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TWO-LINK APPROXIMATION SCHEMES FOR LINEAR LOSS NETWORKS WITHOUT CONTROLS

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ABSTRACT. This paper is concerned with the performance evaluation of loss networks. We shall review the *Erlang Fixed Point* (EFP) method for estimating the blocking probabilities, which is based on an assumption that links are blocked independently. For networks with linear structure, the behaviour of adjacent links can be highly correlated. We shall give particular attention to recently-developed fixed-point methods which specifically account for the dependencies between neighbouring links. For the network considered here, namely a ring network with two types of traffic, these methods produce relative errors typically 10^{-5} of that found using the basic EFP approximation.

1. Introduction

Ever since the work of A. K. Erlang became widely known [2], stochastic models have gained prominence as effective means of predicting the performance of telecommunications systems. For the simplest models there are explicit analytical formulae for the important measures of performance, such as the blocking probabilities. However, these formulae often cannot be computed since, even for networks of moderate size, the number of states can be very large. Under several limiting regimes the *Erlang Fixed Point* (EFP) method provides a good approximation for the blocking probabilities, but when these regimes are not operative the method can produce relative errors of 5% or more. In many cases this is because the key assumption of independent blocking does not

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hold. By accounting for the dependencies between neighbouring links the basic fixed-point method can be improved.

Consider the usual stochastic model for a circuit-switching network with fixed routing. This was introduced as a model for a telephone network, but it also arises in the study of local area networks, multiprocessing architectures, data-base management systems, mobile/cellular radio and broadband packet networks (see for example [3, 8, 12, 17, 20, 23]).

When the usual assumptions are in force, namely that there are no repeated attempts, that lost calls are not held and that there is full availability between adjacent switching nodes, then the model is very accurate in predicting network performance. An example of a circuit-switched network is depicted in Figure 1. If we denote by K the number of links (circuit groups), then any route in the network can be expressed

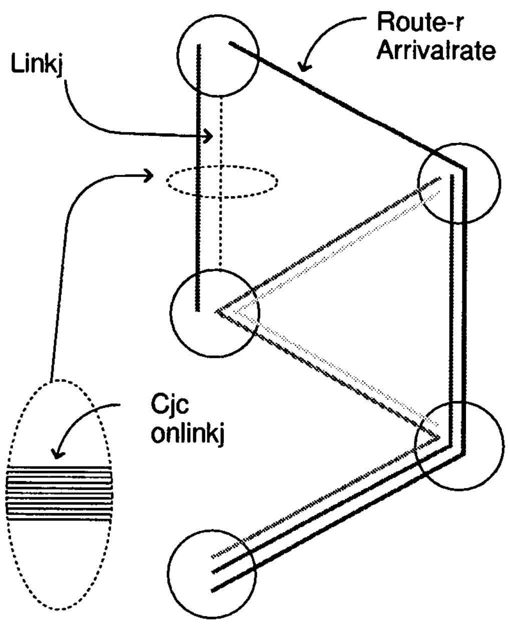

Fig 1. A typical circuit-switched network
(5 nodes, 6 links and 5 routes)

as a subset of $\{1, 2, \ldots, K\}$. Let $\mathcal{R}$ be the set of all routes. Calls using route r are offered at rate ν_r as a Poisson stream, and use $a_{jr}(\geq 0)$ circuits from link j, the total number of circuits on link j being C_j. We assume that $\mathcal{R}$ indexes *independent* Poisson processes. Calls requesting route r are blocked and lost if, on *any* link j, there are fewer than a_{jr} free circuits. Otherwise, the call is connected and simultaneously holds a_{jr} circuits on each link j for the duration of the call. For simplicity, we shall take $a_{jr} \in \{0,1\}$. Call durations are independent and identically distributed exponential random variables with unit mean, and are independent of the arrival processes.

Let $\boldsymbol{n} = (n_r,\ r \in \mathcal{R})$, where n_r is the number of calls in progress using route r, let $\boldsymbol{C} = (C_j,\ j = 1,\ldots,K)$, and let $\boldsymbol{A} = (a_{jr},\ r \in \mathcal{R},\ j = 1,\ldots,K)$. Then, the usual model for a circuit-switched network (see for example [15]) is a continuous-time Markov chain $(\boldsymbol{n}(t),\ t \geq 0)$ taking values in

$$S = S(\boldsymbol{C}) = \left\{\boldsymbol{n} \in \mathrm{Z}_+^{\mathrm{R}} \ :\ \boldsymbol{A}\boldsymbol{n} \leq \boldsymbol{C}\right\}.$$

