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# THE GIBBS MEASURE AND COBOUNDARY CONDITION

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ABSTRACT. We investigate coboundary conditions for two functions defined on a mixing subshift of finite type to have the same Gibbs measure. Also we find conditions for a function to be a coboundary.

## 0. Introduction

We consider a dynamical system together with a real–valued continuous function which is called the energy distribution. The variational principle asserts that the supremum of all measure–theoretic pressures on such a system is equal to the topological pressure. If the dynamical system is a subshift, then the topological pressure is attained as the maximum of measure–theoretic ones, and the maximizing measures are called the *equilibrium states* of the energy distribution. In the special case that the dynamical system is a mixing subshift of finite type and the energy distribution has summable variation, there is a unique equilibrium state, called the *Gibbs measure* of the energy distribution. A necessary and sufficient condition for two energy distributions (with summable variation) on a mixing subshift of finite type to have the same Gibbs measure is expressed in terms of the coboundary operator. Specifically, if two energy distributions $\phi$ and $\psi$ on a mixing subshift of finite type have summable variation, then their Gibbs measures are identical if and only if there are a constant $K$ and a continuous function $\beta$ such that

$$\phi - \psi = K + \partial\beta.$$

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However, the known proof of this theorem is somewhat roundabout. In this article, we investigate the relation between the Ruelle operators and the coboundary operator, and consequently present a direct and rather algebraic proof of the theorem. In fact, we will prove a slightly generalized version of the theorem. Then we will discuss conditions on a function $\phi$ for which there is a function $\beta$ such that $\phi = \partial\beta$.

This paper is organized as follows. In §1, we introduce some notions such as mixing subshifts of finite type, the Ruelle operators and the coboundary operator; obtain some technical results which will be used in the sequel. In §2, we will give a simple proof of the theorem mentioned in the previous paragraph. We conclude this paper with some discussions about conditions for a function to be a coboundary (§3).

## 1. Preliminaries and Notations

Let $X$ be a compact metric space and $T : X \to X$ a continuous surjective mapping. Let $\mathcal{C}(X)$ denote the Banach space of all real–valued continuous functions on $X$ with the supremum norm $\|\cdot\|$. For $\phi \in \mathcal{C}(X)$ let $P_\phi$ denote the topological pressure of $\phi$. A $T$–invariant Borel probability measure $\mu$ is called an *equilibrium state* of $\phi$ if

$$P_\phi = h(\mu) + \int_X \phi\, d\mu,$$

where $h(\mu)$ is the measure–theoretic entropy of $\mu$.

The *coboundary operator* $\partial_T : \mathcal{C}(X) \to \mathcal{C}(X)$ of $T$ is defined by

$$\partial_T\beta = \beta - \beta\circ T \qquad (\beta \in \mathcal{C}(X)).$$

Since

$$\int_X \partial_T\beta\, d\mu = \int_X \beta\, d\mu - \int_X \beta\circ T\, d\mu = 0,$$

for all $\beta \in \mathcal{C}(X)$ and $T$–invariant Borel measures $\mu$ on $X$, it follows that for all $\phi, \beta \in \mathcal{C}(X)$, $\phi$ and $\phi + \partial\beta$ have the same set of equilibrium states.

If $T$ is a local homeomorphism, then for each $\phi \in \mathcal{C}(X)$ the *Ruelle operator* $L_{T,\phi} : \mathcal{C}(X) \to \mathcal{C}(X)$ is defined by

$$(L_{T,\phi}f)(x) = \sum_{y\in T^{-1}(x)} e^{\phi(y)} f(y) \qquad (f \in \mathcal{C}(X), x \in X).$$

In this case the dual operator $L^*_{T,\phi}$ of $L_{T,\phi}$ is defined by

$$\int_X (L_{T,\phi} f) d\mu = \int_X f d(L^*_{T,\phi}\mu)$$

for continuous real–valued functions $f$ on $X$ and signed Borel measures $\mu$ on $X$.

LEMMA 1.1. *Let $X$ be a compact metric space and $T : X \to X$ a surjective local homeomorphism. Then for $\phi, \beta \in \mathcal{C}(X)$ we have the following.*

(a) $L_{T,\phi+\partial_T \beta}(f) = e^{-\beta} L_{T,\phi}(e^{\beta} f) \qquad (f \in \mathcal{C}(X))$.

(b) $f L_{T,\phi}(g) = L_{T,\phi}((f \circ T) \cdot g) \qquad (f, g \in \mathcal{C}(X))$.

