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INVARIANTS OF ONE-DIMENSIONAL DIFFUSION PROCESSES AND APPLICATIONS

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ABSTRACT. One-dimensional diffusion processes are characterized by Feller's data of canonical scales and speed measures and, if we apply the theory of spectral functions of strings developed by M. G. Krein, Feller's data are determined by paris of spectral characteristic functions so that theses pairs may be considered as invariants of diffusions under the homeomorphic change of state spaces. We show by examples how these invariants are useful in the study of one-dimensional diffusion processes.

1. Introduction

The general theory of one-dimensional diffusion processes has been established for more than forty years by the works of W. Feller, K. Itô, H. P. McKean and E. B. Dynkin, among others, (cf. e.g., [3]). The analytical part of the theory consists of such notions as *Feller's generalized second-order differential operator* and *Feller's boundary condition* which are defined by giving a pair of *canonical scale* and *speed measure* (cf. [2], [3], [9]). On the other hand, M. G. Krein ([7]) studied in early fifties a *spectral theory of strings* by generalizing the classical work of Stieltjes ([10]) on continued fractions and moment problems (cf. e.g., [1]). Krein's theory may be said as a spectral theory for Feller's operators; it seems to us, however, that applications of Krein's theory to one-dimensional diffusion processes have not been seriously considered for a long time until a work by Kasahara ([4]) appeared in which

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an application to a limit theorem of the Darling-Kac type for additive functionals of one-dimensional diffusions has been studied.

A main purpose of this note is to give a survey on applications of Krein's theory to one-dimensional generalized diffusion processes or gap diffusions. A key in this study is an observation that a Feller's generalized second-order differential operator with a Feller's boundary condition at a regular boundary point can be given, if the scale is suitably normalized, by a pair of strings so that the corresponding pair of spectral characteristic functions determines the associated diffusion up to a homeomorphism of the state interval. In other words, a pair of spectral characteristic functions plays a role of an invariant of one-dimensional diffusion processes under a homeomorphic change of the state space. Also, the one-to-one homeomorphic property of Krein's correspondence between a string and its spectral characteristic function can be used, for example, to determine the domain of attraction in several limit theorems for one-dimensional diffusion processes in terms of their Feller's data.

2. Krein's correspondence

DEFINITION 1. By a string m, we mean a right-continuous and non-decreasing function m defined on $[0,\ell)$ for some $\ell = \ell(m), 0 < \ell \leq \infty$, with values in $[0,\infty)$.

Thus, m may be identified with the Radon measure dm on $[0,\ell)$, so that there is a one-to-one correspondence: $m \leftrightarrow (dm,\ell)$.

REMARK 1. In this note, by a Radon measure, we always mean a nonnegative Borel measure which is finite on every compact set.

We set $m(0-) = 0$ and $m(x) = \infty$ for $x \geq \ell$ when $\ell < \infty$. Then the function

$$m: \ [0,\infty) \ni x \mapsto m(x) \in [0,\infty]$$

is always right-continuous and non-decreasing. Let $\mathcal{M}$ be the totality of strings and $\mathcal{M}^+ = \mathcal{M} \setminus \{0\}$ where 0 is the string defined by $\ell = \infty$ and $m(x) \equiv 0$. It is sometimes convenient to consider the "infinite string" denoted by ∞ and defined by $\ell = 0$, or equivalently, by $m(x) \equiv \infty$. Let $\overline{\mathcal{M}} = \mathcal{M} \cup \{\infty\}$ and introduce the topology of $\overline{\mathcal{M}}$ as follows:

DEFINITION 2. For $m_n, m \in \overline{\mathcal{M}}$, we define $m_n \to m$ in $\overline{\mathcal{M}}$ as $n \to \infty$ if $m_n(x) \to m(x)$ for every $x \in [0,\infty)$ which is a continuity point of $[0,\infty]$-valued function $m(x)$.

As we shall see, $\overline{\mathcal{M}}$ is compact and metrizable under this topology. Introduce the following space of functions in $\lambda > 0$:

DEFINITION 3.

$$\mathcal{H} = \Big\{ h(\lambda) = c + \int_{[0,\infty)} \frac{1}{\lambda+\xi}\sigma(d\xi) \;\Big|\; c \geq 0 \text{ and } \tag{1}$$
$$\sigma : \text{a Radon measure on } [0,\infty) \text{ such that}$$
$$\int_{[0,\infty)} \frac{1}{1+\xi}\sigma(d\xi) < \infty \Big\}.$$

Set $\mathcal{H}^+ = \mathcal{H} \setminus \{0\}$ and $\overline{\mathcal{H}} = \mathcal{H} \cup \{\infty\}$ where 0 denotes the function $h(\lambda) \equiv 0$ and ∞ the function $h(\lambda) \equiv \infty$. We introduce the topology of $\overline{\mathcal{H}}$ by the pointwise convergence:

DEFINITION 4. For $h_n, h \in \overline{\mathcal{H}}$, we define $h_n \to h$ in $\overline{\mathcal{H}}$ as $n \to \infty$ if $h_n(\lambda) \to h(\lambda)$ for every $\lambda > 0$.

This topology is compact metrizable. To see this, we first note that $h \in \mathcal{H}$ if and only if it is expressed uniquely as

$$h(\lambda) = \int_{[0,\infty]} \frac{1+\xi}{\lambda+\xi} \cdot \mu(d\xi)$$

by some finite Radon measure on $[0,\infty]$. Indeed, $\mu(\{\infty\}) = c$ and $\mu(d\xi) = (1+\xi)^{-1}\sigma(d\xi)$ on $[0,\infty)$. If $h_n, h \in \mathcal{H}$, then $h_n(\lambda) \to h(\lambda)$ for every $\lambda > 0$ if and only if the finite Radon measures μ_n corresponding to h_n converge, as $n \to \infty$, to μ corresponding to h in the Prohorov topology. The Prohorov topology on the set of all finite Radon measures on a compact metrizable set is locally compact and metrizable with a countable open base. Then $\mathcal{H}$ is locally compact and metrizable with a countable base and hence, $\overline{\mathcal{H}}$ as its one-point compactification, is compact and metrizable.

For given $m \in \mathcal{M}$, let $\phi(x,\lambda)$ and $\psi(x,\lambda)$ be the solutions of the following integral equations on the interval $x \in [0,\ell)$:

$$\phi(x,\lambda) = 1 + \lambda \int_0^x dy \int_{0-}^y \phi(z,\lambda)dm(z), \tag{2}$$

$$\psi(x,\lambda) = x + \lambda \int_0^x dy \int_{0-}^y \psi(z,\lambda)dm(z). \tag{3}$$

$\phi(x,\lambda)$ and $\psi(x,\lambda)$ are uniquely determined by these equations and, as functions of λ, can be extended to entire functions on the complex plane for each fixed $x \in [0,\ell)$.