Its transition rates are easy to write down, since the only possible transitions involve either an upward or a downward jump of size 1 in one, and only one, component of $\boldsymbol{n}$. If $q(\boldsymbol{n},\boldsymbol{n}')$ denotes the transition rate from state $\boldsymbol{n}$ to state $\boldsymbol{n}'$, then we have that

$$q(\boldsymbol{n},\boldsymbol{n}+\boldsymbol{e}_r) = \nu_r, \qquad \text{if } \boldsymbol{n},\boldsymbol{n}+\boldsymbol{e}_r \in S,$$

$$q(\boldsymbol{n},\boldsymbol{n}-\boldsymbol{e}_r) = n_r \qquad \text{if } \boldsymbol{n},\boldsymbol{n}-\boldsymbol{e}_r \in S,$$

and $q(\boldsymbol{n},\boldsymbol{n}') = 0$ otherwise; here $\boldsymbol{e}_r$ is the unit vector indicating just one call in progress on route r; its i^{th} entry is 1 or 0 according as $i = r$ or $i \neq r$. It can be shown (see for example [3]) that the unique equilibrium distribution $\pi = (\pi(\boldsymbol{n}),\ \boldsymbol{n} \in S)$ is given by

$$\pi(\boldsymbol{n}) = \Phi^{-1} \prod_{r\in\mathcal{R}} \frac{\nu_r^{n_r}}{n_r!}, \quad \boldsymbol{n} \in S,$$

where

$$\Phi = \Phi(\boldsymbol{C}) = \sum_{\boldsymbol{n}\in S(\boldsymbol{C})} \prod_{r\in\mathcal{R}} \frac{\nu_r^{n_r}}{n_r!}.$$

The stationary probability that a route-r call is blocked is then given by

$$1 - \frac{\Phi(\boldsymbol{C} - \boldsymbol{A}\boldsymbol{e}_r)}{\Phi(\boldsymbol{C})}.$$

However, although we have an explicit expression for the blocking probability in terms of Φ, the latter can't (usually) be computed in polynomial time (see for example [14]). Thus, for networks with even moderate capacity, one is forced to use alternative methods.

The remainder of this paper is organized as follows. In Section 2 we review the basic EFP approximation, citing both conditions under which it is accurate, and situations where it is known to perform badly. Section 3 contains a description of the network which we shall use to illustrate our methods: a symmetric ring network with two types of traffic. By calculating the exact blocking probabilities using an iterative technique, we are able to assess the accuracy of the basic EFP method. The correlation between neighbouring links is also illustrated. Finally, in Section 4, we look at some two-link approximation schemes which take into account this behaviour. These methods produce relative errors typically 10^{-5} of that found using the basic EFP approximation.

2. The EFP approximation

Arguably the most important approximation technique is the EFP method. It is widely used (see [13, 16, 18, 26, 27, 28]) and has received a great deal of attention in recent times, with a variety of associated theoretical issues having been settled (see [13, 15]). Kelly [13] proved that, when $a_{jr} \in \{0,1\}$, there is a unique vector $(B_1, \ldots, B_K) \in [0,1]^K$ satisfying

$$B_j = E(\rho_j, C_j), \tag{1}$$

$$\rho_j = (1 - B_j)^{-1} \sum_r a_{jr} \nu_r (1 - L_r), \tag{2}$$

for $j = 1, \ldots, K$, and

$$L_r = 1 - \prod_i (1 - B_i)^{a_{ir}}, \quad r \in \mathcal{R}, \tag{3}$$

where

$$E(\nu, C) = \frac{\nu^C}{C!} \left(\sum_{n=0}^{C} \frac{\nu^n}{n!} \right)^{-1} .$$

$E(\nu, C)$ is *Erlang's Formula* for the loss probability on a *single link* with C circuits and Poisson traffic offered at rate ν. The EFP approximation is obtained by using B_j to estimate the probability that link j is full, and L_r to estimate the route-r blocking probability.

The rationale for the EFP approximation is one of *independent blocking*. If links along route r were blocked independently (they are clearly not) and if B_j *were* the link-j blocking probability, then L_r would be the route-r blocking probability:

$$L_r = 1 - \prod_{i \in r}(1 - B_i) = 1 - \prod_i (1 - B_i)^{a_{ir}} .$$

Carrying this further, the traffic offered to link j would be Poisson (at rate ρ_j, say) and the *carried traffic* (that which is accepted) on link j would be

$$\sum_r a_{jr}\nu_r(1 - L_r) \; (= (1 - B_j)\rho_j) \; .$$

The approximation therefore stipulates that the link blocking probabilities $(B_1, \ldots, B_K)$ should be consistent with this level of carried traffic. On combining (1), (2) and (3) we obtain a set of equations for $(B_1, \ldots, B_K)$:

$$B_j = E\bigg(\Big(\sum_r a_{jr}\nu_r \prod_{i \in r - \{j\}} (1 - B_i)\Big) \, , \, C_j \bigg) \, .$$

The existence of the Erlang Fixed Point, namely a fixed point of these equations, is easy to prove using the *Brouwer fixed point theorem*; they define a continuous mapping from a compact convex set $[0, 1]^K$ into itself. The uniqueness is considerably more difficult to prove [13]. We note that for more complex systems there may be more than one fixed point. For example, networks with random alternative routing can exhibit bistability; the system fluctuates between a "low-blocking state", where calls are accepted readily, and a "high-blocking state", where the likelihood of a call being accepted can be quite low (see for example [7, 24, 25]).

