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CONTROLLING TRAFFIC LIGHTS AT A BOTTLENECK: THE OBJECTIVE FUNCTION AND ITS PROPERTIES

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ABSTRACT. Controlling traffic lights at a bottleneck, in [5] a time of open passage is called optimal, if it minimizes the first moment of the asymptotic distribution of the queue length. The discussion of the first moment as function of the time of open passage is based on an analysis of the behavior of a fixed point when varying control parameters and delivers theoretical and computational aspects of the traffic problem.

1. Introduction

In [4] and [5] the question of the onset or 'non-onset' of a traffic collapse at a bottleneck controlled by traffic lights (symmetric case) — with the time of open passage being the control variable of the installation — is analyzed.

The onset or 'non-onset' of a traffic collapse is related with the ergodicity of the queueing processes at the entrance points of the bottleneck. A time of open passage is called optimal, if it minimizes the first moment of the limiting distribution of the queueing process. The studying of this objective as a function of the time of open passage not only gives insights to the control characteristics of the installation but is also essential for the computational determination of the optimal time of open passage.

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After the description of the model and the compilation of the main results of [4] and [5] a discussion of the properties of the objective function is delivered. A special role thereby is played by a grid which reveals to be the set of the relative minima of the objective function from the right.

2. The model of a bottleneck controlled by traffic lights

To describe the technical part of a bottleneck controlled by traffic lights (symmetric case) we use the following parameters:

$$\Delta\,,\, t_R. \tag{2.1}$$

Δ in [veh/s] is the passage capacity (for both sides) of the bottleneck. t_R in [s] denotes the clearance times (both sides).

The arrival process $A = (A_t)_{t\in\mathbf{R}}$ for an arbitrary direction is assumed as Poisson process on the probability space $(\Omega,\, \mathcal{A},\, P)$ with parameter I being the traffic intensity in [veh/s] in a traffic-theoretic interpretation.

For $0 \le s \le t$ let be

$$N\big((s;t]\big) \;:=\; A_t - A_s \tag{2.3}$$

the increment of the arrival process A during the time interval $(s,t]$.

Moreover the initial distribution q_0 (for both sides) has to be introduced, which means a probability distribution for the number of waiting vehicles. If $\mathcal{M}^1_e(\mathbb{Z}_+)$ denotes the set of all probability measures on $\mathbb{Z}_+$ with finite support, we require $q_0 \in \mathcal{M}^1_e(\mathbb{Z}_+)$.

While Δ, t_R and I are considered as known, q_0 is fixed but unknown to the installation administrator.

It is sensible to assume the times of open passage (signalized by GREEN and afterwards by YELLOW) to be the same for both sides. The time $t_F > 0$ of open passage is the control variable in the hand of the administrator of the installation.

The system of a bottleneck controlled by traffic lights (symmetric case) may be comprised in a 5-tuple

$$(\Delta\,, t_R\,, I\,, q_0\,, t_F)\,. \tag{2.4}$$

The duration of the closed passage (for both sides) is given by

$$t_C := 2\,t_R + t_F \tag{2.5}$$

while

$$t_U := 2(t_R + t_F) \tag{2.6}$$

represents the length of a full control period.

3. The queueing process, collapse of traffic, independence of the initial distribution

The function $\overline{\alpha} : \mathbb{R}_+ \longrightarrow \mathbb{Z}_+$ is defined by

$$\overline{\alpha}(t) := \begin{cases} 0 & ,0 < t < t_C \\ \left[(t - t_C)\Delta\right] & ,t_C \leq t \leq t_U \end{cases} \tag{3.1}$$

and the condition that $\bar{\alpha}$ is periodic with the period t_U on $\mathbb{R}_+$, represents the maximal number of vehicles, which can pass the bottleneck from the beginning of a control period until the time t of a control period.

(Notice that $[a]$ means the greatest integer number less than or equal to a.)

The number

$$\alpha(t_F) := [t_F \cdot \Delta] = \overline{\alpha}(t_U) \tag{3.2}$$

denotes the maximal number of vehicles, which may pass the bottleneck during one control period respectively during one phase of free passage.

Let be $L_0 : (\Omega, \mathcal{A}) \longrightarrow \mathbb{Z}_+$ a random variable having the distribution q_0, independent of the arrival process A.

