\{G(x)>c\}\cap\{G(x)\leq y\}})d\nu_m(x)\\ &\equiv a_m(y;(-\infty,-c)) + a_m(y;[-c.c]) + a_m(y;(c,+\infty)).\end{aligned}$$

Since $\int_{-c}^{c} a_m(y;(c,+\infty))dy = 0$,

$$\begin{aligned}
\int_{-c}^{c} a_m(y)dy &= \int_{-c}^{c} a_m(y;(-\infty,-c))dy + \int_{-c}^{c} a_m(y;[-c,c])dy \\
&= 2c\int_{\{G(x)<-c\}} d\nu_m(x) + \int_{\{|G(x)|\le c\}} (c-G(x))d\nu_m(x) \\
&= c[\int_{\{G(x)<-c\}} - \int_{\{G(x)>c\}}]d\nu_m(x) - \int_{-c}^{c} G(x)d\nu_m(x).
\end{aligned}$$

Since

$$|c\int_{\{|G(x)|>c\}} d\nu_m(x)| \;\le\; \int |G(x)|H_m(x)d\Phi(x) < \infty,$$

dominated convergence theorem implies

$$\lim_{c\to\infty} c\int_{\{|G(x)|>c\}} d\nu_m(x) = 0$$

Therefore $\lim_{c\to\infty}\int_{-c}^{c} a_m(y)dy = -a_m$. The proof is completed. □

Set $Z_i = F(Y_i)$ and denote by $Z_{n:i}$ the i-th order statistics of $\{Z_i,\cdots, Z_n\}$. Introduce

$$\begin{aligned}
E_n(x) &= n^{-1}\sum_{i=1}^{n} I\{Z_i \le x\}, \\
U_n(y) &= \inf\{u:\ E_n(u)\ge y\} \\
&= Z_{n:k} \text{ if } (k-1)/n < y \le k/n, k=1,\cdots,n.
\end{aligned}$$

LEMMA 3. *Assume* $\inf_{x\in\mathcal{R}} f(x) = \beta > 0$. *Then*

$$\sup_{0<y<1} |Q_n(y) - Q(y)| \le \beta^{-1}\sup_{x\in\mathcal{R}} |F_n(x) - F(x)|. \tag{2.6}$$

Proof. By argument similar to (1.2.7) of Csörgő (1983), we have

$$\sup_{0<y<1} |Q_n(y)-Q(y)| \le \beta^{-1}\sup_{0<y<1}|U_n(y)-y| = \beta^{-1}\sup_{0<y<1}|F_n(y)-F(y)|.$$

□

THEOREM 1. *Assume* (1.1) *holds with* $k^*\alpha < 1$. *If* $f(x)$ *is bounded away from zero and* $f'(\cdot)$ *is bounded,then with probability one*

$$\begin{aligned} Q_n(y) &= Q(y) + \frac{(y - F_n(Q(y)))}{f(Q(y))} + R_n(y), \qquad (2.7) \\ \sup_{0<y<1} |R_n(y)| &= O(n^{-3(1-H)/2}(L(n)\log_2 n)^{3k^*/4}) \end{aligned}$$

with $H = 1 - (k^*\alpha/2)$.

Since $\sigma_n^2(k^*) = \mathrm{var}(\sum_{i=1}^n H_{k^*}(X_i)) = O(L(n)^{k^*} n^{2H})$, we can, setting $H = 1/2, L(n) = 1$ and $k^* = 1$, view (2.7) as a formal extension of the i.i.d case (cf. Kiefer, 1967). Representation Similar to (2.7) with crude error $R_n(y)$ was obtained in [Ho and Hsing, 1996] for long-memory linear processes. The following corollary is immediate from Theorem 1.