DEFINITION 5. To $m \in \mathcal{M}$, we associate a function h given by

$$h(\lambda) = \int_0^\ell \frac{dx}{\phi(x,\lambda)^2} = \lim_{x\uparrow\ell} \frac{\psi(x,\lambda)}{\phi(x,\lambda)}, \quad \lambda > 0. \tag{4}$$

To the infinite string $m = \infty$, we associate $h(\lambda) \equiv 0$. The function h thus associated to $m \in \overline{\mathcal{M}} = \mathcal{M} \cup \{\infty\}$ is called the *spectral characteristic function* of the string m.

Note that $h(\lambda) \equiv \infty$ if $m = 0$. Otherwise, it can be shown that $h \in \mathcal{H}$ so that the map $m \mapsto h$ defined in Definition 5 is a map from $\overline{\mathcal{M}}$ to $\overline{\mathcal{H}}$. More precisely, we have the following key result due to M. G. Krein (cf. [1]):

THEOREM 1. *The mapping $m \longmapsto h$ of* Definition 5 *defines a one-to-one onto correspondence between $\overline{\mathcal{M}}$ and $\overline{\mathcal{H}}$. Furthermore it is a homeomorphism with respect to the topologies introduced in* Definition 2 *and* Definition 4.

The correspondence $m \longleftrightarrow h$ between spaces $\overline{\mathcal{M}}$ and $\overline{\mathcal{H}}$ in Theorem 1 is called *Krein's correspondence.* Note that $\mathcal{M}^+$ corresponds to $\mathcal{H}^+$ exactly and $0 \longleftrightarrow \infty$, $\infty \longleftrightarrow 0$, in this correspondence. Since $\overline{\mathcal{H}}$ is compact and metrizable, so is $\overline{\mathcal{M}}$. Furthermore, if

$$m(x) \leftrightarrow \{dm(x), \ell\} \longleftrightarrow h(\lambda) = c + \int_{[0,\infty)} \frac{1}{\lambda+\xi}\sigma(d\xi),$$

we have

$$\ell = \lim_{\lambda\downarrow 0} h(\lambda) = c + \int_{0-}^{\infty} \frac{d\sigma(\xi)}{\xi},$$

$$c = \sup\{x : m(x) = 0\}, \quad m(0) = \lim_{\lambda\uparrow\infty} \frac{1}{\lambda h(\lambda)}$$

and

$$dm([0,\infty)) = \lim_{\lambda\downarrow 0} \frac{1}{\lambda h(\lambda)},$$

$$\text{when } \ell = \infty, \quad \left(\text{note that } \lim_{\lambda\downarrow 0} \frac{1}{\lambda h(\lambda)} = \infty \text{ if } \ell < \infty\right).$$

Also, the following are important properties of Krein's correspondence: If $m(x) \longleftrightarrow h(\lambda)$, then

$$\frac{b}{a}m(\frac{x}{a}) \longleftrightarrow ah(b\lambda) \quad \text{for} \quad a > 0, b > 0 \quad \text{and} \quad m^{-1}(x) \longleftrightarrow \frac{1}{\lambda h(\lambda)}$$

where $m^{-1}(x)$ is the string defined by $m^{-1}(x) := \inf\{u | m(u) > x\}$, $(\inf \emptyset = \infty)$.

Feller's data and strings

Suppose we are given, on an interval $[0, b), 0 < b \leq \infty$, a strictly increasing continuous function $s(x)$ such that $s(0) = 0$ and a Radon measure $m(dx)$ on $[0, b)$. Following Feller and Itô-McKean (cf. [3]), we call $s(x)$ and $m(dx)$ a *canonical scale* and *a speed measure* on $[0, b)$, respectively. Given such a pair $\{s(x), m(dx)\}$, Feller's generalized differential operator $\frac{d}{dm}\frac{d}{ds}$ is defined as in [3].

The right-boundary point b is called *regular* if

$$s(b-) + m([0, b)) < \infty.$$

When b is regular, *Feller's boundary condition* is given in the form

$$p_1 u(b) + p_2 \frac{du}{ds}(b) + p_3 \frac{d}{dm}\frac{du}{ds}(b) = 0$$

where $p_i \geq 0$ and $p_1 + p_2 + p_3 = 1$.

DEFINITION 6. A pair $\{s(x), m(dx)\}$ together with parameters $\{p_1, p_2, p_3\}$ in a Feller's boundary condition at b when b is regular is called a Feller's data on $[0, b)$.

For a given Feller's data, we can associate a string $\hat{m} \in \mathcal{M}$ as follows:

(1) If b is not regular, define $\hat{m}$ by $\ell (= \ell(\hat{m})) = s(b-)$ and

$$\hat{m}(x) = \begin{cases} \int_{[0, s^{-1}(x)]} m(dy), & x < \ell \\ \infty, & x \geq \ell \quad \text{if} \quad \ell < \infty \end{cases}$$

where $s^{-1}(x) = \min\{u | s(u) = x\}$.

(2) If b is regular, define $\hat{m}$ by

$$\hat{m}(x) = \begin{cases} \int_{[0, s^{-1}(x)]} m(dy), & x < s(b-) \\ \int_{[0, b)} m(dy) + \frac{p_3}{p_2}, & s(b-) \leq x < s(b-) + \frac{p_2}{p_1} := \ell \\ \infty, & x \geq \ell \quad \text{if} \quad \ell < \infty \end{cases}.$$

DEFINITION 7. The spectral characteristic function h corresponding to the string $\hat{m}$ defined as above from a Feller's data $\{s(x), m(dx)\}$ (with $\{p_i\}$ if necessary) is called the spectral characteristic function corresponding to the Feller's data.

In the following, we give several examples of Krein's correspondence:

EXAMPLE 1. (1) For given $y_0 = 0, a_0 \geq 0$ and $y_i > 0, a_i > 0$ for $i = 1, \ldots, N$, let $m \in \mathcal{M}$ be defined by

$$dm(x) = \sum_{i=0}^{N} a_i \delta_{y_0+y_1+\cdots+y_i}, \quad \ell = \infty.$$

Then $h \in \mathcal{H}$ corresponding to m in Krein's correspondence is given by

$$h(\lambda) = \cfrac{1}{a_0\lambda + \cfrac{1}{y_1 + \cfrac{1}{a_1\lambda + \cfrac{1}{\ddots + \cfrac{1}{a_{N-1}\lambda + \cfrac{1}{y_N + \cfrac{1}{a_N\lambda}}}}}}}.$$

(2) For given y_i and a_i as in (1), let $m \in \mathcal{M}$ be defined by

$$dm(x) = \sum_{i=0}^{N-1} a_i \delta_{y_0+y_1+\cdots+y_i}, \quad \ell = y_1 + \ldots + y_N.$$

Then $h \in \mathcal{H}$ corresponding to m in Krein's correspondence is given by

$$h(\lambda) = \cfrac{1}{a_0\lambda + \cfrac{1}{y_1 + \cfrac{1}{a_1\lambda + \cfrac{1}{\ddots + \cfrac{1}{a_{N-1}\lambda + \cfrac{1}{y_N}}}}}}.$$

Thus, when a string is given by a discrete measure, the corresponding spectral characteristic function is a Stieltjes continued fraction.