The process of queue lengths (of vehicles) is defined recursively by

$$L(0) := L_0 \quad \text{and} \tag{3.3}$$

$$L\big((j+1)t_U\big) := \Big(L(j\,t_U) + N\big((j\,t_U; (j+1)t_U]\big) - \alpha(t_F)\Big)^+. \tag{3.4}$$

For $j\,t_U < t < (j+1)\,t_U$ let be

$$L(t) := \Big(L(j\,t_U) + N\big((j\,t_U; t]\big) - \overline{\alpha}(t)\Big)^+ . \tag{3.5}$$

Define

$$\lambda(t_F) := 2I(t_F + t_R) = I \cdot t_U , \tag{3.6}$$

λ is the expectation of the number of vehicles arriving at the bottleneck during a control period t_U.

Denoting the distribution of the random variables $L(j\,t_U) : \Omega \longrightarrow \mathbb{Z}_+$ by q_j, the sequence of distributions $(q_j)_{j=0}^{\infty}$ satisfies the recursion

$$q_{j+1} = T\,q_j \qquad (j = 0, 1, \dots), \tag{3.7}$$

where the operator $T : \mathcal{M}^1(\mathbb{Z}_+) \to \mathcal{M}^1(\mathbb{Z}_+)$ is defined by

$$T\,q(l) := \begin{cases} \sum_{k=0}^{\alpha(t_F)} (q * \pi_{\lambda(t_F)}(k)) & \text{for } l = 0 \\ (q * \pi_{\lambda(t_F)})(l + \alpha(t_F)) & \text{for } l \leq 1 \end{cases} . \tag{3.8}$$

($\mathcal{M}^1(\mathbb{Z}_+)$ denotes the set of probability measures on $\mathbb{Z}_+$; $\pi_{\lambda'}$ denoting the Poisson distribution with parameter $\lambda' > 0$.)

For the justification of (3.7) the reader is referred to [Grycko, Moeschlin 1998a], sec. 1.

A traffic collapse does or does not occur if

$$\sup\left\{\int L(t)\,\mathrm{d}P \,\middle|\, t \in \mathbb{R}_+\right\} = \infty , \tag{3.9}$$

or if

$$\sup\left\{\int L(t)\,\mathrm{d}P \,\middle|\, t \in \mathbb{R}_+\right\} < \infty , \tag{3.10}$$

where $\int L(t)\,\mathrm{d}P$ can be computed for any fixed time $t \in \mathbb{R}_+$ according to (3.7), (3.8).

The following theorem delivers a criterion for the onset or non-onset of a traffic collapse.

THEOREM 3.11. ([4]) *Let* $\big(L(t)\big)_{t\in\mathbb{R}_+}$ *be the process of queue lengths according to (3.3) – (3.5). Moreover let* $\alpha(t_F)$ *and* $\lambda(t_F)$ *be the numbers introduced in* (3.2) *and* (3.6) *and let* $\beta \in [1;\infty)$ *be arbitrary.*

3.11.1. *If* $\alpha(t_F) > \lambda(t_F)$, *then:*

$$\sup\left\{\int L(t)^\beta \,\mathrm{d}P \,|\, t \in \mathbb{R}_+\right\} < \infty ;$$

3.11.2. *if on the contrary* $\alpha(t_F) < \lambda(t_F)$, *then*

$$\lim_{t\to\infty} \int L(t)^\beta \,\mathrm{d}P = \infty .$$

Proof. The proof is given in [4]; it uses essentially the condition $q_0 \in \mathcal{M}^1_e(\mathbb{Z})$ (set of probability measures on $\mathbb{Z}_+$ with finite support). □

Theorem 3.13 states the existence of the distributional limit of the queueing process, which coincides with the unique fixed point of the operator T, thus entailing the independence of the distributional limit from the initial distribution.

Moreover the limit of the expectations of the queueing process coincides with the expectation of the limiting distribution.

Preparing Theorem 3.13 we endow $\mathcal{M}^1(\mathbb{Z}_+)$ with the variational distance:

$$\begin{aligned} d_v(p,q) \;&:=\; 2\cdot \sup_{B\in\mathcal{P}(\mathbb{Z}_+)} \big|p(B)-q(B)\big| = \\ &=\; \sum_{l=0}^{\infty} \big|p(l)-q(l)\big| \end{aligned} \tag{3.12}$$

where $\mathcal{P}(\mathbb{Z}_+)$ denotes the power set of $\mathbb{Z}_+$. Thus, $\big(\mathcal{M}^1(\mathbb{Z}_+), d_v\big)$ is a metric space.

