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WHITE NOISE APPROACH TO FLUCTUATIONS

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ABSTRACT. We are interested in random phenomena that will vary as time goes by, being interfered with by fluctuation. These phenomena are often expressed as functionals of white noise. We therefore discuss the analysis of those functionals, where the white noise is understood as a system of idealized elementary random variables. The system is, in many cases, taken to be the innovation of the given random phenomena. The use of the innovation provides a powerful tool to investigate stochastic processes and random fields in line with white noise analysis.

1. Introduction

A typical mathematical model of fluctuation is certainly a white noise. If one is interested in dealing with actual random phenomena with fluctuation, he is naturally led to discuss functionals of white noise. The analysis of white noise functional has extensively developed and has occupied an important part of the infinite dimensional analysis.

It is now the time to have a review of the white noise analysis and to improve it so as to be situated within the pure mathematics in a most suitable position and to have good applications in various fields of science. For this purpose the following three subjects are now proposed.

1) White noise should be thought of as a system of idealized elementary random variables which are taken to be the variables of white noise functionals.

2) Usually actual random phenomena enjoy complex way of dependency, specifically when the space-time parameter varies. To describe such complex dependency it is sometimes convenient to have random

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fields indexed by a manifold C that runs through the space-time parameter space. Namely, we take random fields $X(C)$ with parameter C.

3) We often meet actual examples where input information is not given, but only output is observed. In order to investigate such random phenomena we have to provide a class of white noise functionals the properties of which are known.

Having proposed the research plans as above, we now proceed to discuss the related mathematical problems in order.

2. Idealized elementary random variables

The name in the above title came from the conversation with Professor John R. Klauder who suggested to call a white noise by this name.

Let $B(t)$ be a Brownian motion, which is Gaussian in distribution and has independent increments as is well known. Take the time derivative of $B(t)$, let it be denoted by $\dot{B}(t)$. It is a realization of a "white noise". The system $\{\dot{B}(t)\}$ is elementary in a sense that each $\dot{B}(t)$ is atomic random variable, although it is an infinitesimal random variable and its sample paths are generalized functions of t. Still, it is fitting to be a system of variables of a random function since it is a collection of independent random variables. (This independent property will be explained in details just below.) Thus we may call the $\{\dot{B}(t)\}$ a system of *idealized elementary random variables* (abbr. i.e.r.v.). Even, it is a Gaussian system, which means that it forms the most important system.

Here is an important remark. The $\{\dot{B}(t)\}$ may simply be called an independent system, but it does not mean that it consists of continuously many independent random variables. We should understand that we associate each $\dot{B}(t)$ with an infinitesimal interval dt. Such an understanding is quite important when we discuss various profound calculus of white noise functionals and when we wish to express the fluctuation arising from the natural phenomena in terms of white noise.

Having fixed the system of variables of random functions, we may write them in the form

$$\varphi(\dot{B}) = \varphi(\dot{B}(t), t \in T), \tag{2.1}$$

T being an interval. Once the variable is chosen, it is natural to define polynomials in $\dot{B}(t)$'s, exponential functions of them and so forth, and to define the space $(S)^*$ of generalized white noise functionals. Then, we come to the differential operator with respect to the variable:

$$\partial_t = \frac{\partial}{\partial \dot{B}(t)}. \tag{2.2}$$

To concretize the above notions together with other concepts we introduce the following notions. Let E be a nuclear space which is a member of a Gel'fand triple:

$$E \subset L^2(R) \subset E^*, \tag{2.3}$$

where E^* is the dual space of E.

The probability distribution μ of the $\{\dot{B}(t)\}$ is introduced in E^* and we are given the complex Hilbert space $(L^2) = L^2(E^*, \mu)$. A member $\varphi(x)$ is called a white noise functional. It is viewed as a realization of a formal expression $\varphi(\dot{B})$.

There is a transformation called S-transform of white noise functionals defined by

$$\begin{aligned}(S\varphi)(\xi) &= \int_{E^*} \varphi(x+\xi)d\mu(x) \\ &= \exp\left[-\frac{1}{2}\|\xi\|^2\right]\int_{E^*} \exp\left[\langle x, \xi\rangle\right]d\mu(x),\end{aligned}$$

which is often called a U-functional associated with φ and is denoted by $U(\xi)$.

We are now ready to define a differential operator ∂_t, that was briefly mentioned in (2.2), with respect to the variable $\dot{B}(t)$:

$$\partial_t \varphi = S^{-1}\left[\frac{\delta}{\delta \xi(t)} S\varphi\right], \tag{2.4}$$

where $\frac{\delta}{\delta\xi(t)}$ is the Fréchet derivative. The operator ∂_t defined by the formula (2.4) is an annihilation operator and its adjoint operator ∂_t^* ia a creation operator.

By using the annihilation and creation operators as well as their combinations we can carry on the white noise analysis. Note that the background of the calculus is based on the i.e.r.v. $\{\dot{B}(t)\}$.

3. Factorization of the Volterra or Gross Laplacian

The Volterra and the Gross Laplacians, acting on the space of white noise functionals, are denoted by Δ_V and Δ_G, respectively. They are essentially the same and are expressed (following H.-H. Kuo) in the form

$$\int \partial_t^2 dt. \tag{3.1}$$

For the study of the action of the Volterra Laplacian, it is helpful to have it factorized. To get an explicit expression we must remind the remark that the $\dot{B}(t)$ is to be associated with dt as was mentioned in the last section.

There is a well-known property of the finite dimensional Laplacian illustrated by using the difference between the value of a function at a point and its mean values on a circle around the point.