COROLLARY 1. *Assume the same conditions as in* Theorem 1. *Suppose* $k^* = 1$. *Then as* $n \to \infty$

$$\frac{n}{\sigma_n(1)}(Q_n(y) - Q(y)) \to \frac{a_1(Q(y))}{f(Q(y))}N(0,1) \qquad (2.8)$$

in the sense of weak convergence in $C([0,1])$ *equipped with the sup-norm.*

Let $EY_i = \mu$ and assume the distribution of Y_i is symmetric about μ. For any $0 < \tau < 1/2$ define the τ-trimmed mean

$$T_{n,\tau} = \frac{1}{n - 2[n\tau]} \sum_{i=[n\tau]+1}^{n-[n\tau]} Y_{n:i}.$$

Clearly,

$$|T_{n,\tau} - \frac{1}{1-2\tau}\int_\tau^{1-\tau} Q_n(y)dy| = O(n^{-1}) \quad \text{a.s.},$$

and

$$\frac{1}{1-2\tau}\int_\tau^{1-\tau} Q(y)dy = \mu.$$

Assume the probability density function $f(\cdot)$ of Y_i is positive on its support. Applying continuous mapping theorem [Billingsley, 1968] to (2.8), we have

$$(2.9)\qquad \frac{n}{\sigma_n}(T_{n,\tau}-\mu)\xrightarrow{d} N(0,1)\frac{1}{1-2\tau}\int_{\tau}^{1-\tau}\frac{a_1(Q(y))}{f(Q(y))}dy$$

where $\sigma_n^2 = \text{var}(\sum_{i=1}^n Y_i)$. (2.9) has an interesting implication that the trimmed mean, a simple type of L-estimator, could be as efficient as or even more efficient than sample mean. Under Gaussian long-memory setting, for example, the α-trimmed mean and sample mean are equally efficient because $a_1(Q(y)) = -\phi(Q(y))$, assured by (2.1), implies that the limiting distribution in (2.9) is just $N(0,1)$. Beran (1991) made similar observation for M-estimator. We present in the next theorem the sample-mean version of representation (2.7). Similar results were obtained for iid random variables (see, Ralescu and Puri, 1984, and references therein). This representation has applications in estimating $F(\mu)$ when both F and the mean μ are unknown or in testing the symmetry of F about the unknown μ (see Gastwirth, 1971). Let $\mu_n = n^{-1}\sum_{i=1}^n G(X_i)$.

THEOREM 2. *Assume $EG^2(X) < \infty$ and (1.1) with $\tilde{k}\alpha < 1$. If $f'(\cdot)$ is bounded, then with probability one*

$$(2.10)\qquad \begin{aligned} F_n(\mu_n) &= F_n(\mu) + (\mu_n-\mu)f(\mu) + \tilde{R}_n,\\ &\quad \tilde{R}_n = O(n^{-3/2}[\sigma_n(k^*)(\log_2 n)^{k^*/2}]\cdot[\sigma_n(\tilde{k})(\log_2 n)^{\tilde{k}/2}]^{1/2}). \end{aligned}$$

Suppose $G(x) = x$. If $\alpha < 1/3$, then with probability one

$$(2.11)\qquad \begin{aligned} F_n(\mu_n) &= F_n(\mu) - (n^{-1}\sum_{i=1}^n H_2(X_i))\frac{\mu_n-\mu}{2}\phi"(\mu)\\ &\quad +(\mu_n-\mu)\phi(\mu) + \frac{2(\mu_n-\mu)^3}{3}\phi''(\mu) + \tilde{R}'_n,\\ &\quad \tilde{R}'_n = O((n^{-\alpha}\log_2 n)^{7/4}). \end{aligned}$$

COROLLARY 2. *Assume the same conditions as in* Theorem 2. *Suppose $\tilde{k} = 1$ and $a_1(\mu) + a_1 f(\mu) \neq 0$. Then as $n\to\infty$*

$$(2.12)\qquad \frac{n}{\sigma_n(1)}(F_n(\mu_n) - F(\mu)) \xrightarrow{d} (a_1(\mu) + a_1 f(\mu))N(0,1).$$

When $G(x) = x$, then as $n \to \infty$,

$$
\begin{aligned}
(2.13) \qquad & n^{3\alpha/2}L^{-3/2}(F_n(\mu_n) - F(\mu)) \\
& \xrightarrow{d} \left(\frac{2(c_1W_1)^3}{3} - \frac{c_3W_3}{6} - \frac{c_1c_2W_1W_2}{2}\right)\phi"(\mu)
\end{aligned}
$$

where $c_k = (2k!(1-k\alpha)^{-1}(2-k\alpha)^{-1})^{1/2}$ and W_k can be represented as double Wiener-Ito integral

$$W_k = \int K(u_1, \cdots, u_k)\tilde{B}(du_1)\cdots\tilde{B}(du_k).$$

The function K is defined in (2.4) with $k^* = k$, and $\tilde{B}$ is a Gaussian complex white noise measure with orthogonal increments.

