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ORBITAL LIPSCHITZ STABILITY AND EXPONENTIAL ASYMPTOTIC STABILITY IN DYNAMICAL SYSTEMS

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ABSTRACT. In this paper we introduce the notions of orbital Lipschitz stability (in variation) and orbital exponential asymptotic stability (in variation) of C^r dynamical systems (or C^r diffeomorphisms) on Riemannian manifolds, and study the embedding problem of those concepts in C^r dynamical systems.

1. Introduction

By a C^r *dynamical system* (or *flow*), $0 \le r \le \infty$, on a C^r manifold M we mean a C^r map $\pi : M \times \mathbb{R} \to M$ such that

(1) $\pi(p, 0) = p$, for $p \in M$;
(2) $\pi(\pi(p, s), t) = \pi(p, s + t)$, for $p \in M$ and $s, t \in \mathbb{R}$.

We can see that for each $t \in \mathbb{R}$ the *transition map* $\pi^t : M \to M$ given by

$$\pi^t(p) = \pi(p, t), \ p \in M$$

is a C^r diffeomorphism, and for each $p \in M$ the orbit map $\pi_p : \mathbb{R} \to M$ given by

$$\pi_p(t) = \pi(p, t), \ t \in \mathbb{R}$$

is a C^r map. If $r = 0$ then π corresponds to a continuous flow on M and each transition map π^t corresponds to a homeomorphism on M.

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We say that a continuous flow π is $C^{\frac{1}{2}}$ if each orbit map π_p is C^1 for each $p \in M$.

For any $p \in M$ the *orbit* of π through p will be denoted by the set

$$O(p) \equiv \{\pi(p,t) : t \in \mathbb{R}\}.$$

A point $p \in M$ is said to be *fixed* under a C^r dynamical system π if $\pi(p,t) = p$ for any $t \in \mathbb{R}$, and $p \in M$ is called *regular* if it is not fixed. We can see that each orbit $O(p)$, $p \in M$, is a 1-dimensional immersed submanifold of M if p is regular and $r \geq 1/2$. For any $p \in M$, we let

$$L^+(p) \equiv \{q \in M : \pi^{t_n}(p) \to q, \text{ for some } t_n \to \infty\}, \text{ and}$$
$$L^-(p) \equiv \{q \in M : \pi^{t_n}(p) \to q, \text{ for some } t_n \to -\infty\}.$$

Elaydi and Farran [5, 6] introduced the concepts of Lipschitz stable dynamical systems and exponential asymptotic stable dynamical systems on Riemannian manifolds, and Chen, Chu and Lee studied the systems in [1, 2, 3, 7]. Moreover Lee [7] introduced the concept of orbital Lipschitz stability of a dynamical system, and Chen obtained some interesting properties for the orbitally Lipschitz stable dynamical systems in [1].

In this paper we will introduce the notions of orbital Lipschitz stability in variation and orbital exponential asymptotic stability in variation of a C^r dynamical system π on M (or a C^r diffeomorphism f on M), using the norm on the tangent bundle TM of M. We can see in [5] that two notions of Lipschitz stability and Lipschitz stability in variation are equivalent; but we will claim that two concepts of orbital Lipschitz stability and orbital Lipschitz stability in variation need not be equivalent (see Example 2.7).

In particular, we will study the embedding problem of the orbital Lipschitz stability (in variation) and orbital exponential asymptotic stability (in variation) of C^r diffeomorphisms in C^r dynamical systems for $r \geq 0$. We claim that if a C^r diffeomorphism f on a compact C^r manifold M, $r \geq 1$, is C^1 embedded in a dynamical system π on M and f is orbitally Lipschitz stable in variation (or orbitally exponentially asymptotic stable in variation) under π, then π is also orbitally Lipschitz stable in variation (or orbitally exponentially asymptotic stable in variation), respectively; but we can see that π need not be orbitally Lipschitz stable (or exponentially asymptotic stable) even if f is orbitally Lipschitz stable under π (or exponentially asymptotic stable), respectively. These facts disprove the Theorem 1.11 and Corollary 1.12 in [6].

Throughout the paper we let M denote a Riemannian manifold with a Riemannian metric g.