The same idea is applied in the case of white noise functionals. Let $Y(t)$ be a system of independent standard Gaussian random variables and let $a(t)$ be a (nonrandom) real valued function of tt. The equality

$$\exp[a(t)Y(t)\partial_t]\varphi(x) = \varphi(x(.) + a(t)\delta_t(.)Y(t)) \tag{3.2}$$

can be proved by applying the S−transform. Similarly the equation

$$E_Y\{\exp[a(t)\partial_t Y(t)]\} = \exp\left[\frac{1}{2}a(t)^2\partial_t^2\right] \tag{3.3}$$

can be shown. The notation E_Y means the expectation on the probability space where $Y(t)$'s are defined. Set, in somewhat formal manner, $a(t) = \epsilon\sqrt{dt}$ and take the products with respect to dt's of both sides in the above equation. Then the left hand side is expressible as

$$E_Y\left\{\exp\left[\epsilon\int \partial_t Y(t)\sqrt{dt}\right]\right\}, \tag{3.4}$$

which is equal to

$$\exp\left[\frac{\epsilon^2}{2}\int \partial_t^2 dt\right] = \exp\left[\frac{\epsilon^2}{2}\Delta_V\right]. \tag{3.5}$$

On the other hand, it is in agreement with

$$\prod_{dt}\varphi(x(.) + \epsilon\delta_t(.)Y(t)\sqrt{dt}) = \varphi(x+\epsilon y), \tag{3.6}$$

where $\prod_{dt}$ means the iteration of the shifts by $\epsilon\delta_t(.)Y(t)\sqrt{dt}$ for all dt's, and where y is a trajectory of a Brownian motion $\int^{\cdot} Y(s)\sqrt{ds}$. Hence we conclude

$$\Delta_V\varphi(x) = \lim \frac{2}{\epsilon^2}E_Y\{\varphi(x+\epsilon y) - \varphi(x)\}, \tag{3.7}$$

as was to be expected.

REMARK. The fact shown above (in particular, (3.4)) suggests to define an integral of operators which may be expressed in the classical notation,

$$A = \int \partial_t dB(t). \tag{3.8}$$

The formula (3.8) is viewed as a stochastic integral of the ∂_t with respect to $dB(t)$, B being a Brownian motion. Now we have

THEOREM. *The Volterra Laplacian Δ_V is factorized in such a way that*

$$\begin{aligned} E(A^2) &= \int\int \partial_t\partial_s\delta(t-s)dtds \\ &= \Delta_V, \end{aligned}$$

where A is given by (3.8).

Intuitively speaking, the operator A is a square root, in the stochastic sense, of the Laplacian Δ_V.

4. Random fields

The next topic is a random field $X(C) = X(C, x)$ which is a white noise functional indexed by a manifold C. We often meet such a random function $X(C)$ fluctuating by the influence of the environment depending on C. In general a random field $X(C)$ behaves in a very complex manner in dependency as C varies.

The innovation approach to $X(C)$ is a powerful tool for the investigation of the probabilistic structure of $X(C)$. In some interesting cases, the innovation $\{Y(s), s \in C\}$ for $X(C)$ can be obtained by the variational calculus for $X(C)$. Once the innovation is obtained, the given field can be expressed as the functional of the innovation. Thus the probabilistic structure of the random field is expressed in a visualized manner.

There is a significant class of random fields (see Si Si [11]), which are discussed in this line. The following random field $X(C)$ illustrates the idea.

Let $X(C)$ be given by

$$X(C) = \int_{(C)^n} F(C,u) : x^{n\otimes}(u) : du, \tag{4.1}$$

where $u \in (C)^n$, and (C) is a domain enclosed by a smooth oval. Assuming the canonical property, the innovation is obtained.

Another approach is a generalization of the Langevin equation for random fields.

PROPOSITION. *Let $X(C)$ be defined by a Langevin type equation.*

$$\begin{aligned} \delta X(C) &= -X(C)\int_C \varphi(s)\delta n(s)ds \\ &\quad +X_0 \int_C \nu(s)\partial_s^* \delta n(s)ds, \\ &\quad (C) \supset (C_0), C \in (\mathbf{C})_0 \end{aligned}$$

where φ and ν are given continuous functions, and where $(\mathbf{C})_0$ is a class of plane circles. Then the solution is given by

$$X(C) = X_0 \int_{(C)} \exp\left[-\rho(C,u)\varphi(u)\right]\partial_u^* \nu(u)du, \tag{4.2}$$

where ρ denotes the distance.

The assertion is proved by using the $S-$transform. It would be nice if a variation of a field $f(X(C))$ is obtained for a smooth function f.

REMARK. It would be fine if a generalization of the Itô formula is established for the field $f(X(C))$.

5. Concluding remark

As the white noise analysis develops, more interesting applications are discovered. Among others, there is a problem to identify a black box that admits white noise input.

white noise $\longrightarrow$ nonlinear system $\longrightarrow$ output

See K.-I. Naka et al [8], where the reaction of the catfish retina is discussed in this line.

While, if the input is unknown and if only output is observed, then we first check if there is a possibility of white noise input. Second, we compare the observed output with known white noise functionals.

For the first step, we should remind significant properties of white noise or Brownian motion; not only Gaussian and independent increment properties, but also the optimality of Brownian paths and irregularity of them (see e.g. P.Lévy [4]), and so forth.

Another technique is to form a random field that comes from the fluctuating phenomena. The variational calculus of the field, as we observed in the last section, tells us much finer way of dependency of the field.

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