THEOREM 3.13. *Let* $\big(L(t)\big)_{t\in\mathbb{R}_+}$ *be the process of queue lengths as introduced in* (3.3) – (3.5). *Let* $t_F > 0$ *be such that*

$$\alpha(t_F) > \lambda(t_F)$$

holds.

3.13.1 . *There exists one and only one fixed point* $q \in \mathcal{M}^1(\mathbb{Z}_+)$ *of the operator* $T : (\mathcal{M}^1(\mathbb{Z}_+), d_v) \to (\mathcal{M}^1(\mathbb{Z}_+), d_v)$, *and,*

$$\lim_{j\to\infty} d_v(q_j, q) = 0 .$$

3.13.2 . *For the sequence of expectations of* $L(j\, t_U)$ *we have:*

$$\lim_{j\to\infty} \int L(j\, t_U) \mathrm{d}P = \lim_{j\to\infty} \sum_{l=0}^{\infty} l\, q_j(l) = \sum_{l=0}^{\infty} l\, q(l) .$$

Theorem 3.13 gives rise to speak of $q = q(t_F)$ as of the ergodic distribution of the process $\big(L(t)\big)_{t\in\mathbf{R}_+}$. In a similar way we are speaking of the respective queue lengths $\sum_{l=0}^{\infty} l\, q(l)$ as the ergodic queue length, while t_F chosen in the above way, i.e., with $\alpha(t_F) > \lambda(t_F)$ is simply called ergodic.

Proof. A proof is given in [5]. The proof is given by a fixed point theorem, cp. [6] and also [3]. It is based on the relative compactness of $\{T^n q_0 \,|\, n \in \mathbb{N}\}$, which follows by Theorem 3.11 applying a theorem of Prohorov, cp. [1], p. 95.

The second part follows by Skorohod's theorem, cp. too [2], p. 333. □

REMARK 3.14. In the sequel we base on the limit of the expected queue length (ergodic queue length) at the end of the closed phase

$$\lim_{j\to\infty} \lim_{t\uparrow j t_U + t_C} \int L(t)\, \mathrm{d}P \tag{3.15}$$

when defining an objective function.

If for a t_F with $\alpha(t_F) > \lambda(t_F)$ the (ergodic) distribution $q := q(t_F)$ is the unique fixed point of the respective operator $T =: T(t_F)$, cp. 3.13.1, the expression (3.15) has the representation

$$\sum_{l=0}^{\infty} l\, q(l) + I(2t_R + t_F), \tag{3.16}$$

which follows from 3.13.2.

4. Optimality concept

A comprehensible optimality concept for the control of traffic lights at a bottleneck is to choose t_F in such a way that the limit of the expected queue length taken at the end of the closed phase, cf. (3.15), exists in $\mathbb{R}$ (ergodicity of the queueing process) and is kept minimal.

Such a definition makes sense in so far as the optimality concept does not depend on the initial distribution q_0, cf. Theorem 3.13. Apart from an intuitive accordance with this optimality requirement, one has to see that the efficiency of all time t_F of open passage, which lead to an ergodic queueing process is the same, meaning that for any such t_F the throughput during one control period is proportional to the length of the control period with the proportionality factor I, cf. (3.6).

Another important requirement is the minimization of the waiting time, e.g. the time which is needed till an average queue (at the end of the closed phase) has disappeared. Thereby it is of less importance if a vehicle may pass at the first or the second position during the same control period. More important is the expected number of periods a new arriving vehicle has to wait till it has the possibility to pass the bottleneck. This number is given by

$$\left\lceil \frac{E(L(jt_U + t_C))}{I\, t_U} \right\rceil .$$

If one takes

$$\left\lceil \frac{E(L(jt_U + t_C))}{I\, t_U} \right\rceil \cdot t_U \approx \frac{E(L(jt_U + t_C))}{I}$$

for the expected waiting time, one indeed is lead to the minimization of the expected queue lengths (ergodic length) at the end of the closed phase, cp. (3.15), (3.16), which is taken as the objective.