*Proof.* It follows directly from the definitions of Ruelle operators and the coboundary operator. □

REMARK. The statement (a) in the above lemma can be rewritten in the form

$$L_{T,\phi+\partial_T \beta} = (L_{I,\beta})^{-1} \circ L_{T,\phi} \circ L_{I,\beta},$$

where $I : X \to X$ is the identity mapping.

LEMMA 1.2. *Let $X$ be a zero dimensional compact metric space and $T : X \to X$ a surjective local homeomorphism. Then for all $\phi, \psi \in \mathcal{C}(X)$, $L_{T,\phi} = L_{T,\psi}$ implies $\phi = \psi$.*

*Proof.* Let $y_0 \in X$ be arbitrary. Since $T$ is a local homeomorphism and $X$ is zero dimensional, there is a clopen neighborhood $U$ of $y_0$ such that $T|_U$ is one–to–one. Let $h_U$ denote the characteristic function of $U$. Then

$$L_{T,\phi} h_U(Ty_0) = \sum_{y \in T^{-1}(Ty_0)} e^{\phi(y)} h_U(y) = e^{\phi(y_0)}.$$

This proves our assertion. □

Throughout this paper, we will be interested in a special kind of dynamical systems, namely the one–sided subshifts of finite type, which are defined as follows. Let $\mathcal{A}$ be a finite set and $M$ a 0–1, $\mathcal{A} \times \mathcal{A}$ matrix, and assume that every column and row of $M$ has a non–zero entry. Define $\Sigma_M$ and $\sigma : \Sigma_M \to \Sigma_M$ by

$$\Sigma_M = \{x = \langle x_i \rangle_{i=0}^{\infty} \in \mathcal{A}^{\mathbb{N}} : M(x_i, x_{i+1}) = 1 \quad \text{for} \quad i = 0, 1, 2, \dots\},$$

and

$$\sigma : \Sigma_M \ni x_0 x_1 x_2 \cdots \mapsto x_1 x_2 x_3 \cdots \in \Sigma_M.$$

Let $d : \Sigma_M \times \Sigma_M \to \mathbb{R}$ be defined by

$$d(x, y) = \sum_{i=0}^{\infty} (1 - \delta(x_i, y_i)) 2^{-i}$$

for $x = x_0 x_1 \dots, y = y_0 y_1 \dots \in \Sigma_M$. Then $d$ is a metric on $\Sigma_M$. With this metric, $\Sigma_M$ becomes a compact metric space and $\sigma$ a positively expansive surjective local homeomorphism. The dynamical system $(\Sigma_M, \sigma)$ is called the *one–sided subshift of finite type* (or *one–sided topological Markov shift*) defined by $M$, and $\sigma$ is called the *shift map.* If the matrix $M$ is primitive, i.e., there is a positive integer $N$ such that every entry of $M^N$ is positive, then the one–sided subshift $(\Sigma_M, \sigma)$ is said to be *mixing.*

Let $(\Sigma_M, \sigma)$ be a one–sided subshift of finite type. For $\phi \in \mathcal{C}(\Sigma_M)$ and $k = 0, 1, 2, \dots$, define the *k–th variation* of $\phi$ by

$$\text{var}_k \phi = \sup\{|\phi(x) - \phi(y)| : x, y \in \Sigma_M,\ x_0 x_1 \dots x_k = y_0 y_1 \dots y_k\}.$$

Since $\phi$ is uniformly continuous, it follows that $\text{var}_k \phi \to 0$ as $k \to 0$. The function $\phi$ is said to have *summable variation* if

$$\sum_{k=0}^{\infty} \text{var}_k \phi < \infty.$$

The set of functions with summable variation is denoted by $\mathcal{S}(\Sigma_M)$. If there are $a > 0$ and $b \in (0, 1)$ such that

$$\text{var}_k \phi \le a b^k \qquad (k = 0, 1, 2, \dots)$$

then $\phi$ is said to be *Hölder*, and we denote by $\mathcal{H}(\Sigma_M)$ the set of Hölder functions. Clearly $\mathcal{C}(\Sigma_M) \subset \mathcal{S}(\Sigma_M) \subset \mathcal{H}(\Sigma_M)$. From here on, we will use the simpler notations $\partial$ and $L_\phi$ to denote the operators $\partial_\sigma$ and $L_{\sigma,\phi}$, respectively.