The EFP approximation performs well under a variety of circumstances, despite the fact that the occupancies of neighbouring links may be highly dependent. Several limit theorems exist which help to explain this, and there are two regimes in particular under which the EFP approximation is known to be exact. The first is one in which the topology of the network is held fixed, while capacities and arrival rates at the links become large [13]; this has become known as the *Kelly limiting regime*, or (somewhat misleadingly) as the *heavy traffic limit*. Kelly considered a sequence of networks indexed by an arbitrary parameter N, with the capacities and arrival rates indexed accordingly:

$$\boldsymbol{C}^{(N)} = (C_j^{(N)},\ j = 1, \ldots, K) \quad \text{and} \quad \boldsymbol{\nu}^{(N)} = (\nu_r^{(N)},\ r \in \mathcal{R}).$$

He proved that if, as $N \to \infty$,

$$\frac{1}{N}\boldsymbol{C}^{(N)} \to \boldsymbol{C} \quad \text{and} \quad \frac{1}{N}\boldsymbol{\nu}^{(N)} \to \boldsymbol{\nu}\,,$$

then

$$L_r^{(N)} = 1 - \prod_i (1 - B_i^{(N)})^{a_{ir}}, \qquad r \in \mathcal{R},$$

where, here, $(B_1^{(N)}, \ldots, B_K^{(N)})$ is the Erlang Fixed Point determined by (1) and (2) using the values $\boldsymbol{\nu}^{(N)}$ and $\boldsymbol{C}^{(N)}$, converges to the route-r loss probability as $N \to \infty$. This result alone may perhaps explain why the EFP method has been used so successfully by practitioners in a wide variety of circumstances, for, roughly speaking, it says that, provided the arrival rates and capacities are large, the method is bound to perform well.

Under the second limiting regime, called *diverse routing*, the number of links, and the number of routes which use those links, become large, while the capacities are held fixed and the arrival rates on multi-link routes become small. Thus for a sequence of networks, now indexed by K, we allow the routing matrix and arrival rates to depend on K,

$$\boldsymbol{A}^{(K)} = (a_{jr}^{(K)},\ r \in \mathcal{R},\ j = 1, \ldots, K), \qquad \boldsymbol{\nu}^{(K)} = (\nu_r^{(K)},\ r \in \mathcal{R}),$$

and suppose that

$$\sum_r a_{jr}^{(K)} a_{kr}^{(K)} \nu_r^{(K)} \to 0, \qquad j, k = 1, \ldots, K, \tag{4}$$

and

$$\sum_r a_{jr}^{(K)} \nu_r^{(K)} \to \nu_j > 0, \qquad j = 1, \ldots, K. \tag{5}$$

With this formulation [9], the traffic along link j will be of order ν_j, but that which is common to *any two* links becomes small as K gets large. There are no general results under (4) and (5). Rather, there are many results for a variety of specific systems. These include *star networks* and *fully-connected networks with alternate routing* [10, 11, 19, 27, 30].

To illustrate the effect of diverse routing we shall consider a *symmetric star network*. This consists of a collection of K outer nodes, which communicate via a single central node (see Figure 2).

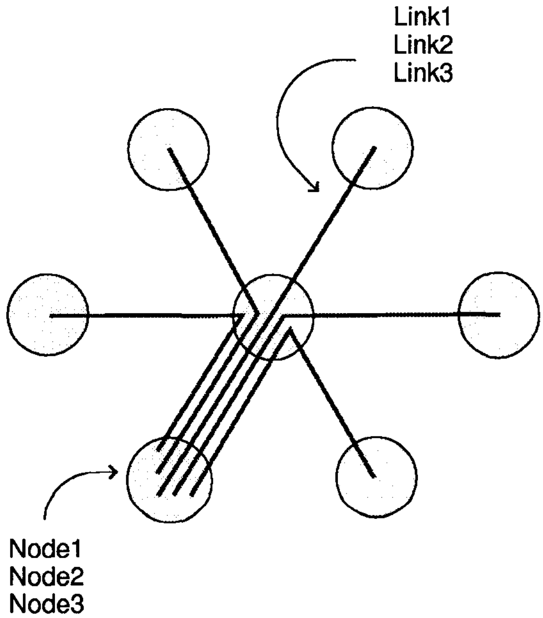

Fig 2. A typical symmetric star network
($K = 6$ outer nodes and 15 routes)

Take $\mathcal{R}$ to be all those routes consisting of a *pair* of links. Fix $\nu > 0$ and take $\nu_r = \nu/(K-1)$ for all $r \in R$, and fix $C_j = C$ for all links $j = 1, \dots, K$. Whitt [27], and Ziedins and Kelly [30] proved that if $L^{(K)}$ is the common route loss probability, then $L^{(K)} \to L$, as $K \to \infty$, where $L = 1-(1-B)^2$, and B, the Erlang Fixed Point (the same for all links), is the unique solution to $B = E(\nu(1-B), C)$.