5. Ergodic times of free passage

Remembered, t_F is called ergodic if $\alpha(t_F) > \lambda(t_F)$.

REMARK 5.1.

5.1.1. In the sequel we assume

$$\Delta > 2I, \tag{5.2}$$

since for $\Delta \leq 2I$ one has $I \geq \frac{\Delta}{2}$ and

$$\alpha(t_F) < \lambda(t_F) \quad \text{for } t_F > 0,$$

with the consequence, cp. 3.11.2, that there exists no time of open passage not leading to a traffic collapse.

5.1.2. On the other hand if there exists an ergodic t_F (with $\alpha(t_F) > \lambda(t_F)$), it follows that

$$t_F > \frac{2It_R}{\Delta - 2I},$$

i.e., $2t_R I(\Delta - 2I)^{-1}$ is a lower bound of the set of ergodic times t_F.

The following definitions turn out to be useful:
The set

$$E := \{t_F \mid \alpha(t_F) > \lambda(t_F)\} \tag{5.3}$$

is called the set of ergodic times; its infimum

$$t_{\text{crit}} := \inf E \tag{5.4}$$

is called the critical time (of open passage).

Moreover we define

$$F := \{t_F \mid t_F \geq t_{\text{crit}}\}, \tag{5.5}$$

$$G := \{\frac{n}{\Delta} \mid n \in \mathbb{N}\}, \tag{5.6}$$

$$D := G \cap F. \tag{5.7}$$

The lemma 5.8 and 5.12 together with the example 5.14 deliver an aspect of the set E of ergodic times of open passage, not only important with regard to computations but also for the theoretical understanding of the control problem.

LEMMA 5.8.

$$t_{\text{crit}} \in E \cap G.$$

Lemma 5.8 especially states that t_{crit} equals $\min E$ and not only $\inf E$; moreover it is an element of the grid G.

Proof. The elements of an antitone sequence $(t_l)_{l=1}^{\infty}$ in E with

$$t_{\mathrm{crit}} := \lim_{l\to\infty} t_l \tag{5.9}$$

have an unique representation

$$t_l = \frac{k_l + \xi_l}{\Delta} \qquad (l = 1, 2, \dots) \tag{5.10}$$

with $k_l \in \mathbb{Z}_+$ and $\xi \in [0; 1)$; the sequence $(k_l)_{l=1}^{\infty}$ being antitone.
Define

$$\overline{k} := \min\{k_l \mid l = 1, 2, \dots\}.$$

As $(k_l)_{l=1}^{\infty}$ is antitone there exists a $\overline{l} \in \mathbb{Z}_+$ with

$$k_l = \overline{k} \quad \text{for } l \geq \overline{l}.$$

Because of (5.10) the sequence $(\xi_l)_{l=\overline{l}}^{\infty}$ is antitone too; define

$$\overline{\xi} := \lim_{l\to\infty} \xi_l.$$

Because of (5.9) we have

$$t_{\mathrm{crit}} = \frac{\overline{k} + \overline{\xi}}{\Delta}$$

with $\overline{\xi} = 0$, since the contrary $\overline{\xi} \neq 0$ entails the contradiction

$$t_F := \frac{\overline{k} + \frac{\overline{\xi}}{2}}{\Delta} < t_{\mathrm{crit}}$$

and $t_F \in E$; i.e.,

$$t_{\mathrm{crit}} = \frac{\overline{k}}{\Delta}. \tag{5.11}$$

Because of $t_{\overline{l}} \in E$ we get

$$\alpha(t_{\mathrm{crit}}) = \overline{k} = k_{\overline{l}} > \lambda(t_{\overline{l}}) \geq \lambda(t_{\mathrm{crit}}),$$

since the function $\lambda(.)$ is isotone; i.e., $t_{\mathrm{crit}} \in E$. Moreover, by (5.11) one recognizes also that $t_{\mathrm{crit}} \in G$. $\square$

LEMMA 5.12. *Every $t_F \in G$ with $t_F \geq t_{\mathrm{crit}}$ is ergodic, i.e., $\alpha(t_F) > \lambda(t_F)$ holds.*

Proof. For $t_F \in G$ the condition

$$\alpha(t_F) > \lambda(t_F)$$

is equivalent to

$$t_F > \frac{2It_R}{\Delta - 2I}, \tag{5.13}$$

which follows recalling the assumption $\Delta > 2I > 0$. By Lemma 5.7 the inequality (5.13) is especially fulfilled for $t_F = t_{\text{crit}}$, therefore also for all $t_F \in G$ with $t_F \geq t_{\text{crit}}$. □

Notice, that times of free passage in $F \backslash D$ need not necessarily be ergodic, which is demonstrated by Example 5.14.