2. Orbital Lipschitz Stability

Let d be the metric on M induced by a Riemannian metric g on M, and let $\|\cdot\|$ be the norm on the tangent bundle TM of M induced by g. A C^1 dynamical system π on M is said to be *Lipschitz stable at* $p \in M$ if there exist $K = K(p) \geq 1$ and $\delta = \delta(p) > 0$ such that

$$d(\pi^t(p), \pi^t(q)) \leq K d(p, q)$$

for any $t \in \mathbb{R}$ and any $q \in M$ with $d(p,q) < \delta$. We say that π is *Lipschitz stable in variation* at $p \in M$ if there are $K = K(p) \geq 1$ and $\delta = \delta(p) > 0$ such that

$$\|D\pi^t(V)\| \leq K\|V\|$$

for any $t \in \mathbb{R}$ and any $V \in T_pM$ with $\|V\| < \delta$, where $D\pi^t$ denotes the derivative map of the transition map π^t. π is said to be *Lipschitz stable* (or *Lipschitz stable in variation*) in a subset A of M if π is Lipschitz stable (or Lipschitz stable in variation) at every point of A, respectively, and one can choose these $K \geq 1$ and $\delta > 0$ independently of the points in A.

In [5], Elaydi and Farran introduced the notion of Lipschitz stable dynamical systems on Riemannian manifolds, and Lee[7] introduced the concept of orbital Lipschitz stability of continuous dynamical systems. In this direction, recently, Chen[1] obtained some interesting properties for the orbital Lipschitz stable dynamical systems.

In this section, we introduce the notions of orbital Lipschitz stability in variation of C^r dynamical system π on M (or C^r diffeomorphism f on M), using the norm on the tangent bundle TM of M, and will study the embedding problem of the orbital Lipschitz stability (in variation) of C^r diffeomorphisms in C^r dynamical systems.

We recall the definitions (see [7]) : A dynamical system π on M is *orbitally Lipschitz stable* at $p \in M$ if there exist $K = K(p) \geq 1$ and $\delta = \delta(p) > 0$ such that

$$d(\pi^t(x), \pi^t(y)) \leq Kd(x, y)$$

for any $t \in \mathbb{R}$ and any $x, y \in O(p)$ with $d(x, y) < \delta$. We say that π is *orbitally Lipschitz stable in a subset* A of M if π is orbitally Lipschitz stable at every point of A and we can choose $K \geq 1$ and $\delta > 0$ independently of the points of A.

Now we will give the notion of orbital Lipschitz stability in variation of C^1 dynamical system on M.

DEFINITION 2.1. A C^1 dynamical system π on M is *orbitally pointwisely Lipschitz stable in variation* at $p \in M$ if p is a regular point, and there exist $K = K(p) \geq 1$ and $\delta = \delta(p) > 0$ such that

$$||D\pi^t(V)|| \leq K||V||$$

for any $t \in \mathbb{R}$ and any $V \in T_pO(p)$ with $||V|| < \delta$, where $T_pO(p)$ is the tangent space of $O(p)$ at p. We say that π is *orbitally pointwisely Lipschitz stable in variation* on a set $A \subseteq M$ if π is orbitally pointwisely Lipschitz stable in variation at every point of A.

LEMMA 2.2. *A C^1 dynamical system π on M is orbitally pointwisely Lipschitz stable in variation at $p \in M$ if and only if it is orbitally pointwisely Lipschitz stable in variation on the orbit set $O(p)$.*

Proof. Suppose π is not orbitally pointwisely Lipschitz stable at a point $q \in O(p)$, and let $q = \pi^t(p)$ for some $t \in \mathbb{R}$. Let K_i be a sequence in $\mathbb{R}^+$ with $K_i \to \infty$. Then for each $i = 1, 2, \cdots$, we can choose $V_i \in T_qO(p)$ and $s_i \in \mathbb{R}$ such that

$$||D\pi^{s_i}(V_i)|| > K_i||V_i||.$$

Since $V_i \in T_qO(p)$, there exists $l_i \in \mathbb{R}$ satisfying $V_i = l_iV_1$ for each $i = 2, 3, \cdots$. For each $i = 1, 2, \cdots$, we let