Proof. Write (2.10) as

$$
\begin{aligned}
& \frac{n}{\sigma_n(1)}(F_n(\mu_n) - F(\mu)) \\
(2.14) \qquad = \; & \frac{n}{\sigma_n(1)}(a_1(\mu) + a_1 f(\mu))(\mu_n - \mu) \\
+ \; & \frac{n}{\sigma_n(1)}\{\sum_{j=2}^{\infty}\frac{a_j}{j!}(n^{-1}\sum_{i=1}^{n}H_j(X_i)) + \tilde{R}_n\}.
\end{aligned}
$$

(2.10) assures that the second part on the right hand side of expansion (2.14) vanishes as $n \to \infty$. Hence (2.12) follows. When $G(x) = x$, i.e., Y_i is itself standard normal, $a_1 = 1$ and, by (2.1), $a_1(\mu) = -\phi(\mu)$ and $a_2(\mu) = \phi'(\mu) = 0$. Note first that $\sigma_n^2(k) = c_k^2 n^{2-k\alpha}L^k(n)$ [Dehling and Taqqu, 1989a, pp. 1769]. Similar to (2.14), (2.11) can be expressed as

$$
\begin{aligned}
& n^{3\alpha/2}L^{-3/2}(n)(F_n(\mu_n) - F(\mu)) \\
= \; & n^{3\alpha/2}L^{-3/2}(n)\phi"(\mu)\{-(n^{-1}\sum_{i=1}^{n}H_3X_i)/6 \\
& \quad + 2(\mu_n - \mu)^3/3 - (n^{-1}\sum_{i=i}^{n}H_2(X_i))(\mu_n - \mu)/2\} \\
+ \; & n^{3\alpha/2}L^{-3/2}(n)\{\sum_{j=4}^{\infty}\frac{a_j}{j!}(n^{-1}\sum_{j=1}^{n}H_j(X_i)) + \tilde{R}'_n\}.
\end{aligned}
$$

(2.13) then follows by the joint convergence in distribution of

$$(n\sigma_n^{-1}(1)(\mu_n - \mu), \sigma_n^{-1}(2)\sum_{i=1}^{n} H_2(X_i))\sigma_n^{-1}(3)\sum_{i=1}^{n} H_3(X_i) \xrightarrow{d} (W_1, W_2, W_3)$$

established in Theorem 2 of Ho and Sun (1990). The proof is complete. □

3. Proofs

Proof of Theorem 1. Define

$$(3.1)\quad G_n(y) = [F_n(Q_n(y)) - F_n(Q(y))] - [F(Q_n(y)) - F(Q(y))],$$

$$D_n(x,y) = [F_n(x) - F_n(Q(y))] - [F(x) - F(Q(y))],$$

$$d_n = \beta^{-1}\sigma_n(k^*)n^{-1}(k^*!)^{-1/2}(\log_2 n)^{k^*/2}(1+\epsilon),\ \epsilon > 0,$$

and

$$I_n(y) = [Q(y) - d_n, Q(y) + d_n].$$

Lemma 1 and 3 imply that with probability one

$$(3.2)\qquad |G_n(y)| = O(\sup_{x\in I_n(y)} |D_n(x,y)|).$$

Write

$$\begin{aligned} D_n(x,y) &= \frac{a_{k^*}(x) - a_{k^*}(Q(y))}{k^*!}\left(\frac{1}{n}\sum_{i=1}^{n} H_{k^*}(X_i)\right) \\ &\quad + \sum_{j=k^*+1}^{\infty} \frac{a_j(x) - a_j(Q(y))}{j!}\left(\frac{1}{n}\sum_{i=1}^{n} H_j(X_i)\right) \\ &\equiv A_n(x, Q(y)) + B_n(x, Q(y)). \end{aligned}$$