THEOREM 1.3. *Let $(\Sigma_M, \sigma)$ be a mixing one-sided subshift of finite type and $\beta \in \mathcal{C}(\Sigma_M)$. Then $\beta \in \mathcal{H}(\Sigma_M)$ if and only if $\partial\beta \in \mathcal{H}(\Sigma_M)$.*

*Proof.* It is clear that if $\beta \in \mathcal{H}(\Sigma_M)$ then $\partial\beta \in \mathcal{H}(\Sigma_M)$. Conversely, assume that $\partial\beta \in \mathcal{H}(\Sigma_M)$. For simplicity, write $\phi = \partial\beta$. Since $\phi \in \mathcal{H}(\Sigma_M)$, there are $a > 0$ and $b \in (0,1)$ such that

$$\mathrm{var}_k\phi \le ab^k \qquad (k = 0, 1, 2, \dots).$$

Since $(\Sigma_M, \sigma)$ is mixing, there is a point $z = z_0 z_1 z_2 \dots$ in $\Sigma_M$ such that the forward orbit $\Gamma = \{z, \sigma z, \sigma^2 z, \dots\}$ of $z$ is dense in $\Sigma_M$.

Let $k \in \mathbb{N}$, $x, y \in \Gamma$, and assume that $x_0 x_1 \dots x_k = y_0 y_1 \dots y_k$. We will show that

$$|\beta(x) - \beta(y)| \le \frac{ab}{1-b} b^k,$$

from which our assertion follows because $\Gamma$ is dense in $\Sigma_M$ and $\beta$ is continuous.

Since $x, y \in \Gamma$, there are $i, j \in \mathbb{N}$ such that $x = \sigma^i z$ and $y = \sigma^j z$. We may assume that $i < j$. From the assumption that $x_0 x_1 \dots x_k = y_0 y_1 \dots y_k$, we have $z_i z_{i+1} \dots z_{i+k} = z_j z_{j+1} \dots z_{j+k}$, and hence there is a point $w$ in $\Sigma_M$ such that

$$\sigma^{j-i} w = w \quad \text{and} \quad z_i z_{i+1} \dots z_{j+k} = w_0 w_1 \dots w_{j-i+k}.$$

Then we have

$$\sum_{l=0}^{j-i-1} \phi(\sigma^l w) = 0,$$

and

$$|\phi(\sigma^{i+l} z) - \phi(\sigma^l w)| \le ab^{j-i+k-l} \qquad (l = 0, 1, \dots, j-i-1).$$

Therefore

$$\begin{aligned}|\beta(x)-\beta(y)| &= |\beta(\sigma^i z)-\beta(\sigma^j z)|\\ &= \left|\sum_{l=0}^{j-i-1}\phi(\sigma^{i+l}z)\right|\\ &\le \sum_{l=0}^{j-i-1}|\phi(\sigma^{i+l}z)-\phi(\sigma^l w)|\\ &\le \sum_{l=0}^{j-i-1} ab^{j-i+k-l}\\ &\le \frac{ab}{1-b}b^k.\end{aligned}$$

□

If $\partial\beta \in \mathcal{S}(\Sigma_M)$, $\beta$ may not necessarily lie in $\mathcal{S}(\Sigma_M)$ as we see in the following example.

EXAMPLE 1.4. Let $\beta : \{0,1\}^{\mathbb{N}} \to \mathbb{R}$ be defined by

$$\beta(x) = \sum_{n=0}^{\infty}\frac{x_n}{(n+1)^2} \qquad (x = x_0x_1x_2\cdots \in \{0,1\}^{\mathbb{N}}).$$

If $x, y \in \{0,1\}^{\mathbb{N}}$ and $x_0x_1\ldots x_k = y_0y_1\ldots y_k$, then

$$\beta(x)-\beta(y) = \sum_{n=k+1}^{\infty}\frac{x_n-y_n}{(n+1)^2}.$$

Hence

$$\mathrm{var}_k\beta = \sum_{n=k+1}^{\infty}\frac{1}{(n+1)^2} \ge \int_{k+1}^{\infty}\frac{1}{(v+1)^2}dv = \frac{1}{k+2},$$

so that $\beta$ is not in $\mathcal{S}(\{0,1\}^{\mathbb{N}})$. On the other hand,

$$\begin{aligned}|\partial\beta(x)-\partial\beta(y)| &\le \sum_{n=k+1}^{\infty}\frac{2n+1}{n^2(n+1)^2}|x_n-y_n|\\ &\le \sum_{n=k+1}^{\infty}\frac{2n+1}{n^2(n+1)^2}\\ &\sim \frac{1}{(k+1)^3},\end{aligned}$$

whenever $x, y \in \{0,1\}^{\mathbb{N}}$ and $x_0x_1 \dots x_k = y_0y_1 \dots y_k$. Therefore $\partial\beta \in \mathcal{S}(\{0,1\}^{\mathbb{N}})$.