EXAMPLE 2. (Bessel operator) Let $m(x) = x^{1/\alpha - 1}$, $\ell = \infty$ for $0 < \alpha < 1$. Then

$$h(\lambda) = \frac{1}{[\alpha(1-\alpha)]^\alpha} \frac{\Gamma(1+\alpha)}{\Gamma(1-\alpha)} \cdot \frac{1}{\lambda^\alpha}.$$

An essentially same statement is that, if $(s(x), m(dx))$ is a pair of canonical scale and speed measure on $[0, \infty)$ given by

$$s(x) = \int_0^x y^{1-\delta} dy, \quad m(dx) = 2x^{\delta-1}dx, \quad (0 < \delta < 2), \tag{5}$$

then the spectral characteristic function corresponding to this Feller's data is given by

$$h(\lambda) = \frac{\Gamma(1-\frac{\delta}{2})}{2^{\frac{\delta}{2}}\Gamma(\frac{\delta}{2})} \cdot \frac{1}{\lambda^{1-\frac{\delta}{2}}}. \tag{6}$$

Note that Feller's differential operator $\frac{d}{dm} \cdot \frac{d}{ds}$ is just the Bessel differential operator

$$L^{(\delta)} = \frac{1}{2}\frac{d^2}{dx^2} + \frac{\delta - 1}{2x}\frac{d}{dx}. \tag{7}$$

EXAMPLE 3. (Bessel operator with drift) The function $h(\lambda)$ in (6) is a particular case of the following family of functions:

(8)

$$h(\lambda) = \frac{\Gamma(1-\frac{\delta}{2})}{2^{\frac{\delta}{2}}\Gamma(\frac{\delta}{2})} \cdot \frac{1}{(\lambda + c)^{1-\frac{\delta}{2}} + c^{1-\frac{\delta}{2}}\theta}, \quad 0 < \delta < 2, \quad c \geq 0, \quad \theta \geq -1.$$

This class of functions is contained in $\mathcal{H}$; $h(\lambda)$ in (8) is indeed the spectral characteristic function corresponding to the Feller's data on $[0, \infty)$ given by

$$s(x) = \int_0^x y^{1-\delta} \rho_{\delta,c,\theta}^{-2}(y) dy \quad \text{and} \quad m(dx) = 2x^{\delta-1}\rho_{\delta,c,\theta}^2(x)dx \tag{9}$$

where

$$\rho_{\delta,c,\theta}(x) = \varphi_\delta(x; c) + \theta\psi_\delta(x; c). \tag{10}$$

Here, we set generally for $\lambda > 0$ and $x \in (0, \infty)$,

$$\varphi_\delta(x;\lambda) = \Gamma\left(\frac{\delta}{2}\right)\left(\frac{\sqrt{2\lambda}x}{2}\right)^{1-\frac{\delta}{2}} I_{\frac{\delta}{2}-1}(\sqrt{2\lambda}x), \tag{11}$$

$$\psi_\delta(x;\lambda) = \Gamma\left(\frac{\delta}{2}\right)\left(\frac{\sqrt{2\lambda}x}{2}\right)^{1-\frac{\delta}{2}} I_{1-\frac{\delta}{2}}(\sqrt{2\lambda}x), \tag{12}$$

$I_\nu(x)$ being the modified Bessel functions.

Note that Feller's differential operator $\frac{d}{dm}\cdot\frac{d}{ds}$ is given by the following differential operator which we call a *Bessel differential operator with drift*:

$$L^{(\delta,c,\theta)} = \frac{1}{2}\frac{d^2}{dx^2} + \left[\frac{\delta-1}{2x} + \frac{\rho'_{\delta,c,\theta}(x)}{\rho_{\delta,c,\theta}(x)}\right]\frac{d}{dx}. \tag{13}$$

3. Generalized diffusion on a line corresponding to a pair of strings

Given a pair $\mathbf{m} = (m_+, m_-)$, $m_\pm \in \mathcal{M}$, such that $m_-(0) = 0$, we define the Radon measure $\mathbf{m}(dx)$ on $(-l_-, l_+)$ by

$$\mathbf{m}(dx) = \begin{cases} dm_+(x) & \text{on} \quad [0, \ell_+) \\ d\check{m}_-(x) & \text{on} \quad (-\ell_-, 0) \end{cases} \tag{14}$$

where $d\check{m}_-$ is the image measure of dm_- under the reflexion $x \longrightarrow -x$. Let E_m be the union of the support in $(-\ell_-, \ell_+)$ of $\mathbf{m}(dx)$ and the boundary point $-\ell_-$ or ℓ_+ whichever is finite.

DEFINITION 8. By the generalized diffusion process $\mathbf{X} = (X_t, P_x)$ corresponding to the pair $\mathbf{m}$, we mean a Hunt process (a time-homogeneous strong Markov process with càdlàg and quasi-càg paths) on the E_m, with point $-\ell_-$ or ℓ_+ as the trap, obtained from the Brownian motion $B(t)$ on $\mathbf{R}$ (starting at $x \in E_m$) by the time change:

$$X_t = B(\phi_t^{-1}).$$

Here,

$$\phi_t = \begin{cases} \int_{(-\ell_-,\ell_+)} \ell(t,x)\mathbf{m}(dx), & t < \sigma_{-\ell_-} \wedge \sigma_{\ell_+}, \\ \infty, & t \geq \sigma_{-\ell_-} \wedge \sigma_{\ell_+}, \end{cases}$$

$$\phi_t^{-1} = \inf\{u \mid \phi_u > t\},$$

$\ell(t,x)$ is the sojourn time density of the process $B(t)$ with respect to the measure $2dx$, i.e.,

$$\ell(t,x) = \lim_{\epsilon \downarrow 0} \frac{1}{4\epsilon} \int_0^t 1_{(x-\epsilon, x+\epsilon)}(B(s))ds, \quad t \geq 0, \quad x \in \mathbf{R}$$

and

$$\sigma_a = \inf\{t \mid B(t) = a\}, \quad a \in \mathbf{R}.$$

Since the measure $\mathbf{m}(dx)$ need not be everywhere positive on $(-\ell_-, \ell_+)$, X_t may jump to a neighboring point in E_m (gap diffusion) so that when $\mathbf{m}(dx)$ is a discrete measure, X_t is a birth and death process. The process $\mathbf{X}$ is transient if and only if $\ell_+ \wedge \ell_- < \infty$ and, in this case,

$$P_x(\lim_{t \uparrow \zeta} X_t = \ell_+) = 1 - P_x(\lim_{t \uparrow \zeta} X_t = -\ell_-) = \frac{x + \ell_-}{\ell_+ + \ell_-}, \quad -\ell_- < x < \ell_+,$$

where ζ is the first hitting time to the boundary points. ζ is identified with the life time of the process so that the boundary points may be identified with the terminal point the process.