We note by Hölder's inequality that for all $j \geq 1$

$$(3.3)\qquad |a_j(x_1) - a_j(x_2)| \leq [j!|x_1 - x_2|(\sup_{x\in\mathcal{R}} f(x))]^{1/2},$$

which yields

$$\sup_{0<y<1}\sup_{x\in I_n(y)} |a_{k^*}(x) - a_{k^*}(Q(y))| = O(d_n^{1/2}).$$

Hence from (2.2) of Lemma 1 we get, with probability one,

$$(3.4)\qquad \tilde{A}_n \equiv \sup_{0<y<1} \sup_{x \in I_n(y)} |A_n(x, Q(y))| = O((n^{-1}\sigma_n(k^*))^{3/2}(\log_2 n)^{3k^*/4}).$$

The same argument as in the proof of Theorem 3.1 of Dehling and Taqqu (1989a) shows

$$(3.5)\qquad \sup_{0<y<1} \sup_{x \in I_n(y)} |B_n(x, Q(y))| = o(\tilde{A}_n).$$

It is clear that uniformly for all $0 < y < 1$

$$(3.6)\qquad |F_n(Q_n(y)) - y| = O(1/n) \quad \text{a.s.}$$

In view of (2.2) and (2.6),

$$(3.7)\qquad \sup_{0<y<1} |Q_n(y) - Q(y)|^2 = o((n^{-1}\sigma_n(k^*))^{3/2}(\log_2 n)^{3k^*/4}).$$

Apply Taylor's theorem to $F(Q_n(y)) - F(Q(y))$ in (3.1). Then, using (3.6),

$$\begin{aligned} & Q_n(y) \\ = \; & Q(y) + \frac{(y - F_n(Q(y)))}{f(Q(y))} + O(1/n) - G_n(y) + O(|Q_n(y) - Q(y)|^2). \end{aligned}$$

(2.7) follows from (3.2) combined with (3.4) through (3.7).

The proof of Theorem 2 is very similar to that of Theorem 1. We shall only sketch it. □

Proof of Theorem 2. The main points that are different from the proof of Theorem 1 are: (1) μ_n and μ are in place of $Q_n(y)$ and $Q(y)$ respectively, (2) k^* in the expression of d_n is replaced by $\tilde{k}$, and (3) there are two Hermite ranks k^* and $\tilde{k}$ involved. First, Lemma 2 assures that $k^*\alpha < 1$. Replace $I_n(y)$ by

$$I_n = [\mu - d_n, \mu + d_n].$$

Then, (3.3), and (2.2) and (2.3) of Lemma 1 give, with probability one,

$$\sup_{x \in I_n} |A_n(x, \mu)| = O(n^{-3/2}[\sigma_n(k^*)(\log_2 n)^{k^*/2}] \cdot [\sigma_n(\tilde{k})(\log_2 n)^{\tilde{k}/2}]^{1/2}),$$

which leads to (2.10). To prove (2.11), redefine

$$\begin{aligned} G_n &= [F_n(\mu_n) - F_n(\mu) - \frac{(\mu_n - \mu)}{2}(n^{-1}\sum_{i=1}^{n} H_2(X_i))\phi''(\mu)] \\ &\quad -[\Phi(\mu_n) - \Phi(\mu) - \frac{(\mu_n - \mu)^2}{2}(n^{-1}\sum_{i=1}^{n} X_i)\phi''(\mu)] \end{aligned}$$

and

$$\begin{aligned} D_n(x,\mu) &= [F_n(x) - F_n(\mu)] - [\Phi(x) - \Phi(\mu)] \\ &\quad +\frac{(x-\mu)}{2}(n^{-1}\sum_{i=1}^{n} H_2(X_i))\phi''(\mu) \\ &\quad +\frac{(x-\mu)^2}{2}(n^{-1}\sum_{i=1}^{n} X_i)\phi''(\mu). \end{aligned}$$

Note by (2.1) that $a_1(x) = -\phi(x)$ and $a_2(x) = \phi'(x)$. Therefore, the Hermite expansion of $D_n(x,\mu)$ is,

$$\begin{aligned} D_n(x,\mu) &= -\frac{\phi''(x) - \phi''(\mu)}{\sigma}(\frac{1}{n}\sum_{i=1}^{n} H_3(X_i)) \\ &\quad +\sum_{j=4}^{\infty} \frac{(-1)^j(\phi'(x) - \phi'(\mu))}{j!}(\frac{1}{n}\sum_{i=1}^{n} H_j(X_i)) \\ &\quad +(n^{-1}\sum_{i=1}^{n} X_i)O(\mid x-\mu\mid^3) + (n^{-1}\sum_{i=1}^{n} H_2(X_i))O(\mid x-\mu\mid^2). \end{aligned}$$