If neither of the above regimes is operative, the EFP method may not perform as well: in particular, in highly linear networks and/or networks with low capacities. Further, adding controls to the network may cause the method to perform badly under otherwise favourable regimes. A particularly useful control is *trunk reservation*. Here, traffic streams are assigned priorities and calls are accepted only if the occupancies of links along their route are below a given threshold, the level of which depends on the type of call. This widely used control mechanism is typically very robust to fluctuations in arrival rate and has the added advantage of eliminating pathological behaviour such as bistability [7]. With such a control operating in a network of reasonable size, the equilibrium distribution will no longer have a product form, as it does for the corresponding uncontrolled network. We can then no longer afford to ignore dependencies. For an excellent review of the theory of loss networks, and in particular the basic EFP method, see Kelly [15].

We shall now focus attention on simple, highly linear networks, since here the EFP approximation is expected to perform relatively poorly.

3. A symmetric ring network

Consider a loss network with K links forming a loop, and each link having the same capacity C. Such a network is depicted in Figures 3 and 4. There are two types of traffic: 1-link routes (type-1 traffic) and 2-link routes comprising pairs of adjacent links (type-2 traffic). Type-t traffic is offered at rate ν_t on each type-t route.

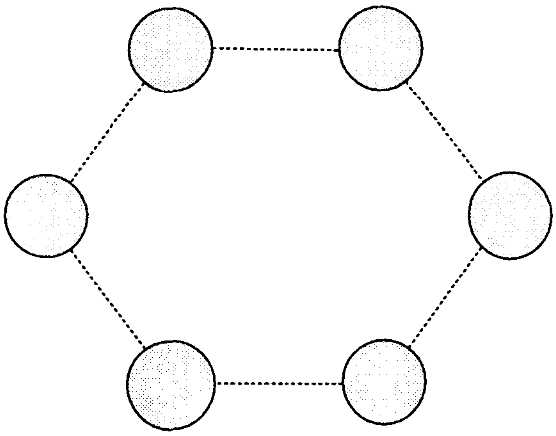

Fig 3. A ring network (6 nodes)

If L_t is the EFP approximation for the loss probability of type-t calls, then it is easy to show that

$$L_1 = B \quad \text{and} \quad L_2 = 1 - (1 - B)^2,$$

where the Erlang Fixed Point B is the unique solution to

$$B = E(\nu_1 + 2\nu_2(1 - B), C),$$

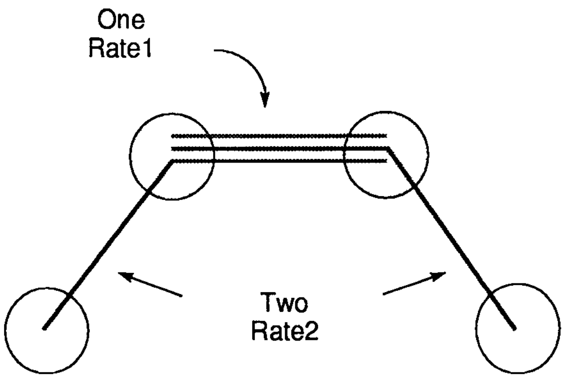

Fig 4. One- and two-link traffic using a given link

where recall that $E(\nu, C)$ is Erlang's Formula. Figure 5 shows the EFP approximation for the blocking probability of type-1 calls in a network with $C = 10$, $K = 10$ and $\nu_1 = \nu_2 (= \nu)$.

In order to assess the accuracy of the EFP approximation, as well as the improved methods described below, we shall need to evaluate the exact blocking probabilities. This will be done using a recursive technique; related methods are discussed by Ziedins and Kelly [30], and Bean and Stewart [1]. The state space for the ring network is given by

$$S_K = \{\boldsymbol{n} : n_i + n_{i-1,i} + n_{i,i+1} \leq C, i = 1, \ldots, K\},$$

where, in a convenient notation, route $\{K, 1\}$ is denoted by $\{K, K+1\}$; since we shall be varying K, it will be necessary to make any dependence on K explicit in our notation. The equilibrium distribution is given by

$$\pi_K(\boldsymbol{n}) = \Phi_K^{-1} \frac{\nu_1^{\sum_i n_i} \nu_2^{\sum_i n_{i,i+1}}}{\prod_i n_i! \, n_{i,i+1}!}, \quad \boldsymbol{n} \in S_K,$$

where

$$\Phi_K = \sum_{\boldsymbol{n} \in S_K} \frac{\nu_1^{\sum_i n_i} \nu_2^{\sum_i n_{i,i+1}}}{\prod_i n_i! \, n_{i,i+1}!}$$

is the normalizing constant for the network with K links.