$$U_i = D\pi^{-t}(V_i).$$

Since π is orbitally pointwisely Lipschitz stable in variation at p, there exists $K \geq 1$ such that

$$||D\pi^t(U)|| \leq K||U||$$

for any $t \in \mathbb{R}$ and any $U \in T_pO(p)$. For each $i = 1, 2, \cdots$, we have

$$||D\pi^{s_i}(V_i)|| = ||D\pi^{s_i+t}(U_i)|| \leq K||U_i||$$

Consequently we get

$$\frac{K_i||V_i||}{||U_i||} \leq \frac{||D\pi^{s_i}(V_i)||}{||U_i||} \leq K,$$

for each $i = 1, 2, \cdots$. However we have

$$\frac{||V_i||}{||U_i||} = \frac{l_i||V_1||}{l_i||D\phi^{-t}(V_1)||} = \frac{||V_1||}{||U_1||}$$

The contradiction implies that π is orbitally Lipschitz stable in variation on the orbit set $O(p)$. □

DEFINITION 2.3. A C^1 dynamical system π on M is said to be *orbitally Lipschitz stable in variation at* $p \in M$ if p is a regular point, and there exist $K = K(p) \geq 1$ and $\delta = \delta(p) > 0$ such that

$$||D\pi^t(V)|| \leq K||V||$$

for any $t \in \mathbb{R}$ and any $V \in TO(p)$ with $||V|| < \delta$, where $TO(p) = \bigcup_{x \in O(p)} T_xO(p)$. π is said to be *globally orbitally Lipschitz stable in variation* at $p \in M$ if $\delta = \infty$. We say that π is *orbitally Lipchitz stable in variation* if π is orbitally Lipschitz stable in variation at every point of M, and one can choose $K \geq 1$ and $\delta > 0$ independently of the points of M.

Lemma 2.4. The property of orbital Lipschitz stability in variation of a C^1 dynamical system on M is independent of the choice of the Riemannian metric on M if M is compact.

Proof. Let g, g' be any two Riemannian metrics on M. Then there exist $\alpha, \beta > 0$ such that

$$\alpha||V||_g \leq ||V||_{g'} \leq \beta||V||_g,$$

for any $V \in TM$. Suppose π is orbitally Lipschitz stable in variation at a regular point $p \in M$ under the metric g. Then there exists $K \geq 1$ such that

$$||D\pi^t(V)||_g \leq K||V||_g,$$

for any $t \in \mathbb{R}$ and any $V \in TO(p)$. For any $V \in TO(p)$ and $t \in \mathbb{R}$, we have

$$||D\pi^t(V)||_{g'} \leq \beta||D\pi^t(V)||_g \leq \beta K||V||_g \leq \frac{\beta K}{\alpha}||V||_{g'}.$$

This means that π is orbitally Lipschitz stable in variation under the metric g'. □

It is clear that if π is Lipschitz stable (or Lipschitz stable in variation) then it is orbitally Lipschitz stable (or orbitally Lipschitz stable in variation), respectively. However the converse does not hold.

LEMMA 2.5. *A C^1 dynamical system π on M is orbitally Lipschitz stable in variation at a regular point $p \in M$ if and only if it is globally orbitally Lipschitz stable in variation at $p \in M$.*

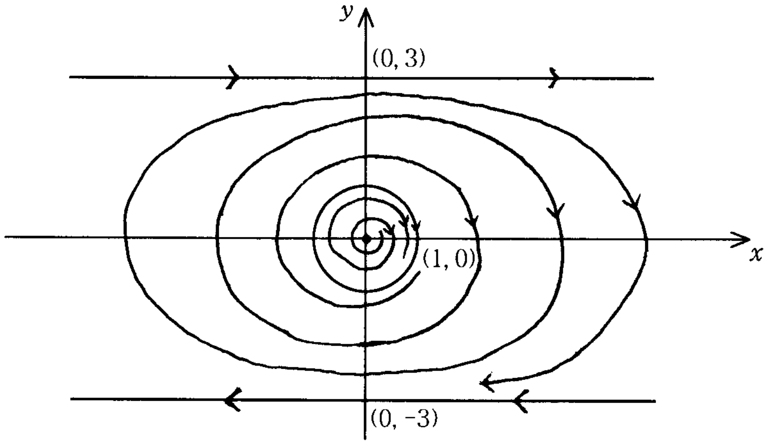
Proof. It is straightforward. □

In the following theorem which is a main result of [5], we can see that two concepts of Lipschitz stability and Lipschitz stability in variation of a C^1 dynamical system on a connected, complete Riemannian manifold are equivalent.