Like in proving (2.10), applying Taylor's theorem to $\Phi(\mu_n) - \Phi(\mu)$, we get

$$\Phi(\mu_n) - \Phi(\mu) = (\mu_n - \mu)\phi(\mu) + (\mu_n - \mu)^3\phi''(\mu)/6 + O(|\mu_n - \mu|^4).$$

By imitating the proof of (2.10), the remaining details to conclude (2.11) can be straightforwardly filled up and are thus omitted. □

References

[1] Bahadur, R. R., A note on quantiles in large samples, Ann. Math. Statist. **37** 577-580, (1966).

[2] Beran, J., *M-estimator of location for data with slowly decaying serial correlations*, J. Amer. Statist. Assoc. **86** 704-708, (1991).

[3] ———, *Statistics for Long-Memory Processes*, Chapman and Hall, New York, 1994.

[4] Billingsley, P., *Convergence of Probability Measures*, Wiley, New York, 1968.

[5] Cox, D. R., *Long -range dependence: a review*, In statistics: An Appraisal. Proceedings 50th Anniversary Conference, Eds. H. A. David and H. T. David, pp. 55-74, The Iowa State University Press, 1984.

[6] Csörgő, M., *Quantile Processes with Statistical Applications*, CBMS-NSF Regional Conference Series in Applied Mathematics vol. 42, SIAM, Philadelphia, 1983.

[7] Dehling, H. and Taqqu, M. S., *The empirical processes of some long-range dependent sequences with an application to U-statistics*, Ann. Statist. **17** 1767-1783, (1989a).

[8] ———, *The functional law of the iterated logarithm for the empirical process of some long-range dependent sequences*, Statistics & Probability Letters **7** 81-85, (1989b).

[9] Dobrushin, R. L. and Major, P., *Non-central limit theorems for nonlinear functionals of Gaussian fields*, Z. Wahrsch. Verw. Gebiete **50** 27-52, (1979).

[10] Gastwirth, J. L., *On the sign test for symmetry*, J. Amer. Staist. Assoc. **66** 821-823, (1971).

[11] Hesse, C. H., *A Bahadur-type representation for empirical quantiles of a large class of stationary, possibly infinite-variance, linear processes*, Ann. Statist. **18** 1188-1202, (1990).

[12] Ho, H. C. and Sun, T. C., *Limit distributions of non-linear vector functions of stationary Gaussian processes*, Ann. Probab. **18** 1159-1173, (1990).

[13] Ho, H. C. and Hsing, T., *On the asymptotic expansion of the empirical process of long-memory moving averages*, Ann. Statist. **24** 992-1024, (1996).

[14] Kiefer, J., *On Bahadur's representation of sample quantiles*, Ann. Math. Statist. **38** 1323-1342, (1967).

[15] Ralescu, S. and Puri, M. L., *On the Berry-Esseen rates, a law of the iterated logarithm and an invariance principle for the proportion of the sample below the sample mean*, J. Multivariate Analysis **14** 231-247, (1984).

[16] Robinson, P. M., *Time series with strong dependence*, Invited Paper, 6th World Congress of the Econometrics Society, Barcelona, Adv. in Econometrics, Sixth World Congress, Cambridge University Press, Cambridge, pp. 47-96, 1994.

[17] Sen, P. K., *On the Bahadur representation of sample quantiles for sequences of phi-mixing random variables*, J. Multivariate Analysis **2** 77-95, (1972).

[18] Stout, W., *Almost Sure Convergence*, Academic Press, New York, 1979.

[19] Taqqu, M. S., *Convergence of integrated processes of arbitrary Hermite rank*, Z. Wahrsch. verw. Gebiete **50** 53-83, (1979).

Institute of Statistical Science
Academia Sinica
Taipei 115, Taiwan
Republic of China
E-mail: hcho@stat.sinica.edu.tw