Now consider a *line network* consisting of a series of K links. This is obtained if the ring is disconnected at one node. In a similar fashion we define the normalizing constant Ψ_K for this network. We also define

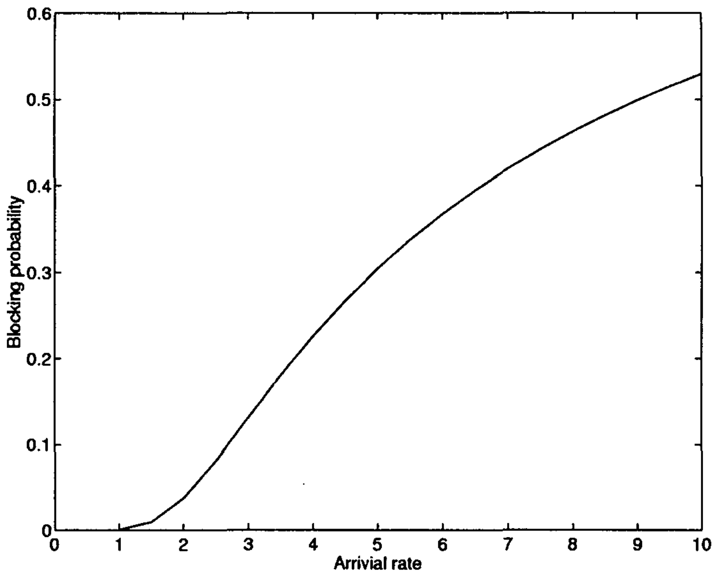

Fig 5. EFP approximation for the blocking probability of type-1 calls ($C = 10$, $K = 10$, $\nu_1 = \nu_2 =$ arrival rate)

$\Psi_K^{(i,j)}$ to be the normalizing constant for the line network with $C_1 = i$, $C_k = C$, for $1 < k < K$, and $C_K = j$. Note that $\Psi_K^{(C,C)} = \Psi_K$. Note also that $\Psi_K^{(i,j)} = \Psi_K^{(j,i)}$. Then, the $\Psi_K^{(i,j)}$ satisfy the following recursion:

$$\Psi_K^{(i,j)} = \sum_{\beta=0}^{i} \left[\sum_{\alpha=0}^{i-\beta} \frac{\nu_1^\alpha}{\alpha!} \frac{\nu_2^\beta}{\beta!} \Psi_{K-1}^{(C-\beta,j)} \right]$$

with

$$\begin{aligned} \Psi_1^{(i,j)} &= \sum_{\alpha=0}^{\min(i,j)} \frac{\nu_1^\alpha}{\alpha!}, \\ \Psi_0^{(i,j)} &= 1. \end{aligned}$$

This recursion is obtained by considering the number of one-link and two-link calls on link 1. Consider the contribution to the normalizing constant made by some fixed configuration of calls. Suppose that this configuration has $n_{12} = \beta$, where β must lie between 0 and i inclusive.

Then we must have $n_1 \leq \beta - i$. Thus, the contribution from these two routes for this particular configuration is exactly

$$\frac{\nu_1^{n_1}}{n_1!}\frac{\nu_2^{\beta}}{\beta!}.$$

We now consider the remaining routes; they use links $\{2, \ldots, K\}$ and form a $(K-1)$-link network. Since there are β calls on route $\{1, 2\}$, link 2 only has $C - \beta$ free circuits, and so the contribution from the remaining routes is $\Psi_{K-1}^{(C-\beta,j)}$.

Let us return to the ring network. An expression for Φ_K in terms of the Ψ_K is obtained as follows. Consider links K and 1. By conditioning on n_{K1}, we can break the ring network into a line network, and write

$$\Phi_K = \sum_{n_{K1}=0}^{C} \frac{\nu_2^{n_{K1}}}{n_{K1}!}\Psi_K^{(C-n_{K1},C-n_{K1})}.$$

Note that links K and 1 have been chosen as the reference links here, but of course the recursion would be the same if any other pair of adjacent links had been chosen.

The blocking probabilities can now be written in terms of the normalizing constants. To do this, we introduce some further notation. Let $\Phi_K^{(i)}$ denote the normalizing constant for the ring network in which all the links have capacity C, except for one link, which has capacity i. Similarly, let $\Phi_K^{(i,j)}$ be the normalizing constant for the ring network in which all links, except two, have capacity C; the exceptions have capacities i and j, and are adjacent. Then, the probability that a one-link call is accepted (which is also the probability that a link has free capacity) is given by

$$\frac{\Phi_K^{(C-1)}}{\Phi_K},$$

and the probability that a two-link call is accepted is given by

$$\frac{\Phi_K^{(C-1,C-1)}}{\Phi_K},$$

where, just as for Φ_K, we can write

$$\Phi_K^{(C-1)} = \sum_{n_{K1}=0}^{C-1} \frac{\nu_2^{n_{K1}}}{n_{K1}!}\Psi_K^{(C-n_{K1},C-1-n_{K1})}$$

and

$$\Phi_K^{(C-1,C-1)} = \sum_{n_{K1}=0}^{C-1} \frac{\nu_2^{n_{K1}}}{n_{K1}!} \Psi_K^{(C-1-n_{K1},C-1-n_{K1})}.$$

These recursions are easily implemented to obtain the exact blocking probabilities numerically.