## 2. Coincidence of the Gibbs Measures

Let $(\Sigma_M, \sigma)$ be a mixing one–sided subshift of finite type and let $\phi \in \mathcal{S}(\Sigma_M)$. Then it is known that there is one and only one equilibrium state $\mu_\phi$ of $\phi$, called the *Gibbs measure* of $\phi$. We can easily extend the result when $\phi \in \mathcal{S}(\Sigma_M) + \partial\mathcal{C}(\Sigma_M)$. The following theorem is called *Ruelle's operator theorem.*

THEOREM 2.1. *Let $(\Sigma_M, \sigma)$ be a mixing one–sided subshift of finite type, and let $\phi \in \mathcal{S}(\Sigma_M) + \partial\mathcal{C}(\Sigma_M)$. Then there are a positive real number $\lambda$, a continuous function $f$ which is positive everywhere on $\Sigma_M$, and a positive Borel measure $\mu$ such that*
(a) *$L_\phi f = \lambda f, L_\phi^* \mu = \lambda\mu$, and $\int_{\Sigma_M} f d\mu = 1$,*
(b) *for all $g \in \mathcal{C}(\Sigma_M)$, $\|\lambda^{-n} L_\phi^n g - (\int_{\Sigma_M} g d\mu) f\| \to 0$ as $n \to \infty$, and*
(c) *the Gibbs measure of $\phi$ is given by $f d\mu$.*
*Moreover, the $\lambda$–eigenspaces of $L_\phi$ and $L_\phi^*$ are 1–dimensional.*

*Proof.* For the special case when $\phi \in \mathcal{S}(\Sigma_M)$, see [4]. The general case follows from the special case, because for all $\phi, \beta \in \mathcal{C}(\Sigma_M)$ we have

$$L_{\phi+\partial\beta} = L_{I,\beta}^{-1} \circ L_\phi \circ L_{I,\beta},$$

and $\phi$ and $\phi + \partial\beta$ have the same set of equilibrium states. □

From here on, any triple $(\lambda, f, d\mu)$ which satisfies the conditions (a) and (b) of the above theorem will be called a *Ruelle triple* of $\phi (\in \mathcal{S}(\Sigma_M) + \partial\mathcal{C}(\Sigma_M))$.

LEMMA 2.2. *Let $(\Sigma_M, \sigma)$ be a mixing one–sided subshift of finite type, and let $(\lambda, f, d\mu)$ be a Ruelle triple of $\phi \in \mathcal{S}(\Sigma_M) + \partial\mathcal{C}(\Sigma_M)$. Then for all nonempty clopen subset $U$ of $\Sigma_M$, we have $\mu(U) > 0$.*

*Proof.* Let $U$ be an arbitrary clopen subset of $\Sigma_M$ and let $h_U$ denote the characteristic function of $U$. Since $(\Sigma_M, \sigma)$ is mixing, there is a positive integer $N$ such that

$$L_\phi^N h_U(x) = \sum_{y \in \sigma^{-N}(x)} \exp(\phi(y) + \phi(\sigma y) + \cdots + \phi(\sigma^{N-1} y)) h_U(y) > 0$$

for all $x \in \Sigma_M$. Since $\Sigma_M$ is compact, the continuous function $L_\phi^N h_U$ has a positive minimum, so that

$$\mu(U) = \int_{\Sigma_M} h_U d\mu = \lambda^{-N} \int_{\Sigma_M} L_\phi^N h_U d\mu > 0. \qquad \square$$

THEOREM 2.3. *Let $(\Sigma_M, \sigma)$ be a mixing one–sided subshift of finite type, $\phi, \psi \in \mathcal{S}(\Sigma_M) + \partial\mathcal{C}(\Sigma_M)$, and assume that $(1, f, d\mu)$ and $(1, g, d\nu)$ are Ruelle triples of $\phi$ and $\psi$, respectively. Then the following are equivalent:*

(a) $\phi = \psi$.
(b) $\mu = c\nu$ *for some positive constant* $c$.
(c) $L_\phi = L_\psi$.