It is important to remark that every non-singular diffusions process on an interval can be obtained, up to a homeomorphic change of the state interval, as a generalized diffusion process corresponding to a pair of strings. Consider an interval Q and let $X = \{X(t), P_x\}_{x \in Q}$ be a time-homogeneous strong Markov process on Q with continuous paths up to the life time ζ. We denote $\overline{Q} = [a, b]$ and $Q^\circ = (a, b)$ so that $Q^\circ \subset Q \subset \overline{Q}$ and call a and b boundary points of Q. These points may, or may not, belong to Q. X is called *regular* if every $x \in Q^\circ$ is a regular point in Feller's sense; i.e., $P_x(m_{x+} = m_{x-} = 0) = 1$ where m_x is the hitting time to x and

$$m_{x+} = \lim_{y \downarrow x} m_y, \quad m_{x-} = \lim_{y \uparrow x} m_y.$$

X is called *conservative* if $P_x(\zeta = \infty) = 1$ for all $x \in Q$ and *semi-conservative* if there is no killing inside Q; i.e., $P_x(\zeta \geq m_{a+} \wedge m_{b-}) = 1$ for all $x \in Q^\circ$.

In this note, we mainly consider a time homogeneous, regular, semi-conservative diffusion on an interval Q with the property $P_x(m_y < \infty) > 0$ or $P_x(m_y < \infty) > 0$ for all $x, y \in Q$ and call such a diffusion a *non-singular diffusion*. Suppose we are given a non-singular diffusion $X = \{X(t), P_x\}_{x \in Q}$ on an interval Q. Since X is regular, there are associated a canonical scale $s(x)$ which is a strictly increasing continuous function on Q° and a speed measure $m(dx)$ which is an everywhere positive Radon measure on Q°, (cf. [3]). Take a point c such that $a < c < b$ and define a pair $\{s_+(x), m_+(dx)\}$ of the scale and measure on $[0, b-c)$ and another pair $\{s_-(x), m_-(dx)\}$ on $[0, c-a)$ by

$$s_+(x) = s(c+x) - s(c),\ 0 \le x < b-c, \qquad m_+(dx) = m(c+dx)|_{[0,b-c)}$$

and

$$s_-(x) = s(c) - s(c-x),\ 0 \le x < c-a, \qquad m_-(dx) = \check{m}(dx-c)|_{(0,c-a)},$$

respectively. If a boundary point a or b is regular, i.e., $b-c$ is regular with respect to the pair $\{s_+(x), m_+(dx)\}$ or $c-a$ is regular with respect to the pair $\{s_-(x), m_-(dx)\}$, then we have a Feller's boundary condition there associated to the process X with parameters $\{p_i^+\}$ or $\{p_i^-\}$, (cf. [3]), so that we have two Feller's data: $\{s_+(x), m_+(dx)\}$ (together with $\{p_i^+\}$ if necessary) and $\{s_-(x), m_-(dx)\}$ (together with $\{p_i^-\}$ if necessary). Conversely, by the general theory of Feller and Itô-McKean, these two Feller's data are enough to determine the process X. Thus, analytically, it is equivalent to give a non-singular diffusion and a pair of Feller's data.

Finally, let $\hat{m}_+$ and $\hat{m}_-$ be strings corresponding to the Feller's data $\{s_+(x), m_+(dx)\}$ (together with $\{p_i^+\}$ if necessary) and to $\{s_-(x), m_-(dx)\}$ (together with $\{p_i^-\}$ if necessary), respectively. Let $\hat{\mathbf{X}} = \{\hat{X}(t), \hat{P}_x\}$ be the generalized diffusion process corresponding to the pair $\hat{\mathfrak{m}} = (\hat{m}_+, \hat{m}_-)$ of strings. The following theorem is a way of rephrasing sample paths construction, due to Itô-McKean ([3]), of a non-singular diffusion from its analytical data:

THEOREM 2.

$$\{s(X(t)) - s(c), P_x\}_{x \in \tilde{Q}} \stackrel{d}{=} \left\{\hat{X}(t), \hat{P}_{s(x)}\right\}_{x \in \tilde{Q}}$$

where $\tilde{Q}$ is obtained from Q by deleting the boundary point a or b whichever is in Q and at which the scale $s(x)$ is unbounded, i.e., $s(a+) = -\infty$ or $s(b-) = \infty$.

As for the restriction of the interval Q to $\tilde{Q}$, see Remark 2,(ii) below. In this way, we see that every non-singular diffusion can be essentially obtained as a generalized diffusion corresponding to a pair of strings. The pair of spectral characteristic functions corresponding to this pair of strings (or the pair of Feller's data) may be considered as an invariant of non-singular diffusion X; indeed, we have the following theorem which is essentially based on the uniqueness part of Krein's correspondence:

THEOREM 3. *Let X and X' be two non-singular diffusions on intervals Q and Q', respectively, and let $(h_+(\lambda), h_-(\lambda))$ and $(h'_+(\lambda), h'_-(\lambda))$ be pairs of spectral characteristic functions associated to them, respectively. If there exists some constant $k > 0$ such that*

$$(h_+(\lambda), h_-(\lambda)) = k(h'_+(\lambda), h'_-(\lambda)),$$

then there exists a homeomorphism $H : Q \to Q'$ such that $X = H^{-1}(X')$.

Note that $H(c) = c'$ where c and c' are points in Q° and Q'°, respectively, with respect to which, pairs of Feller's data are defined as above.

REMARK 2. Consider a non-singular diffusion X on an interval $Q = [a, b)$.
(i) If a is a regular boundary point and Feller's boundary condition at a is reflecting, i.e., $p_1 = p_3 = 0$, then it is essentially a generalized diffusion corresponding to a pair of strings $\mathbf{m} = (m_+, m_-)$ with $m_- = 0$ and m_+ is the string associated to the scale $s_+(x) = s(x) - s(a)$ and the measure $m_+(dx) = m(a + dx)$ on $[0, b - a)$ together with parameters $\{p_i^+\}$ in Feller's boundary condition when b is regular. So X is given essentially by a single string m_+.
(ii) There is a case of X when a is not a regular boundary point but an entrance boundary point in the sense of Feller (cf. [3]), e.g., $Q = [0, \infty)$ with the scale $s'(x) = x^{1-\delta}$ and the measure $m(dx)$ given by (5) for $\delta \geq 2$. We can construct the process starting at a so that it enters into (a, b) immediately and can not return to a. However, for the corresponding generalized diffusion $\hat{\mathbf{X}}$, the point $-\ell_-$ corresponding to a is now $-\infty$ so that it is not in the state interval of $\hat{\mathbf{X}}$. This suggests that there should be some extension of Krein's theory for strings with entrance left-end points. This problem was studied by Kotani ([5]); its application to theory of diffusions, however, seems to remain still open.