THEOREM 2.6. *Suppose M is a connected, complete Riemannian manifold. A C^1 dynamical system π on M is Lipschitz stable if and only if it is Lipschitz stable in variation.*

However the following example shows that two notions of orbital Lipschitz stability and orbital Lipschitz stability in variation of a C^1 dynamical system on a connected, complete Riemannian manifold need not be equivalent.

EXAMPLE 2.7. Let $M = \{(x,y) \in \mathbb{R}^2 : -3 \le y \le 3\}$ and $D = \{(x,y) \in M : x^2 + y^2 < 1\}$. Consider a C^1 dynamical system π on the space M with the constant speed at every point of $M - D$, which is given by the following figure :



Let $A = \{(x,y) \in M : y = -3 \text{ or } 3\}$. For any $x \in M - (A \cup \overline{D})$, we have

$$L^+(x) = A \text{ and } L^-(x) = S^1.$$

For any $x \in D - \{(0,0)\}$, we get

$$L^+(x) = S^1 \text{ and } L^-(x) = \{(0,0)\}.$$

If $x \in A$, then we have

$$L^+(x) = L^-(x) = \emptyset.$$

Every point of S^1 is periodic , and $(0,0)$ is the unique fixed point of π. It is clear that π is orbitally Lipschitz stable in variation at $p = (0,2)$. But we can see that π is not orbitally Lipschitz stable at p. To show this, we choose two sequences $\{x_n\}$, $\{y_n\}$ in

$$O(p) \cap \{(0,y) : 1 < y < 3\}$$

which are converging to $(0,3)$, and $x_n \ne y_n$ for each $n = 1, 2, \cdots$. Then we have

$$d(x_n, y_n) \to 0 \text{ as } n \to \infty$$

Moreover we can choose a sequence $\{t_n\}$ in $\mathbb{R}^+$ such that

$$\frac{d(\pi^{t_n}(x_n), \pi^{t_n}(y_n))}{d(x_n, y_n)} \to \infty$$

as $n \to \infty$. This means that π is not orbitally Lipchitz stable at p.

Let f be a diffeomorphism on a smooth manifold M. If there exists a C^r dynamical system π on M such that $\pi^t = f$ for some $t \in (0,1]$ then we say that f is C^r *embedded* in the dynamical system π. The embedding problem in dynamic theory is the study of the existence of such dynamical system π (see [8]). Elaydi and Farran claimed that if a diffeomorphism f on a complete, connected Riemannian manifold M is C^∞ embedded in a dynamical system π on M and f is Lipchitz stable then π is also Lipschitz stable ([4], Theorems 2.3 and 2.5). In [3], Chu and Lee gave an example to show that the above result by Elaydi and Farran does not hold.

Here we will study the embedding problem of orbital Lipschitz stability and orbital Lipschitz stability in variation. For our purpose, we introduce the notions of orbital Lipchitz stability and orbital Lipschitz stability in variation of C^r diffeomorphisms f on M.

DEFINITION 2.8. Suppose a homeomorphism f on M is C^0 embedded in a $C^{\frac{1}{2}}$ dynamical system π on M. We say that f is $\{$*positively*$\}$ *orbitally Lipschitz stable at* $p \in M$ under π if there exist $K \geq 1$ and $\delta > 0$ such that

$$d(f^n(x), f^n(y)) \leq K d(x,y)$$

for any $n \in \{\mathbb{Z}^+\}\mathbb{Z}$ and any $x, y \in O(p)$ with $d(x,y) < \delta$, where $O(p)$ is the orbit of π through p.