With a view to comparing the EFP approximation with the exact blocking probabilities, let us focus our attention on type-1 calls. Figure 6 shows the relative error in using the EFP approximation in a network with $C = 10$, $K = 10$ and $\nu_1 = \nu_2 (= \nu)$. Notice that the exact blocking

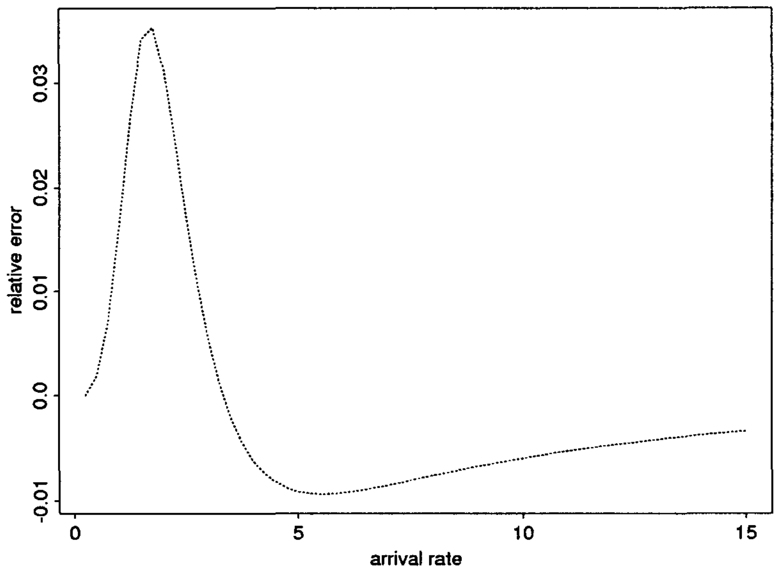

Fig 6. Relative error in the EFP approximation for the blocking probability of type-1 calls ($C = 10$, $K = 10$, $\nu_1 = \nu_2 =$ arrival rate)

probabilities are overestimated for small values of the arrival rate ν and underestimated for larger values, and, that the accuracy improves as the arrival rate becomes very large. An intriguing feature of Figure 6 is that the approximation is *precise* (and also most sensitive) near the point of "critical loading", namely when $\nu_1 + 2\nu_2 = C$; for the parameter values used, this is when $\nu = 10/3$, a point just before the graph crosses the x-axis.

To illustrate why we might expect the EFP approximation not to perform as well in the present context, we shall assess the dependence between two adjacent links in the ring network. Figure 7 shows the

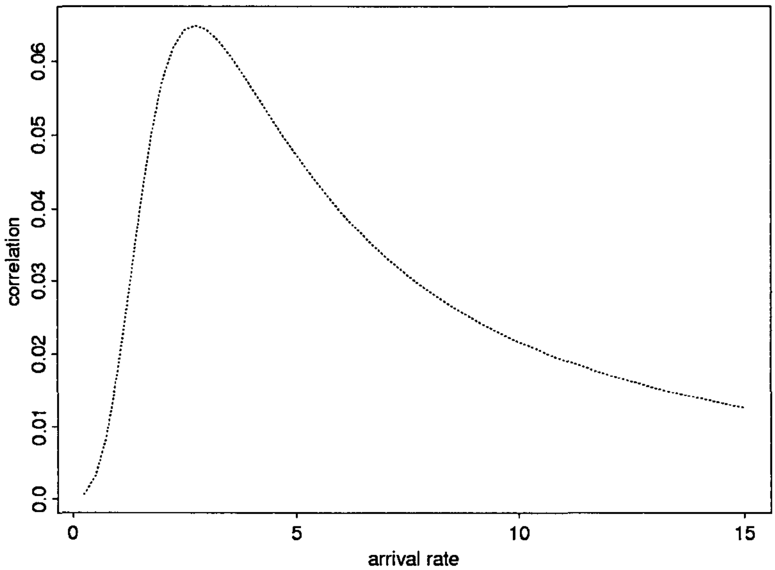

Fig 7. Correlation of spare capacity on adjacent links ($C = 10$, $K = 10$, $\nu_1 = \nu_2 =$ arrival rate)

correlation between links 1 and 2 for the network with $C = 10$, $K = 10$ and $\nu_1 = \nu_2 (= \nu)$; to be precise, we have plotted

$$\text{Corr}\left(I_{\{n_{K1}+n_1+n_{12}<C\}}, I_{\{n_{12}+n_2+n_{23}<C\}}\right)$$

against the arrival rate ν. Notice that the correlation is greatest at values of the arrival rate near where the EFP approximation is least accurate. Notice also that, as the arrival rate becomes large, both the correlation and the relative error in the EFP approximation tend to 0.