So far, we considered the case of diffusions. Since $m(dx)$ in (14) need not be everywhere positive, a generalized diffusion corresponding to a pair of strings may have discontinuous sample functions. The following is a typical application of such a case to the study of random walks:

Imbedding of a space-dependent random walk into a generalized diffusion

Let $\Xi = \{\xi_n, P_i^{\Xi}\}_{i\in\mathbf{Z}}$ be a space-dependent and time-homogeneous simple random walk on the one-dimensional lattice $\mathbf{Z}$ with one-step transition probability given by

$$P_\cdot^{\Xi}(\xi_{n+1} = j|\xi_n = i) = p_i\delta_{j,i+1} + q_i\delta_{j,i-1} + r_i\delta_{i,i}, \quad i, j \in \mathbf{Z},$$

where

$$p_i > 0,\ q_i > 0,\ r_i \geq 0, \quad p_i + q_i + r_i = 1.$$

Define two sequences $\{s_n\}_{n=0}^{\infty}$ and $\{t_n\}_{n=0}^{\infty}$ by

$$s_0 = 0, \quad s_n = \sum_{k=0}^{n-1}\prod_{i=0}^{k} q_i/p_i,\ n = 1, 2, \ldots \tag{15}$$

and

$$t_0 = 0, \quad t_1 = 1, \quad t_n = 1 + \sum_{k=1}^{n-1}\prod_{i=1}^{k} p_{-i}/q_{-i},\ n = 2, 3, \ldots. \tag{16}$$

Then we define $m_+, m_- \in \mathcal{M}$ by

(17)

$$m_+(x) = \begin{cases} \sum_{k=0}^{n} 1/p_k \prod_{i=0}^{k} p_i/q_i, & s_n \leq x < s_{n+1}, \quad n = 0, 1, \ldots \\ \infty, & x \geq s_\infty (= \lim_{n\uparrow\infty} s_n) := l_+ \end{cases}$$

and

(18)

$$m_-(x) = \begin{cases} 0, & t_0 = 0 \leq x < t_1 = 1 \\ \sum_{k=1}^{n} 1/q_{-k} \prod_{i=1}^{k} q_{-i}/p_{-i}, & t_n \leq x < t_{n+1}, \quad n = 1, 2, \ldots \\ \infty, & x \geq t_\infty (= \lim_{n\uparrow\infty} t_n) := \ell_-. \end{cases}$$

Let $\mathbf{X} = \{X(t), P_x\}$ be the generalized diffusion process (actually a birth and death process) corresponding to the pair $\mathbf{m} = \{m_+, m_-\}$ given by (17) and (18). The state space of $\mathbf{X}$ is the following discrete set:

$$\begin{aligned} E = \{\ldots < -t_{n+1} < -t_n < \ldots < -t_1 &= -1 < 0 \\ &= s_0 < \ldots < s_n < s_{n+1} < \ldots\}. \end{aligned}$$

Define a map $\Phi : E \to \mathbf{Z}$ by $\Phi(-t_n) = -n$, $n = 1, 2, \ldots$ and $\Phi(s_n) = n$, $n = 0, 1, \ldots$. We define a sequence $\{T_n\}$ of stopping times for $\mathbf{X}$ inductively as follows: Take a sequence of mutually independent exponential times $\{e_i^{(n)}\}$ which is also independent of Ξ such that

$$E(e_i^{(n)}) = 1/r_i, \quad i \in \mathbf{Z},\ n = 1, 2, \ldots, \quad (\ e_i^{(n)} = \infty\ a.s. \text{ if } r_i = 0).$$

Let

$$\tilde{T}^{(1)} = \inf\{t \geq 0 \mid X(t) \neq X(0)\} \quad \text{and} \quad T_1 = e^{(1)}_{\Phi(X(0))} \wedge \tilde{T}^{(1)}.$$

If $T_1, \cdots, T_n$ are already defined, set

$$\tilde{T}^{(n+1)} = \inf\{t \geq T_n \mid X(t) \neq X(T_n)\} \quad \text{and}$$
$$T_{n+1} = T_n + e^{(n+1)}_{\Phi(X(T_n))} \wedge (\tilde{T}^{(n+1)} - T_n).$$

It is easy to deduce that $T_1, T_2 - T_1, \ldots, T_n - T_{n-1}, \ldots$ are i.i.d. random variables with mean 1 exponential distribution.

THEOREM 4.

$$\{\xi_n, P_i^{\Xi}\}_{i \in \mathbf{Z}} \stackrel{d}{=} \{\Phi(X(T_n)), P_{\Phi^{-1}(i)}\}_{i \in \mathbf{Z}}.$$

4. Applications

(1) Generalized arc-sine laws

We consider a generalization of the classical arc-sine law for the ratio of occupation times on the positive side for one-dimensional Brownian motion and simple random walk (cf. [3], p.57 and p.40) to one-dimensional diffusions and space-dependent simple random walks. In order to describe possible limit random variables, we introduce the following

DEFINITION 9. For $0 \leq \alpha \leq 1$ and $0 \leq p \leq 1$, let $Y_{\alpha,p}$ be a $[0,1]$-valued random variable with the Stieltjes transform given by

$$E\left[\frac{1}{\lambda + Y_{\alpha,p}}\right] = \frac{p(\lambda+1)^{\alpha-1} + (1-p)\lambda^{\alpha-1}}{p(\lambda+1)^{\alpha} + (1-p)\lambda^{\alpha}}, \ \lambda > 0. \tag{19}$$

The family $\{Y_{\alpha,p}, 0 \leq \alpha \leq 1, 0 \leq p \leq 1\}$ was introduced by Lamperti ([8]) and is called *the Lamperti class of random variables.*

If $\alpha = 0$, $Y_{0,p}$ is the two valued random variable with values 0 and 1 such that

$$P(Y_{0,p} = 1) = 1 - P(Y_{0,p} = 0) = p.$$

If $\alpha = 1$, $Y_{1,p}$ is the constant random variable such that

$$P(Y_{1,p} = p) = 1.$$

When $0 < \alpha < 1$ and $0 < p < 1$, the law of $Y_{\alpha,p}$ has the density $f_{\alpha,p}(x)$, $0 \leq x \leq 1$, which can be obtained easily by inverting the Stieltjes transform:

(20)

$$f_{\alpha,p}(x) = \frac{\sin \alpha\pi}{\pi} \frac{p(1-p)x^{\alpha-1}(1-x)^{\alpha-1}}{p^2(1-x)^{2\alpha} + (1-p)^2 x^{2\alpha} + 2p(1-p)x^{\alpha}(1-x)^{\alpha}\cos \alpha\pi}.$$

Hence, $Y_{1/2,1/2}$ is arc-sine distributed, more generally,

$$P(Y_{1/2,p} \leq x) = \frac{2}{\pi}\sin^{-1}\sqrt{\frac{x}{x + (\frac{p}{1-p})^2(1-x)}}, \quad 0 \leq x \leq 1. \tag{21}$$

Barlow, Pitman and Yor [2] noticed the following remarkable expression of $Y_{\alpha,p}$:

$$Y_{\alpha,p} \stackrel{d}{=} \frac{p^{1/\alpha}S_\alpha}{p^{1/\alpha}S_\alpha + (1-p)^{1/\alpha}S'_\alpha} \tag{22}$$

where S_α and S'_α are independent copies of the positive normalized stable random variable with exponent α; $E(e^{-\lambda S_\alpha}) = e^{-\lambda^\alpha}$, $\lambda > 0$.