DEFINITION 2.9. Suppose a diffeomorphism f on M is C^1 embedded in a dynamical system π on M. We say that f is $\{$*positively*$\}$ *orbitally Lipschitz stable in variation* at $p \in M$ under π if there exists $K \geq 1$ such that

$$||Df^n(V)|| \leq K||V||$$

for any $n \in \{\mathbb{Z}^+\}\mathbb{Z}$ and any $V \in TO(p)$, where Df^n denotes the derivative map of $f^n : M \to M$.

Now we have a question: *Suppose a C^r diffeomorphism f on M is C^r embedded in a dynamical system π on M and f is orbitally Lipschitz stable (or orbitally Lipschitz stable in variation) under π. Is π orbitally Lipschitz stable (or orbitally Lipschitz stable in variation), respectively?*

The following example which is a slight modification of Example 2.4 in [2] shows that the embedding problem of orbital Lipschitz stability does not well behave even if the phase space is compact.

EXAMPLE 2.10. Let us consider the dynamical system ϕ on the unit disc $M = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$, generated by the differential system (polar coordinates)

$$\begin{cases} r^3 \cos^2\theta(r\sin\theta - \frac{r'}{2\pi}\cos\theta) = \sin\theta(r\cos\theta + \frac{r'}{2\pi}\sin\theta)^2 \\ \theta' = 2\pi \end{cases}$$

Then the orbit $O(a,0)$ of ϕ passing through a point $(a,0)$, $0 < a \leq 1$, in M is the ellipse

$$\{(x,y) \in M : \frac{x^2}{a^2} + \frac{y^2}{a^4} = 1\},$$

and $(0,0)$ is the unique fixed point of ϕ. Under a small perturbation of the system ϕ on M, we can obtain the $C^{\frac{1}{2}}$ dynamical system π on M satisfying the following properties;

(1) the orbit $O(1,0)$ of π through the point $(1,0)$ is periodic;
(2) for any point $p = (a,0)$, $0 < a < 1$, in M,

$$\pi^{\frac{1}{4}}(p) = (a^2, \frac{\pi}{2}),\ \pi^{\frac{1}{2}}(p) = (a,\pi),\ \pi^{\frac{3}{4}}(p) = (a^2, \frac{3}{2}\pi),$$
$$\pi^1(p) = (a^2, 0),\ L^+(p) = \{(0,0)\} \text{ and } L^-(p) = S^1.$$

Let $\pi^1 = f$. Then it is clear that f is positively orbitally Lipschitz stable at $p = (\frac{1}{2}, 0)$ under π. However π is not positively orbitally Lipschitz stable at p. To show this, we let

$$x_n = ((\frac{1}{2})^{2n}, \frac{\pi}{2}),\quad y_n = ((\frac{1}{2})^{4n}, \frac{\pi}{2})$$

for each $n = 1, 2, \cdots$. Then we have

$$x_n, y_n \in O(p), \text{ and } d(x_n, y_n) \to 0 \text{ as } n \to \infty.$$

Moreover we get

$$\lim_{n\to\infty} \frac{d(\pi^{\frac{1}{4}}(x_n), \pi^{\frac{1}{4}}(y_n))}{d(x_n, y_n)} = \lim_{n\to\infty} \frac{1}{(\frac{1}{2})^n + (\frac{1}{2})^{2n}} = \infty.$$

This means that π is not positively orbitally Lipschitz stable at p.

However the following theorem shows that the embedding problem of orbital Lipschitz stability in variation does well behave whenever the phase space is compact.

THEOREM 2.11. *Let M be a compact C^1 manifold. If a diffeomorphism f on M is C^1 embedded in a dynamical system π on M, then π is orbitally Lipschitz stable in variation if and only if f is orbitally Lipschitz stable in variation under π.*