*Proof.* The implication (a)⇒(b) follows from Ruelle's operator theorem, and (c)⇒(a) follows from Lemma 1.2. To prove (b)⇒(c), suppose that $\mu = c\nu$ for some positive constant $c$. Then Lemma 1.1 implies that

$$\begin{aligned}
\int_{\Sigma_M} hL_\phi h' d\mu &= \int_{\Sigma_M} L_\phi((h \circ \sigma)h') d\mu \\
&= \int_{\Sigma_M} (h \circ \sigma)h' d\mu \\
&= \int_{\Sigma_M} L_\psi((h \circ \sigma)h') d\mu \\
&= \int_{\Sigma_M} hL_\psi h' d\mu \qquad (h, h' \in \mathcal{C}(\Sigma_M)).
\end{aligned}$$

Therefore we obtain $L_\phi = L_\psi$ by Lemma 2.2. $\square$

Under the assumptions of Theorem 2.3, it is not true in general that $f = g$ implies $\phi = \psi$, as we see in the following example.

EXAMPLE 2.4. Let $h_0$ and $h_1$ be the characteristic functions of $U_0 = \{x \in \{0,1\}^{\mathbb{N}} : x_0 = 0\}$ and $U_1 = \{x \in \{0,1\}^{\mathbb{N}} : x_0 = 1\}$ respectively, and let $a, b \in \mathbb{R}$ satisfy $e^a + e^b = 1$. Define $\phi$ and $\psi$ by

$$\phi = ah_0 + bh_1, \quad \psi = bh_0 + ah_1.$$

Then $\phi, \psi \in \mathcal{H}(\{0,1\}^{\mathbb{N}})$. Let $\mu$ and $\nu$ denote the Bernoulli measures on $\{0,1\}^{\mathbb{N}}$ defined by

$$\mu(U_0) = e^a, \quad \mu(U_1) = e^b, \quad \nu(U_0) = e^b, \quad \nu(U_1) = e^a.$$

Then it is easy to see that $(1, 1, \mu)$ and $(1, 1, \nu)$ are Ruelle triples of $\phi$ and $\psi$ respectively.

Now we prove the main theorem of this section in a simple and direct manner. In [1] and [4], one can find other proofs of the theorem in weaker forms.

THEOREM 2.5. *Let $(\Sigma_M, \sigma)$ be a mixing one–sided subshift of finite type, and let $\phi, \psi \in \mathcal{S}(\Sigma_M) + \partial\mathcal{C}(\Sigma_M)$. Then $\phi$ and $\psi$ have the same Gibbs measure if and only if there are a constant $K$ and a continuous function $\beta : \Sigma_M \to \mathbb{R}$ such that*

$$\phi = \psi + K + \partial\beta.$$

*Moreover, if $\phi, \psi \in \mathcal{H}(\Sigma_M)$ then $\beta \in \mathcal{H}(\Sigma_M)$.*

*Proof.* Let $(\lambda, f, d\mu)$ and $(\lambda', g, d\nu)$ denote Ruelle triples of $\phi$ and $\psi$ respectively. Then, by Ruelle's operator theorem, $fd\mu$ is the Gibbs measure of $\phi$ and $gd\nu$ is that of $\psi$.

Suppose that $\phi = \psi + K + \partial\beta$ for some constant $K$ and some continuous $\beta$. Then it is easy to see that $(\lambda' e^K, e^{-\beta} g, e^{\beta} d\nu)$ is a Ruelle triple of $\phi$. Therefore we have

$$fd\mu = e^{-\beta} g e^{\beta} d\nu = gd\nu,$$

by the uniqueness of the Gibbs measure.

Conversely, suppose that $fd\mu = gd\nu$. Then we will show that

$$\phi = \psi + (\log\lambda - \log\lambda') + \partial(\log g - \log f).$$

First of all, we may assume that $\lambda = \lambda' = 1$ since otherwise we can replace $\phi$ and $\psi$ by $\phi - \log\lambda$ and $\psi - \log\lambda'$ respectively. Since $(1, g, d\nu)$ is a Ruelle triple of $\psi$, it follows that

$$(1, \frac{f}{g} g, \frac{g}{f} d\nu)$$

is a Ruelle triple of $\psi + \partial(\log g - \log f)$. Now the assumption that $fd\mu = gd\nu$ implies that $(g/f)d\nu = d\nu$. Therefore by Theorem 2.3 we see that $\phi = \psi + \partial(\log g - \log f)$. Finally, the last statement follows from Theorem 1.3. □

EXAMPLE 2.6. Let $h_0$ and $h_1$ be the characteristic functions defined in Example 2.4. For $i, j \in \{0,1\}$ let $U_{ij} = \{x \in \{0,1\}^{\mathbb{N}} : x_0 x_1 = ij\}$ and let $h_{ij}$ be the characteristic function of $U_{ij}$. Let $a$ and $b$ be arbitrary real numbers and define $\phi$ and $\psi$ by

$$\phi = ah_0 + bh_1, \qquad \psi = ah_{00} + bh_{01} + ah_{10} + bh_{11}.$$

Then it is easy to see that

$$\phi - \psi = \partial\beta,$$

where $\beta = -bh_0 - ah_1$. Therefore the Gibbs measures of $\phi$ and $\psi$ are the same.