EXAMPLE 5.14. t_{crit} is given by the smallest number $t_F \in G$ fulfilling

$$t_F > \frac{2It_R}{\Delta - 2I}.$$

Therefore t_{crit} can be calculated according to the formula

$$t_{\text{crit}} = \left(\left\lceil \frac{2t_R \cdot I \cdot \Delta}{\Delta - 2I} \right\rceil + 1\right) \cdot \frac{1}{\Delta}\,. \tag{5.15}$$

For

$$\Delta := \frac{1250}{3600}\ [\text{veh/s}], \quad I := \frac{400}{3600}\ [\text{veh/s}], \quad t_R := 50\ [\text{s}]$$

we have $t_{\text{crit}} = 89.28$ [s]. Taking t_F as

$$t_F := t_{\text{crit}} + \frac{0.9\ [\text{veh}]}{\Delta}$$

it follows that t_F is not ergodic, because of

$$\alpha(t_F) = \lfloor t_F \cdot \Delta \rfloor = 31\ [\text{veh}] < 31.5271\ [\text{veh}] = \lambda(t_F),$$

although $t_F > t_{\text{crit}}$.

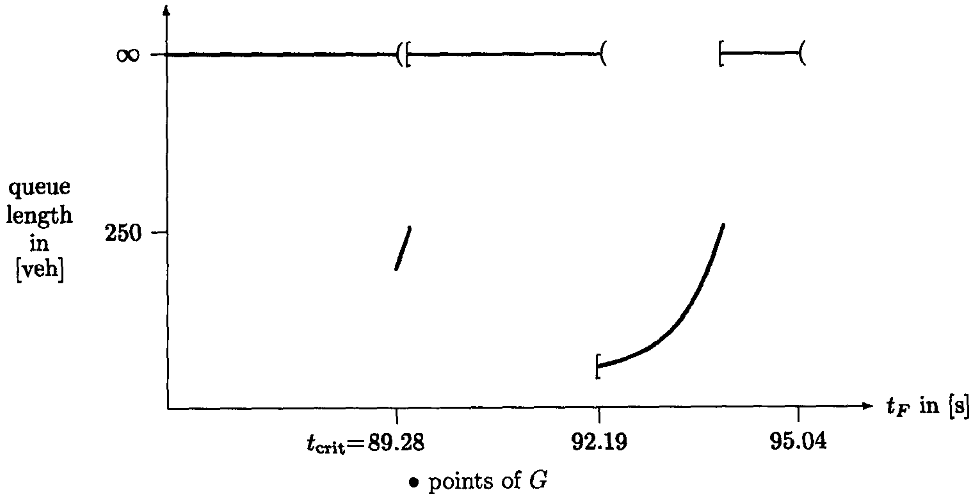

DIAGRAM 1. Queue Length as Function of t_F in the Neighborhood of t_{crit}

6. Relative minima of the objective function

To treat the numerical problem to determine the optimal time of free passage it is important to know the set of relative minima of the objective function.

Showing that the restrictions of the objective function to the intervals

$$[n \cdot \frac{1}{\Delta}; (n+1) \cdot \frac{1}{\Delta}) \qquad (t_{crit} \cdot \Delta \leq n \in \mathbb{Z}_+)$$

are increasing, proves that the grid D is a subset of the set of the relative minima of the objective function.

By reasons of effectiveness our problem of determining the optimal time of free passage is solved when minimizing the objective function over D.

To this end we consider the set

$$C := \{(\alpha', \lambda') \in \mathbb{Z}_+ \times (0; \infty) \,|\, \alpha' > \lambda'\},$$

and define the function

$$\Lambda : C \longrightarrow \mathbb{R}_+, \qquad \Lambda(\alpha', \lambda') := \sum_{l=0}^{\infty} l q_{\alpha',\lambda'}(l)$$

(cf. 3.13.2).