Proof. Let $u > 0$ be such that $\pi^u = f$. Suppose f is orbitally Lipschitz stable in variation at p under π. Then there exists $K \geq 1$ such that

$$||Df^n(W)|| \leq K||W||$$

for any $n \in \mathbb{Z}$ and any $W \in TO(p)$. Since the phase space M is compact and the tangent map $D\pi^t : TM \to TM$ is continuous, we have that

$$\sup\{||D\pi^t(V)|| : ||V|| \leq 1, t \in [0, u]\} \equiv L$$

is finite. For any $t \in \mathbb{R}$, we can choose $n \in \mathbb{Z}$ and $s \in [0, u)$ satisfying $t = nu + s$. Then for any $V \in TO(p)$, we have

$$||D\pi^t(V)|| \leq K||D\pi^s(V)|| \leq KL||V||.$$

This implies that π is orbitally Lipschitz stable in variation at p. □

3. Exponential Asymptotic Stability

In [6], Elaydi and Farran introduced the notion of exponential asymptotic stable dynamical systems on Riemannian manifolds, in which some general properties were investigated. In particular, they claimed that if an exponential asymptotic stable discrete system is embedded in a continuous system, then the continuous system is also exponentially asymptotic stable. As a corollary, they obtained that if π^t is a contraction for some $t \in \mathbb{R}^+$ then π^s is a contraction for all $t \in \mathbb{R}^+$(see [6], Theorem 1.11 and Corollary 1.12).

In this section we claim that the above results do not hold, i.e. a C^1 dynamical system on a Riemannian manifold which is embedded by an exponential asymptotic stable diffeomorphism need not be exponentially asymptotic stable. However we show that a C^1 dynamical system on a Riemannian manifold which is embedded by a diffeomorphism which satisfies the exponential asymptotic stability in variation is also exponentially asymptotic stable in variation.

We say that a dynamical system π on M is *exponentially asymptotic stable* if there exist $K \geq 1$, $\alpha > 0$ and $\delta > 0$ such that

$$d(\pi^t(x), \pi^t(y)) \leq Ke^{-\alpha t} d(x, y)$$

for any $t \in \mathbb{R}^+$ and any $x, y \in M$ with $d(x, y) < \delta$. A C^1 dynamical system π on M is said to be *exponentially asymptotic stable in variation* if there exist $K \geq 1$ and $\alpha > 0$ such that

$$||D\pi^t(V)|| \leq Ke^{-\alpha t}||V||$$

for any $t \in \mathbb{R}^+$ and any $V \in TM$.

Similarly we can give the concepts of exponential asymptotic stability and exponential asymptotic stability in variation of C^r diffeomorphisms f on M (for more details, see [6]).

First of all, we study the embedding problem of exponential asymptotic stability of a diffeomorphism in a dynamical system; which was investigated in [6].

As in Section 2, we can introduce the notions of orbital exponential asymptotic stability and orbital exponential asymptotic stability in variation of a dynamical system π (or a diffeomorphism f) on M.

DEFINITION 3.1. A dynamical system π on M is *orbitally exponentially asymptotic stable at* $p \in M$ if there exist $K \geq 1, \alpha > 0$ and $\delta > 0$ such that

$$d(\pi^t(x), \pi^t(y)) \leq Ke^{-\alpha t} d(x, y)$$

for any $t \in \mathbb{R}^+$ and any $x, y \in O(p)$ with $d(x, y) < \delta$. We say that π is *orbitally exponentially asymptotic stable* if π is orbitally exponentially asymptotic stable at every point of M and one can choose $K \geq 1$, α and $\delta > 0$ independently of the points of M.

DEFINITION 3.2. A C^1 dynamical system π on M is *orbitally exponentially asymptotic stable in variation at* $p \in M$ if there exist $K \geq 1$ and $\alpha > 0$ such that

$$||D\pi^t(V)|| \leq Ke^{-\alpha t}||V||$$

for any $t \in \mathbb{R}^+$ and any $V \in TO(p)$, where $O(p)$ is the orbit of π through p. We say that π is *orbitally exponentially asymptotic stable in variation* if π is orbitally exponentially asymptotic stable in variation at every point of M, and one can choose $K \geq 1$ and $\alpha > 0$ independently of the points of M.

The following example which is a small perturbation of Example 1 in [3] shows that a dynamical system on a Riemannian manifold which is embedded by an exponential asymptotic stable diffeomorphism need not be exponentially asymptotic stable. This fact disproves Theorem 1.11 in [6].