## 3. Conditions for a Function to be a Coboundary

In this section, we discuss conditions for a function $\phi$ to be a coboundary, i.e., $\phi = \partial\beta$ for some function $\beta$.

Let $(\Sigma_M, \sigma)$ be a mixing one–sided subshift of finite type, and let $\phi \in \mathcal{C}(\Sigma_M)$. If there is a function $\beta$ such that $\phi = \partial\beta$ then we have

$$\sum_{i=0}^{p-1} \phi(\sigma^i x) = 0,$$

for all $p = 1, 2, \dots$ and for all $x \in \Sigma_M$ satisfying $\sigma^p x = x$. The following theorem shows that the converse is true provided that $\phi \in \mathcal{S}(\Sigma_M)$ and $\beta \in \mathcal{C}(\Sigma_M)$.

THEOREM 3.1. *Let $(\Sigma_M, \sigma)$ be a mixing one–sided subshift of finite type, and let $\phi \in \mathcal{S}(\Sigma_M)$. Then there is a function $\beta \in \mathcal{C}(\Sigma_M)$ such that $\phi = \partial\beta$ if and only if*

$$\sum_{i=0}^{p-1} \phi(\sigma^i x) = 0,$$

*for all $p = 1, 2, \dots$ and for all $x \in \Sigma_M$ satisfying $\sigma^p x = x$. Moreover, if $\phi \in \mathcal{H}(\Sigma_M)$ then $\beta \in \mathcal{H}(\Sigma_M)$ also.*

*Proof.* First of all, recall that we have already proved the last statement in Theorem 1.3. Then we need only to prove the 'if' part. So assume that $\phi$ satisfies

$$\sum_{i=0}^{p-1} \phi(\sigma^i x) = 0,$$

for all $p = 1, 2, \ldots$ and for all $x \in \Sigma_M$ satisfying $\sigma^p x = x$.

Since $(\Sigma_M, \sigma)$ is mixing, there is a point $z = z_0 z_1 z_2 \ldots$ in $\Sigma_M$ such that the forward orbit $\Gamma = \{z, \sigma z, \sigma^2 z, \ldots\}$ of $z$ is dense in $\Sigma_M$. Then define $\beta_0 : \Gamma \to \mathbb{R}$ by

$$\beta_0(\sigma^j z) = -\sum_{l=0}^{j-1} \phi(\sigma^l z) \qquad (j = 0, 1, 2, \ldots).$$

It is obvious that $\phi(x) = \beta_0(x) - \beta_0(\sigma x)$ for all $x \in \Gamma$. Now a similar argument as in the proof of Theorem 1.3 shows that if $x, y \in \Gamma$ satisfies

$$x_0 x_1 \ldots x_k = y_0 y_1 \ldots y_k$$

then we have

$$|\beta_0(x) - \beta_0(y)| \leq \sum_{l=k+1}^{\infty} \mathrm{var}_l \phi.$$

Therefore $\beta_0$ is uniformly continuous on $\Gamma$, so that there is a continuous function $\beta$ on $\Sigma_M$ such that $\beta|_\Gamma = \beta_0$. Now it is clear that $\phi = \partial\beta$. □

REMARK. Using the same method as in the above proof, we can show that for $\phi \in \mathcal{S}(\Sigma_M)$ and $K \in \mathbb{R}$ there is a function $\beta \in \mathcal{C}(\Sigma_M)$ such that $\phi = K + \partial\beta$ if and only if

$$\sum_{i=0}^{p-1} \phi(\sigma^i x) = pK,$$

for all $p = 1, 2, \ldots$ and for all $x \in \Sigma_M$ satisfying $\sigma^p x = x$.