If $p = 0$ or $p = 1$, they are *trivial* in the sense that $Y_{\alpha,0} = 0, a.s.$ and $Y_{\alpha,1} = 1, a.s.$ In the nontrivial case of $0 < p < 1$, $Y_{\alpha,p} \stackrel{d}{=} Y_{\alpha',p'}$ if and only if $\alpha = \alpha'$ and $p = p'$. Also it is clear that

$$Y_{\alpha,1-p} \stackrel{d}{=} 1 - Y_{\alpha,p}.$$

Let $m_+, m_- \in \mathcal{M}$ and $\mathbf{X} = (X(t), P_x)_{x \in E_m}$ be the generalized diffusion process corresponding to the pair $\mathbf{m} = (m_+, m_-)$, (cf. Def. 8). We assume that $0 \in E_m$ and $X(0) = 0$ (that is, we consider under P_0), for simplicity. Consider the occupation times of $\mathbf{X}$ on the positive and negative sides:

$$A_+(t) = \int_0^t 1_{[0,\infty)}(X(s))ds, \quad A_-(t) = \int_0^t 1_{(-\infty,0)}(X(s))ds$$

and ask the following question: *What are possible limit random variables in law of $A_+(t)/t$ as $t \to \infty$ and when the limit exists in law?* An answer is given by the following

THEOREM 5. (1) *The class of possible random variables in law of $A_+(t)/t$ as $t \to \infty$ is the Lamperti class $\{Y_{\alpha,p}, 0 \leq \alpha \leq 1, 0 \leq p \leq 1\}$.*

(2) *The convergence in law*

$$\frac{1}{t}A_+(t) \xrightarrow{d} Y_{\alpha,p}, \quad t \to \infty$$

holds if (and only if when $0 < p < 1$) the pair $\mathbf{m}$ *satisfies the following conditions:*

(i) *If $0 < \alpha \leq 1$, then $\ell_+ = \ell_- = \infty$, $m_+(x)$ and $m_-(x)$ are regularly varying at ∞ with exponent $1/\alpha - 1$; that is,*

(23) $$m_\pm(x) = x^{1/\alpha - 1}K_\pm(x),$$

with slowly varying functions $K_\pm(x)$ at $x = \infty$ and furthermore

(24) $$\lim_{x\uparrow\infty} \frac{K_+(x)}{K_-(x)} = \frac{p^{1/\alpha}}{(1-p)^{1/\alpha}}.$$

(ii) *If $\alpha = 0$, then either* $\mathbf{X}$ *is transient, i.e. $\ell_+ \wedge \ell_- < \infty$ with*

$$\frac{\ell_-}{\ell_+ + \ell_-} = p$$

or $\ell_+ = \ell_- = \infty$ and there exists a continuous increasing function $f(x)$ of $x \in (0,\infty)$ such that $f(0) > 0$, $\lim_{x\uparrow\infty} f(x) = \infty$, its inverse $f^{-1}(s)$ is slowly varying at $s = \infty$ and satisfies both

(25) $$\lim_{s\uparrow\infty} \frac{m_+(xs)}{f(sp)} = \begin{cases} 0, & \text{if } 0 < x < 1 \\ \infty, & \text{if } x > 1 \end{cases}$$

and

(26) $$\lim_{s\uparrow\infty} \frac{m_-(xs)}{f(s(1-p))} = \begin{cases} 0, & \text{if } 0 < x < 1 \\ \infty, & \text{if } x > 1. \end{cases}$$

In the proof, a key role is played by a theorem due to Kasahara ([4] Th. 2, [6] Cor. to Th. 2.1) concerning a necessary and sufficient condition on a string under which its spectral characteristic function is regularly varying. This theorem of Kasahara is essentially based on Theorem 2 of Section 2. If we apply the imbedding result, Theorem 4, for a space-dependent random walk $\Xi = (\xi_n, P_i^\Xi)$, then we can deduce the following (cf. [12]):
Let $A_n = \sum_{k=1}^n 1_{\{\xi_k \geq 0\}}$. Then, as for the convergence in law of A_n/n as $n \to \infty$ under P_0^Ξ, we have the same conclusion as Theorem 4 *in terms of strings defined by* (17) *and* (18).

A remarkable fact in Theorem 5 is that we can determine the domain of attraction in the limit theorem in terms of strings, or equivalently, in terms of Feller's data. A similar result in the case of a limit theorem of the Darling-Kac type for occupation times of generalized diffusions has been obtained by Kasahara [4].

(2) Time inversion and bilateral Bessel diffusion processes with drift

We start with the following simple examples:

EXAMPLE 4. If $B(t)$ is a $BM^0(1)$ (1-dim. Brownian motion starting at 0), then

$$\left\{tB\left(\frac{1}{t}\right)\right\}_{t>0} \stackrel{d}{=} \{B(t)\}_{t>0}.$$

Indeed, the both are centered Gaussian processes with the same covariance $s \wedge t$.

EXAMPLE 5. Let $a, b \in \mathbf{R}$ and $B(t)$ be a $BM^0(1)$. If

$$X(t) = a + bt + B(t) \quad \text{and} \quad X'(t) = b + at + B(t),$$

then

$$\left\{tX\left(\frac{1}{t}\right)\right\}_{t>0} \stackrel{d}{=} \{X'(t)\}_{t>0}.$$

This can be deduced immediately from Example 4. Being motivated by these examples, we ask the following question: *What are the classes of non-singular diffusions X and X' on intervals Q and Q' and their initial laws μ and μ', respectively, such that, for a family $\{G_t, t > 0\}$ of homeomorphisms from Q onto Q';*

$$G_t:\ Q \ni x \longmapsto G(t,x) \in Q',$$

the following invariance property under the time inversion holds?

$$\left\{G\left(t, X\left(\frac{1}{t}\right)\right), P_\mu\right\}_{t>0} \stackrel{d}{=} \left\{X'(t), P'_{\mu'}\right\}_{t>0} \tag{27}$$

This problem has been discussed in [11] for unilateral diffusions on $Q = [a, b)$ and $Q' = [a', b')$ with the left boundaries a and a' regular and reflecting. Examples 4 and 5 are the case of $Q = Q' = \mathbf{R}$, $G(t,x) = tx$ with $\mu = \mu' = \delta_0$ and $\mu = \delta_a, \mu' = \delta_b$, respectively.