THEOREM 6.1.

(i) *Let be $\alpha' \in \mathbb{Z}_+$ and $\lambda'_1, \lambda'_2 \in (0;\infty)$ with $\alpha' > \lambda'_1 > \lambda'_2$. Then*

$$q_{\alpha',\lambda'_1}\Big([l;\infty)\Big) \geq q_{\alpha',\lambda'_2}\Big([l;\infty)\Big)$$

holds for $l \in \mathbb{Z}_+$.

(ii) *The function Λ is antitone in the first and isotone in the second argument.*

Notice that function Λ is even strictly antitone in α' and strictly isotone in λ'. The proof requires more preparations and is dealt with in a separate paper.

Applying now Theorem 6.1 to the restrictions of the objective function (3.16) to the intervals

$$[n \cdot \frac{1}{\Delta}; (n+1) \cdot \frac{1}{\Delta}) \qquad (t_{\text{crit}} \cdot \Delta \leq n \in \mathbb{Z}_+),$$

it becomes clear that these restrictions are isotone as functions of time of free passage since $\alpha(.)$ is kept constant and $\lambda(.)$ is increasing within any such interval, which proves our statement about D being a subset of the relative minima of the objective function.

It remains to prove Theorem 6.1.

proof of Theorem 6.1. Define

(i) $$\lambda' := \lambda'_1 - \lambda'_2.$$

Let $(\psi_j^{(2)})_{j=0}^{\infty}$ be an independent sequence of random variables on a probability space $(\Omega, \mathcal{A}, P)$ distributed according to the Poisson distribution $\pi_{\lambda'_2}$ with parameter λ'_2.

Moreover, let $(Z_j)_{j=0}^{\infty}$ be an independent sequence of random variables on $(\Omega, \mathcal{A}, P)$ distributed according to $\pi_{\lambda'}$ such that $(Z_j)_{j=0}^{\infty}$ and $(\psi_j^{(2)})_{j=0}^{\infty}$ are independent.

Let then $(\psi_j^{(1)})_{j=0}^{\infty}$ with

$$\psi_j^{(1)} := \psi_j^{(2)} + Z_j \qquad (j = 0, 1, \dots) \tag{6.2}$$

be an independent sequence of random variables distributed according to $\pi_{\lambda_1'}$.

Define then the random variables

$$L_0^{(1)} := L_0^{(2)} : \Omega \to \mathbb{Z}_+$$

by $L^{(i)}(\omega) := 0$ $(\omega \in \Omega,\ i = 1, 2)$ and

$$L_{j+1}^{(i)} := (L_j^{(i)} + \psi_j^{(i)} - \alpha')^+ \qquad (j = 0, 1, \dots;\ i = 1, 2). \tag{6.3}$$

As shown in [5], Sec. 1 and Remark 4.3, the sequence $(q_j^{(i)})_{j=0}^{\infty}$ of distributions of $(L_j^{(i)})_{j=0}^{\infty}$ fulfills the condition

$$q_j^{(i)} = T_{\alpha',\lambda_i'}^j \delta_0 \qquad (j = 0, 1, \dots;\ i = 1, 2),$$

where δ_0 denotes the Dirac measure at 0.

By (6.3) and (6.2) it follows inductively that

$$L_j^{(1)} \geq L_j^{(2)} \qquad (j = 0, 1, \dots), \tag{6.4}$$

which implies the inequality

$$q_{\alpha',\lambda_1'}\Big([l;\infty)\Big) = \lim_{j\to\infty} q_j^{(1)}\Big([l;\infty)\Big) \geq \lim_{j\to\infty} q_j^{(2)}\Big([l;\infty)\Big) = q_{\alpha',\lambda_2'}\Big([l;\infty)\Big).$$

As

$$\sum_{l=1}^{\infty} l\, q_{\alpha',\lambda_i'}(l)$$

is finite (cf. Theorem 3.13.2), it follows from (i) and from the representation of $\Lambda(\alpha', \lambda')$ as

$$\Lambda(\alpha', \lambda') = \sum_{l=1}^{\infty} q_{\alpha',\lambda'}\Big([l;\infty)\Big) \qquad ((\alpha', \lambda') \in C)$$

that Λ is isotone in the second argument. The fact that Λ is antitone in the first argument follows analogously. □

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