Finally, we consider coboundary relations on functions defined on two–sided subshifts of finite type. Let $(\Sigma_M, \sigma)$ be a mixing one–sided subshift of finite type. Then the *natural extension* of $(\Sigma_M, \sigma)$ is the mixing two–sided subshift of finite type $(\hat{\Sigma}_M, \hat{\sigma})$ defined by

$$\hat{\Sigma}_M = \{x = \langle x_i \rangle_{i\in\mathbb{Z}} : M(x_i, x_{i+1}) = 1 \quad \text{for} \quad i \in \mathbb{Z}\},$$

and

$$\hat{\sigma}(x)_i = x_{i+1} \qquad (x = \langle x_i \rangle_{i\in\mathbb{Z}} \in \hat{\Sigma}_M, i \in \mathbb{Z}).$$

In this case, the natural factoring $\pi : \hat{\Sigma}_M \to \Sigma_M$, defined by

$$\pi : \hat{\Sigma}_M \ni \ldots x_{-2}x_{-1}x_0x_1x_2 \cdots \mapsto x_0x_1x_2 \cdots \in \Sigma_M,$$

is a continuous open surjective mapping and satisfies

$$\pi \circ \hat{\sigma} = \sigma \circ \pi.$$

For each $\sigma$–invariant Borel probability measure $\mu$ on $\Sigma_M$ there is a unique $\hat{\sigma}$–invariant Borel probability measure $\hat{\mu}$ on $\hat{\Sigma}_M$ such that

$$\int_{\Sigma_M} f\, d\mu = \int_{\hat{\Sigma}_M} f \circ \pi\, d\hat{\mu} \qquad (f \in \mathcal{C}(\Sigma_M)).$$

It is easy to see that $\mu \mapsto \hat{\mu}$ is a one–to–one correspondence between the set of all $\sigma$–invariant Borel probability measures on $\Sigma_M$ and the set of all $\hat{\sigma}$–invariant Borel probability measures on $\hat{\Sigma}_M$. Moreover, for each $\phi \in \mathcal{C}(\Sigma_M)$, a $\sigma$–invariant measure $\mu$ is an equilibrium state of $\phi$ if and only if $\hat{\mu}$ is an equilibrium state of $\phi \circ \pi$.

For $\phi \in \mathcal{C}(\hat{\Sigma}_M)$, and $k = 0, 1, 2, \ldots$, define the $k$–th variation of $\phi$ by

$$\mathrm{var}_k\phi = \sup\{|\phi(x) - \phi(y)| : x, y \in \hat{\Sigma}_M, \quad x_i = y_i \quad \text{for} \quad |i| \le k\}.$$

Then define $\mathcal{S}(\hat{\Sigma}_M)$ and $\mathcal{H}(\hat{\Sigma}_M)$ in the obvious way. From the discussions given in [1, Chapter 1], it follows that for all $\phi \in \mathcal{S}(\hat{\Sigma}_M) + \partial\mathcal{C}(\hat{\Sigma}_M)$ there is one and only one equilibrium state of $\phi$, which is also called the Gibbs measure of $\phi$. Moreover, Theorem 2.5 holds for $(\hat{\Sigma}_M, \hat{\sigma})$ also.

The *Gibbs equivalence relation* $\sim$ on $\hat{\Sigma}_M$ is defined as follows: for $x, y \in \hat{\Sigma}_M$, $x \sim y$ if and only if there is a positive integer $n$ such that

$x_i = y_i$ whenever $|i| \geq n$. For $x \in \hat{\Sigma}_M$ let $[x]$ denote the equivalence class of $x$, i.e., $[x] = \{y \in \hat{\Sigma}_M : x \sim y\}$.

Let $\phi \in \mathcal{S}(\hat{\Sigma}_M)$. If there are a constant $K$ and a function $\beta \in \mathcal{C}(\hat{\Sigma}_M)$ such that $\phi = K + \partial\beta$, then it is easy to see that

$$\sum_{l=0}^{\infty} \left(\phi(\hat{\sigma}^l x) - \phi(\hat{\sigma}^l y)\right) = \sum_{l=1}^{\infty} \left(\phi(\hat{\sigma}^{-l} y) - \phi(\hat{\sigma}^{-l} x)\right),$$

whenever $x, y \in \hat{\Sigma}_M$ and $x \sim y$. Note that since $\phi \in \mathcal{S}(\hat{\Sigma}_M)$, the infinite sums in the above equation converge absolutely. The following theorem shows that the converse is also true.