EXAMPLE 3.3. Let us consider the smooth dynamical system ϕ on $M = \{(x, y) \in \mathbb{R}^2 : y > 0\}$ generated by the differential system

$$\begin{cases} \dot{x} = -y + \frac{x^2+y^2+1}{2y} \\ \dot{y} = 2\pi x \end{cases}$$

Then each orbit of ϕ is periodic with the period 1, and the orbit $O(0, b)$ of ϕ passing through a point $(0, b)$, $b > 0$, in M is the circle

$$\{(x, y) \in M : x^2 + (y - \frac{1+b^2}{2b})^2 = (\frac{1-b^2}{2b})^2\}.$$

Under a small perturbation of the system ϕ on M, we can obtain a C^1 dynamical system π on M satisfying the following properties;

(1) $(0, 1)$ is the unique fixed point of π;
(2) for any point $p = (0, b), 0 < b < 1$, of M,

$$L^+(p) = \{(0,1)\}, \ \ L^-(p) = \phi, \ \text{ and } \pi^{\frac{1}{2}}(p) \in \{(0,y) : y > 0\};$$

(3) if $A = \{(0,b) : 0 < b \leq 1\}$ then $O(A) = M$; and

(4) π^1 is a contraction.

Let $\pi^1 = f$. Then it is clear that f is exponentially asymptotic stable. However we can see that the system π is not exponentially asymptotic stable. To show this, we let

$$x_n = (0, (\frac{1}{2})^n), \quad y_n = (0, (\frac{1}{2})^{n+1})$$

for each $n = 1, 2, \cdots$. Then we have

$$\lim_{n\to\infty} \frac{d(\pi^{\frac{1}{2}}(x_n), \pi^{\frac{1}{2}}(y_n))}{d(x_n, y_n)} = \infty.$$

This implies that π is not exponentially asymptotic stable.

REMARKS 3.4. It is worthwhile to notice that there is a gap in the proof of [6, Theorem 1.11]. The fact that invalidates the proof is the following statement in that proof: "*ϕ is globally positively Lipschitz stable in variation. It follows from* Theorem 2.5 *in* [4] *that π is globally positively Lipschitz stable in variation*" (Here [4] is the reference [5] in the present paper). However we can see that above statement does not hold if M is not compact, as we can see in the above Example 3.3.

The following theorem shows that the embedding problem of exponential asymptotic stability in variation does well behave whenever the phase space is compact.

THEOREM 3.5. *Let M be a compact Riemannian manifold, and suppose a diffeomorphism f on M is C^1 embedded in a dynamical system π on M. Then π is orbitally exponentially asymptotic stable in variation (or exponentially asymptotic stable in variation) if and only if f is orbitally exponentially asymptotic stable in variation under π (or exponentially asymptotic stable in variation), respectively.*

Proof. Let $u > 0$ be such that $\pi^u = f$, and suppose f is orbitally exponentially asymptotic stable in variation at $p \in M$ under π. Then there exist $K \geq 1$ and $\alpha > 0$ such that

$$||Df^n(V)|| \leq Ke^{-\alpha n}||V||$$

for any $n \in \mathbb{Z}^+$ and any $V \in TO(p)$. For any $t \in \mathbb{R}^+$, we choose $n \in \mathbb{Z}^+$ and $s \in [0, u)$ such that $t = nu + s$. Then for any $V \in TO(p)$ we have

$$\begin{aligned} \|D\pi^t(V)\| &\leq Ke^{-\alpha n}\|D\pi^s(V)\| \\ &\leq KLe^{-\alpha n}\|V\| \\ &\leq KLe^{\alpha s}e^{-\alpha t}\|V\|, \end{aligned}$$

where $L = \sup\{\|D\pi^t(W)\| : \|W\| = 1,\ 0 \leq t \leq u\}$. This implies that π is orbitally exponentially asymptotic stable in variation at p.

Similarly we can show that π is exponentially asymptotic stable in variation if and only if f is exponentially asymptotic stable in variation. □

REMARKS 3.6. The Corollary 1.12 in [6] says that if π^s is a contraction for some $s \in \mathbb{R}^+$ then each π^t is a contraction for all $t \in \mathbb{R}^+$. However the property does not hold. To show this, we let π be the dynamical system on the unit disc given in Example 2.10. Then π^1 is a contraction, but $\pi^{\frac{1}{4}}$ is not a contraction.

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