In order to discuss this problem, we introduce the following family of non-singulardiffusion processes:

DEFINITION 10. (1) (Bilateral case) Let $0 < \delta < 2, c \geq 0, \theta_+ \geq -1, \theta_- \geq -1$ and $0 < p < 1$. By the bilateral Bessel diffusion process with drift, of the dimension parameter δ, the drift parameter (c, θ_+, θ_-) and the skew parameter p, and denote it by $biBESD(\delta, c, \theta_+, \theta_-, p)$, we mean a conservative non-singular diffusion process $X(t)$ on $\mathbf{R}$ determined by the canonical scale

$$s(x) = \begin{cases} p^{-1} \int_0^x y^{1-\delta} \rho_{\delta,c,\theta_+}^{-2}(y) dy, & x \geq 0 \\ -(1-p)^{-1} \int_0^{|x|} y^{1-\delta} \rho_{\delta,c,\theta_-}^{-2}(y) dy, & x < 0 \end{cases} \tag{28}$$

and the speed measure

$$m(dx) = \begin{cases} 2px^{\delta-1} \rho_{\delta,c,\theta_+}^2(x) dx, & x \geq 0 \\ 2(1-p)|x|^{\delta-1} \rho_{\delta,c,\theta_-}^2(|x|) dx, & x < 0. \end{cases} \tag{29}$$

(2) (Unilateral case) This is the extreme case of $p = 0$ and $p = 1$. So let $0 < \delta < 2, c \geq 0, \theta \geq -1$. By a (unilateral) Bessel diffusion process with drift on the positive (negative) side with the dimension parameter δ and the drift parameter (c, θ), and denote it by $BESD^+(\delta, c, \theta)$ (resp. $BESD^-(\delta, c, \theta)$), we mean a conservative non-singular diffusion on $\mathbf{R}^+ = [0, \infty)$ (resp. on $\mathbf{R}^- = (-\infty, 0]$), with the scale and speed measure given by

$$s(x) = \int_0^x y^{1-\delta} \rho_{\delta,c,\theta}^{-2}(y) dy, \quad x \geq 0, \tag{30}$$

$$(\text{resp.} \quad s(x) = -\int_0^{|x|} y^{1-\delta} \rho_{\delta,c,\theta}^{-2}(y) dy, \quad x < 0)$$

and

$$m(dx) = 2x^{\delta-1} \rho_{\delta,c,\theta}^2(x) dx, \quad x \geq 0, \tag{31}$$

$$(\text{resp.} \quad m(dx) = 2|x|^{\delta} \rho_{\delta,c,\theta}^2(|x|) dx, \quad x < 0).$$

Furthermore, we would like to define unilateral Bessel diffusion processes with drift for $\delta \geq 2$. In order for the boundary 0 to be entrance, we assume in this case $\theta = 0$. That is, we only consider $BESD^+(\delta, c, 0)$ or $BESD^-(\delta, c, 0)$ when $\delta \geq 2$. To be precise, by $BESD^+(\delta, c, 0)$ (resp. $BESD^-(\delta, c, 0)$) for $\delta \geq 2$ and $c \geq 0$, we mean a conservative non-singular diffusion on $\mathbf{R}^+ = [0, \infty)$ (resp. on $\mathbf{R}^- = (-\infty, 0]$), with the scale and speed measure given by

(32)
$$s(x) = \int_1^x y^{1-\delta}\varphi_\delta(y;c)^{-2}dy, \; x \geq 0,$$
$$(\text{resp.} \quad s(x) = -\int_1^{|x|} y^{1-\delta}\varphi_\delta(y;c)^{-2}dy, \; x < 0)$$

and

(33)
$$m(dx) = 2x^{\delta-1}\varphi_\delta(x;c)^2dx, \; x \geq 0,$$
$$(\text{resp.} \quad m(dx) = 2|x|^{\delta-1}\varphi_\delta(|x|;c)^2dx, \; x < 0).$$

Cf. (10), (11) and (12) for the definition of $\rho_{\delta,c,\theta}$ and $\varphi_\delta(x;c)$. Thus, in the unilateral case, the boundary 0 is regular and reflecting for $0 < \delta < 2$ while non-regular and entrance if $\delta \geq 2$. In the latter case, a path starting at 0 enters immediately into $(0,\infty)$ (resp. $(-\infty,0)$) and can never come back to 0.

REMARK 3. When $c = 0$, $biBESD$, ($BESD^+$, $BESD^-$) does not depend on $\theta_\pm$ (resp. θ). In this case, it is simply called a bilateral skew Bessel process and denoted by $skewBES(\delta,p)$, (resp. unilateral Bessel process and denoted by $BES^+(\delta)$ or $BES^-(\delta)$).

The following theorems solve our problem in a particular case of $G(t,x) = tx$. First we consider the bilateral case:

THEOREM 6. *Let $X = (X(t))$ and $X' = (X'(t))$ be non-singular diffusion on* $\mathbf{R}$ *with the initial distributions μ and μ', respectively. In order for the following invariance under the time inversion:*

(34)
$$\left\{tX\left(\frac{1}{t}\right)\right\}_{t>0} \stackrel{d}{=} \{X'(t)\}_{t>0}$$

to hold, it is necessary and sufficient that the following (i), (ii) and (iii) are satisfied:

(i)
$$X = biBESD(\delta,c,\theta_+,\theta_-,p) \quad \text{and} \quad X' = biBESD(\delta,c',\theta'_+,\theta'_-,p)$$
with
$$0 < \delta < 2, \; c \geq 0, \; c' \geq 0, \; \theta_\pm \geq -1, \; \theta'_\pm \geq -1, \; 0 < p < 1.$$

(ii) *These parameters satisfy*
$$p\theta_+ + (1-p)\theta_- = 0 \quad \text{and} \quad p\theta'_+ + (1-p)\theta'_- = 0.$$

(iii) *The initial distributions* μ *and* μ' *are given by*

$$\begin{aligned}\mu &= K\left[p(1+\theta'_+)(\phi+\theta_+\psi)\delta_{\sqrt{2c'}} + (1-p)(1+\theta'_-)(\phi+\theta_-\psi)\delta_{-\sqrt{2c'}}\right],\\ \mu' &= K\left[p(1+\theta_+)(\phi+\theta'_+\psi)\delta_{\sqrt{2c}} + (1-p)(1+\theta_-)(\phi+\theta'_-\psi)\delta_{-\sqrt{2c}}\right],\end{aligned}$$

where δ_a *is the unit mass at* $a \in \mathbf{R}$,

$$\begin{aligned}\phi &= \varphi_\delta(\sqrt{2c'};c) = \varphi_\delta(\sqrt{2c};c') = \Gamma\left(\frac{\delta}{2}\right)\sqrt{cc'}^{1-\frac{\delta}{2}} I_{\frac{\delta}{2}-1}(2\sqrt{cc'}),\\ \psi &= \psi_\delta(\sqrt{2c'};c) = \psi_\delta(\sqrt{2c};c') = \Gamma\left(\frac{\delta}{2}\right)\sqrt{cc'}^{1-\frac{\delta}{2}} I_{1-\frac{\delta}{2}}(2\sqrt{cc'}),\end{aligned}$$