THEOREM 3.2. *Let $(\hat{\Sigma}_M, \hat{\sigma})$ be a mixing two–sided subshift of finite type and $\phi \in \mathcal{S}(\hat{\Sigma}_M)$. Then there are a constant $K$ and a function $\beta \in \mathcal{C}(\hat{\Sigma}_M)$ such that $\phi = K + \partial\beta$ if and only if*

$$\sum_{l=0}^{\infty} \left(\phi(\hat{\sigma}^l x) - \phi(\hat{\sigma}^l y)\right) = \sum_{l=1}^{\infty} \left(\phi(\hat{\sigma}^{-l} y) - \phi(\hat{\sigma}^{-l} x)\right),$$

*whenever $x, y \in \hat{\Sigma}_M$ and $x \sim y$. Moreover, if $\phi \in \mathcal{H}(\hat{\Sigma}_M)$ then $\beta \in \mathcal{H}(\hat{\Sigma}_M)$ also.*

*Proof.* Again, we need only to prove the 'if' part. We first prove the theorem in the special case that $(\hat{\Sigma}_M, \hat{\sigma})$ has a fixed point. So assume that there is a point $z \in \hat{\Sigma}_M$ such that $\hat{\sigma} z = z$, and that $\phi$ satisfies the condition stated in the theorem. Since $(\hat{\Sigma}_M, \hat{\sigma})$ is mixing, it follows that $[z]$ is dense in $\hat{\Sigma}_M$.

Let $K = -\phi(z)$, and define a function $\beta_0 : [z] \to \mathbb{R}$ by

$$\beta_0(x) = \sum_{l=0}^{\infty} \left(\phi(\hat{\sigma}^l x) + K\right) \qquad (x \in [z]).$$

Since $z$ is a fixed point of $\hat{\sigma}$, $\beta_0$ is a well–defined function on $[z]$. Then it is clear that

$$\phi(x) = K + \beta_0(x) - \beta_0(\hat{\sigma} x) \qquad (x \in [z]).$$

Moreover, the assumption implies

$$\beta_0(x) = \sum_{l=0}^{\infty} \big(\phi(\hat{\sigma}^l x) + K\big) = -\sum_{l=1}^{\infty} \big(\phi(\hat{\sigma}^{-l} x) + K\big) \qquad (x \in [z]).$$

Now assume that $k$ is a positive integer, $x, y \in [z]$, and that $x_i = y_i$ for $|i| \le k$. Then there is a point $w \in [z]$ such that $w_i = x_i$ for $i \ge -k$ and $w_i = y_i$ for $i \le k$ so that

$$\begin{aligned}|\beta_0(x) - \beta_0(y)| &\le |\beta_0(x) - \beta_0(w)| + |\beta_0(w) - \beta_0(y)| \\ &\le \sum_{l=0}^{\infty} \big|\phi(\hat{\sigma}^l x) - \phi(\hat{\sigma}^l w)\big| + \sum_{l=1}^{\infty} \big|\phi(\hat{\sigma}^{-l} y) - \phi(\hat{\sigma}^{-l} w)\big| \\ &\le \sum_{l=k}^{\infty} \mathrm{var}_l \phi + \sum_{l=k+1}^{\infty} \mathrm{var}_l \phi.\end{aligned}$$

Since $[z]$ is dense in $\hat{\Sigma}_M$, this inequality implies our assertion.

In the general case, there is a positive integer $p$ and a point $z \in \hat{\Sigma}_M$ such that $\hat{\sigma}^p z = z$. Then $(\hat{\Sigma}_M, \hat{\sigma}^p)$ has a fixed point $z$, and therefore there is a constant $C$ and a function $\beta \in \hat{\Sigma}_M$ such that

$$\phi(x) + \phi(\hat{\sigma} x) + \cdots + \phi(\hat{\sigma}^{p-1} x) = C + \beta(x) - \beta(\hat{\sigma}^p x) \qquad (x \in \hat{\Sigma}_M).$$

Moreover, if $\phi$ is Hölder then $\beta$ is Hölder. Let

$$\psi(x) = \phi(x) - \beta(x) + \beta(\hat{\sigma} x) \qquad (x \in \hat{\Sigma}_M).$$

Then we must show that $\psi$ is a constant function. From the definition of $\psi$, we have

$$\psi(x) + \psi(\hat{\sigma} x) + \cdots + \psi(\hat{\sigma}^{p-1} x) = C \qquad (x \in \hat{\Sigma}_M),$$

so that

$$\psi(x) = \psi(\hat{\sigma}^p x) \qquad (x \in \hat{\Sigma}_M).$$

Since $(\hat{\Sigma}_M, \hat{\sigma})$ is mixing, there exists a point $x \in \hat{\Sigma}_M$ whose forward orbit $\{x, \hat{\sigma}^p x, \hat{\sigma}^{2p} x, \dots\}$ is dense in $\hat{\Sigma}_M$. Therefore we conclude that $\psi$ is a constant function. □

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