4. Two-link approximation schemes

We can estimate the blocking probabilities more accurately by considering subnetworks of the original network, thus specifically accounting for the dependencies between adjacent links. Take links 1 and 2 as reference links and consider the subnetwork depicted in Figure 8. We identify

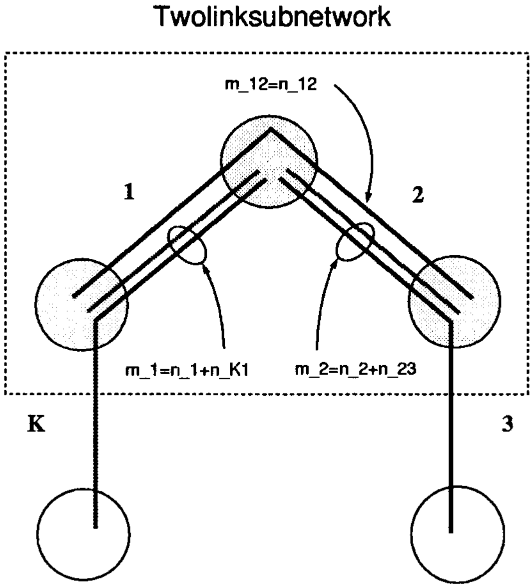

Fig 8. Definition of m_1, m_2 and m_{12} for the symmetric ring network

three routes: $\{1\}$, $\{2\}$ and $\{1,2\}$. If m_r denotes the number of calls on route r, then m_1 is the number of calls occupying capacity on link 1 *but not* on link 2, that is $m_1 = n_1 + n_{K1}$, m_2 is the number occupying capacity on link 2 *but not* on link 1, that is $m_2 = n_2 + n_{23}$, and, $m_{12}(= n_{12})$ is the number of calls occupying capacity on both links.

Our first approximation (Approximation I) is obtained by adapting the method of Pallant [21]. In Pallant's method, the network is decomposed into independent subnetworks and the stationary distribution is evaluated for each. For example, if we take our subnetwork to be the one depicted in Figure 8, then its state space will be

$$S = \{(m_1, m_2, m_{12}) : m_i + m_{12} \le C, i = 1, 2\}$$

and its stationary distribution will be

$$\pi(\boldsymbol{m}) = \Phi^{-1} \frac{(\nu_1 + \nu_2(1 - B))^{m_1+m_2} \nu_2^{m_{12}}}{m_1!\, m_2!\, m_{12}!},$$

where Φ is a normalizing constant. We then estimate B, the probability that a link adjacent to the two-link subnetwork is fully occupied, using the subnetwork itself; set

$$\begin{aligned} B &= \sum_{m : m_1 + m_{12} = C} \pi(m_1, m_2, m_{12}) \\ &= \sum_{m_{12}=0}^{C} \sum_{m_2=0}^{C-m_{12}} \pi(C - m_{12}, m_2, m_{12}) . \end{aligned}$$

These expressions are used iteratively to determine a fixed point B, and we then set $L_1 = B$ and

$$L_2 = 2L_1 - \sum_{m_{12}=0}^{C} \pi(C - m_{12}, C - m_{12}, m_{12}).$$

Our second, and more accurate, approximation (Approximation II) uses additional knowledge of the state of a given link in estimating the probability that the adjacent link is full. We use *state-dependent* arrival rates, $\rho_n = \nu_1 + \nu_2(1 - b_n)$, $n \in \{0, 1, \ldots, C - 1\}$, where b_n is the probability that link K is fully occupied, conditional on $m_1 = n$ (b_n is also the probability that link 3 is fully occupied, *conditional on* $m_2 = n$), so that

$$\pi(\boldsymbol{m}) = \Phi^{-1} \frac{\nu_2^{m_{12}} \left(\prod_{n=0}^{m_1-1} \rho_n\right) \left(\prod_{n=0}^{m_2-1} \rho_n\right)}{m_1!\, m_2!\, m_{12}!} .$$

Once b_n is estimated and π determined, we set L_1 and L_2 as for Approximation I. An estimate of b_n is found by assuming that b_n does not depend on m_{12}. For $n = 0, \ldots, C - 1$, we set

$$b_n = \frac{\sum_{m=0}^{n} p(n - m, C - m, m)}{\sum_{m=0}^{n} \sum_{r=0}^{C-m} p(n - m, r, m)},$$

where

$$p(n_1, m_K, n_{K1}) = \frac{\nu_1^{n_1} \nu_2^{n_{K1}} \left(\prod_{s=0}^{m_K-1} \rho_s\right)}{n_1!\, n_{K1}!\, m_K!} .$$

The form of $p(n_1, m_K, n_{K1})$ can be explained with reference to Figure 8 as an invariant measure (an unnormalized equilibrium distribution) of the two-link subnetwork comprising links K and 1 of that diagram, with $m_K = n_{K-1,K} + n_K$; b_n is then calculated from this distribution by identifying the configurations with $m_1 (= n_1 + n_{K1}) = n$, and those that also have link K full. The dependence of b_n on m_{12} is due to the cyclic nature of the network, but is expected to be slight for large networks.