and

$$\begin{aligned}K &= \{p(1+\theta'_+)(\phi+\theta_+\psi) + (1-p)(1+\theta'_-)(\phi+\theta_-\psi)\}^{-1}\\ &= \{p(1+\theta_+)(\phi+\theta'_+\psi) + (1-p)(1+\theta_-)(\phi+\theta'_-\psi)\}^{-1}\\ &= \{\phi + [p\theta_+\theta'_+ + (1-p)\theta_-\theta'_-]\psi\}^{-1}\end{aligned}$$

so that μ *and* μ' *are probability measures.*

In the unilateral case, we consider only the case of the interval $\mathbf{R}^+ = [0,\infty)$; the case of the interval $\mathbf{R}^- = (-\infty, 0]$ being essentially the same.

THEOREM 7. *Let* $X = (X(t))$ *and* $X' = (X'(t))$ *be non-singular diffusions on* $\mathbf{R}^+ = [0,\infty)$ *with the initial distributions* μ *and* μ', *respectively. In order for the invariance property* (34) *to hold, it is necessary and sufficient that*

$$X = BESD^+(\delta, c, 0) \quad \text{and} \quad X' = BESD^+(\delta, c', 0)$$

with

$$\delta > 0,\ c \geq 0,\ c' \geq 0$$

and the initial distributions μ *and* μ' *are given by*

$$\mu = \delta_{\sqrt{2c'}} \quad \text{and} \quad \mu' = \delta_{\sqrt{2c}}.$$

We could not answer our question proposed above in full generality. However, under a certain restriction on the family $\{G_t, t > 0\}$ of homeomorphisms, we can show that Theorems 6 and 7 describe essentially all possible solutions. We consider two cases of state intervals: $Q = (a, b)$, $Q' = (a', b')$ (the bilateral case) and $Q = [a, b)$, $Q' = [a', b')$ (the unilateral case). In the bilateral case, we assume that the family $\{G_t, t > 0\}$ of homeomorphisms; $G_t : Q \ni x \to G(t, x) \in Q'$, satisfies $G(t, c) = c'$ for all $t > 0$, for some c, c' such that $a < c < b$ and $a' < c' < b'$. In the unilateral case, it should satisfy that $G(t,a) = a'$ for all $t > 0$. Under such a restriction on the family $\{G_t, t > 0\}$ of homeomorphisms, we consider non-singular diffusions Y and Y' on Q and Q' with the initial laws ν and ν', respectively, for which the following invariance property under the time inversion holds for some family $\{G_t, t > 0\}$ of homeomorphisms; $G_t : Q \ni x \to G(t,x) \in Q'$;

$$(35) \qquad \left\{G\left(t, Y\left(\frac{1}{t}\right)\right), P_\mu\right\}_{t>0} \stackrel{d}{=} \{Y'(t), P'_{\mu'}\}_{t>0}.$$

THEOREM 8. *The all possible class of such Y and Y' with ν and ν' is given by*

$$Y = H^{-1}(X), \quad Y' = H'^{-1}(X') \quad \text{and} \quad \nu = \mu \circ H, \quad \nu' = \mu' \circ H'$$

and $\{G_t, t > 0\}$ is given by

$$G(t,x) = H'^{-1}(t \cdot H(x)).$$

Here H and H' are homeomorphisms; $H : Q \to \mathbf{R}$ and $H' : Q' \to \mathbf{R}$ with $H(c) = H'(c') = 0$ in the bilateral case, and $H : Q \to \mathbf{R}^+$ and $H' : Q' \to \mathbf{R}^+$ with $H(a) = H'(a') = 0$ in the unilateral case. And, X, X', μ, μ' are exactly those given by Theorem 6 *in the bilateral case, and by* Theorem 7 *in the unilateral case.*

For the proof in the bilateral case (and similarly for the unilateral case with a slight modification), it is important to notice that $biBESD(\delta, c, \theta_+, \theta_-, p)$ is, in the context of what we have discussed before Theorem 2, a non-singular diffusion process on $\mathbf{R}$ determined by two pairs $\{s_+(x), m_+(dx)\}$ and $\{s_-(x), m_-(dx)\}$ of Feller's data on $[0,\infty)$ given by

$$s_+(x) = s(x)|_{[0,\infty)} \quad m_+(dx) = m(dx)|_{[0,\infty)}$$

and

$$s_-(x) = -s(-x)|_{[0,\infty)} \quad m_-(dx) = \check{m}(dx)|_{[0,\infty)}$$

where $s(x), m(dx)$ are defined by (28) and (29). The pair of spectral characteristic functions corresponding to these pairs is given by

$$(36) \quad (h_+(\lambda), h_-(\lambda)) = \frac{\Gamma(1-\frac{\delta}{2})}{2^{\frac{\delta}{2}}\Gamma(\frac{\delta}{2})} \left(\frac{1}{p[(\lambda+c)^{1-\frac{\delta}{2}} + c^{1-\frac{\delta}{2}}\theta_+]}, \frac{1}{(1-p)[(\lambda+c)^{1-\frac{\delta}{2}} + c^{1-\frac{\delta}{2}}\theta_-]}\right).$$

Then the idea is to deduce from the invariance property (35) that each of the pairs of spectral characteristic functions $(h_+[Y](\lambda), h_-[Y](\lambda))$ and $(h_+[Y'](\lambda), h_-[Y'](\lambda))$ associated to Y and Y', respectively, must be a constant multiple of the pair $(h_+(\lambda), h_-(\lambda))$ defined by (36). We can furthermore deduce from (35) again that the function $h[Y](\lambda)$ defined by

$$\frac{1}{h[Y](\lambda)} = \frac{1}{h_+[Y](\lambda)} + \frac{1}{h_-[Y](\lambda)}$$

must be given in the form

$$h[Y](\lambda) = \text{const} \cdot \frac{1}{(\lambda + c)^{1-\frac{\delta}{2}}},$$

and similarly for $h_+[Y'], h_-[Y']$. Then it is easy to deduce that, unless $c = 0$, parameters in these spectral characteristic functions must satisfy the condition (ii) of Theorem 6. Now appealing to Theorem 3, we can conclude that Y and Y' are obtained from *biBESD*'s by homeomorphisms H and H' as above. Thus, we see that Krein's correspondence plays a crucial role in the proof that Y and Y' should be so given as described in Theorem 8. The converse part that such Y and Y' really satisfy the inversion property, equivalently, the sufficiency part of Theorem 6, can be proved by using the explicit form of transition densities of *biBESD*'s which can be computed under the condition (ii) in Theorem 6 of parameters. For details, cf. [13].

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