This approximation is exact for the infinite line network, as shown by Zachary [29] for an equivalent network with $\nu_1 = 0$ (that is, no one-link traffic). Our expression for b_n is the same as that obtained in his paper for the infinite line network, although written in a different form. Kelly [12] also considers an equivalent system with no one-link traffic. State-dependent arrival rates such as we have here are also discussed by Pallant and Taylor [22] (see also [4, 5, 6]).

Figures 9 and 10 show the relative error in using each of the three approximations to estimate the blocking probability of type-1 and type-2 calls, respectively, in a network with $C = 10$, $K = 10$ and $\nu_1 = \nu_2 (= \nu)$. Notice that, while Approximation I gives some improvement in accuracy over the EFP approximation, the improvement obtained using Approximation II is considerable. Indeed, the maximum relative error for Approximation II is of order 10^{-8} for both types of traffic.

We are presently working on extending our methods to deal with trunk reservation and networks with a more general topology.

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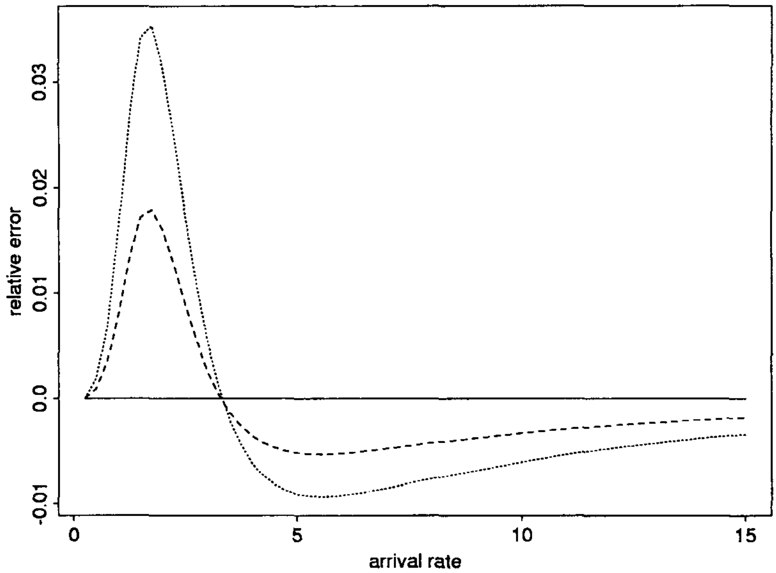

Fig 9. Relative error in the estimated blocking probability of type-1 calls ($C = 10$, $K = 10$, $\nu_1 = \nu_2 =$ arrival rate)
.... EFP - - - - - Approx. I
________________ Approx. II

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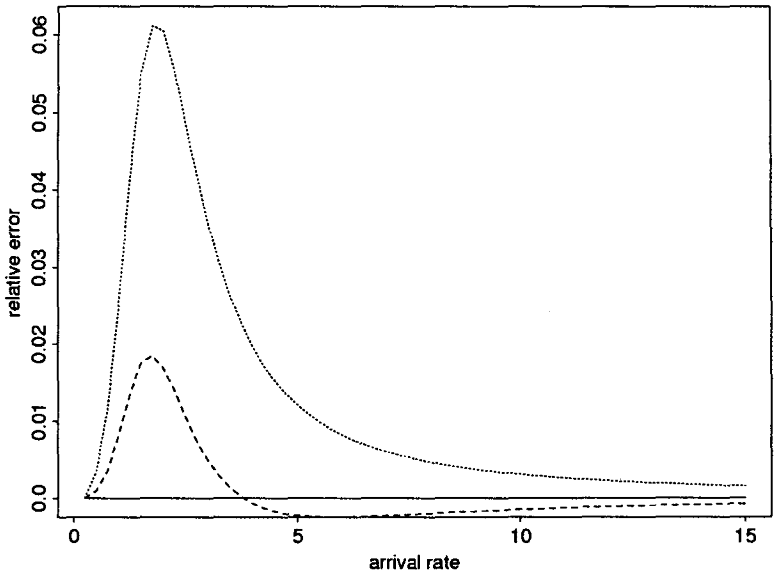

Fig 10. Relative error in the estimated blocking probability of type-2 calls ($C = 10$, $K = 10$, $\nu_1 = \nu_2 =$ arrival rate)
············· EFP - - - - - Approx. I
_______________ Approx. II

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