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NODAL SOLUTIONS OF NONLINEAR ELLIPTIC EQUATIONS IN ANNULAR DOMAINS

SOON YEUN JANG[1] AND DAE HYEON PAHK[2]

ABSTRACT. We investigate the existence of radial nodal solutions of the elliptic equation $\Delta u + h(|x|)f(u) = 0$, in annular domains. It is proved that for each integer $k \geq 1$, there exist at least one radially symmetric solution which has exactly k nodes.

1. Introduction

In this paper we consider the existence of radial solutions, which have exactly k-nodes for any given integer k, of the equation

$$\begin{aligned} \Delta u + h(|x|)f(u) &= 0 \qquad \text{in} \quad \Omega(a,b), \\ u &= 0 \qquad \text{on} \quad \partial\Omega, \end{aligned} \tag{1.1}$$

where $\Omega(a,b) = \{x \in \mathbb{R}^n \mid a < |x| < b\}, n \geq 2$ and the functions f and h satisfy :

$$(A0) \quad h \in C^1((0,\infty)),\ h(r) > 0 \quad \text{for} \quad r > 0$$

$$(A1) \quad f \in C^1(\mathbb{R}),\ u \cdot f(u) \geq 0$$

$$(A2) \quad \lim_{u \to \infty} \frac{f(u)}{u} = \infty$$

$$(A3) \quad \lim_{u \to 0} \frac{f(u)}{u} = 0.$$

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This model was studied by several authors (see e.g. [1]~[8]). The existence of positive solutions of (1.1) with various nonlinearities was studied by Bandle, Coffman and Marcus [1], Coffman [2], Garazier [3], Lin [6], and Ni and Nussbaum [7]. Also the uniqueness of solutions of this model was studied by Kwong and Zhang [5] and Ni and Nussbaum [7].

Our paper is motivated by the recent work of Lin [6] who proved the existence of positive solutions of (1.1) on an annulus. We are interested in the existence of solutions of (1.1) which change sign. We are going to show that, for each integer $k > 0$, there exists a radial solution of (1.1) which changes sign exactly k times. The method used here is the backward shooting method combined with the Sturm comparision theorem after a suitable change of variables.

Our main result is stated as follows :

THEOREM. *For each positive integer k, (1.1) has at least one radial solution which has exactly k-nodes.*

2. Proof of Theorem

Since we are interested in a radial solution $u = u(r)$, we shall write (1.1) in the form

$$u'' + \frac{n-1}{r}u' + h(r)f(u) = 0 \quad \text{in} \quad (a,b). \tag{2.0}$$

Thus, for $n \geq 3$, in terms of variables

$$s = \{(n-2)r^{(n-2)}\}^{-1} \quad \text{and} \quad u(s) = u(r)$$

(2.0) can be rewritten as

$$u''(s) + \rho(s)f(u(s)) = 0 \quad \text{in} \quad (s_*, s^*), \tag{2.1}$$

where

$$\rho(s) = \{(n-2)s\}^{-k}h((n-2)s^{-\frac{1}{n-2}}), \quad k = \frac{2n-2}{n-2},$$

$$s_* = \{(n-2)b^{(n-2)}\}^{-1}, \quad s^* = \{(n-2)a^{(n-2)}\}^{-1}.$$

As for $n = 2$, in terms of variables

$$s = \frac{1}{2} - \log a + \log r \quad \text{and} \quad u(s) = u(r),$$

Eq. (2.0) can also be written as (2.1) with

$$\rho(s) = a^2 e^{2s-1} h(ae^{s-1/2}),$$

$$s_* = \frac{1}{2}, \quad \text{and} \quad s^* = \frac{1}{2} - \log a + \log b.$$

Using backward shooting method, we consider the family of solutions of the initial value problem

$$\begin{aligned} u''(s) + \rho(s)f(u(s)) = 0, \quad &\text{for } s < s^*, \\ u(s^*) = 0, \quad u'(s^*) = -\gamma, \end{aligned} \tag{2.2}$$

where $\gamma > 0$ is the shooting parameter.

Here $s^* > 0$ will be kept fixed throughout the paper.

For every $\gamma > 0$, the problem (2.2) has a unique solution $u(\cdot) \equiv u(\cdot, \gamma)$ with the maximal domain of definition $(s(\gamma), s^*)$.

It is easy to check that (2.2) is equivalent to the integral equation

$$u(s) = \gamma(s^* - s) - \int_s^{s^*} (t-s)\rho(t)f(u(t))dt, \quad \text{for} \quad s < s^* \tag{2.3}$$

and the solution u also satisfies

$$u(s) = u(\bar{s}) + u'(\bar{s})(s - \bar{s}) + \int_{\bar{s}}^{s} (t-s)\rho(t)f(u(t))dt, \tag{2.4}$$

for s and $\bar{s} \in (s(\gamma), s^*)$.

From (2.3), if u is positive in some interval (α, s^*) with $\alpha \geq 0$, then

$$u(s) \leq \gamma(s^* - s) \quad \text{in} \quad (\alpha, s^*). \tag{2.5}$$

If u have k zeros in $(s(\gamma), s^*)$, we set

$$\begin{aligned} s_1(\gamma) &= \inf\{s_1 \mid u(s,\gamma) > 0 \quad \text{in} \quad (s_1, s^*)\}, \\ s_2(\gamma) &= \inf\{s_2 < s_1 \mid u(s,\gamma) < 0 \quad \text{in} \quad (s_2, s_1)\}, \\ &\vdots \\ s_k(\gamma) &= \inf\{s_k < s_{k-1} \mid (-1)^k u(s,\gamma) > 0 \quad \text{in} \quad (s_k, s_{k-1})\}. \end{aligned}$$

By standard results in ordinary differential equations, the function $(s,\gamma) \to u(s,\gamma)$ and $(s,\gamma) \to u'(s,\gamma)$ are continuously differentiable in the set

$$\{(s,\gamma) \mid \gamma > 0 \quad \text{and} \quad s \in (s(\gamma), s^*)\}.$$

Since $u'(s_i(\gamma),\gamma) \neq 0, i = 1, 2, \cdots k$, by the implicit function theorem, the sets

$$\begin{aligned} I_1 &\equiv \{\gamma > 0 \mid s_1(\gamma) > 0\}, \\ I_2 &\equiv \{\gamma > 0 \mid s_2(\gamma) > 0\}, \\ &\vdots \\ I_k &\equiv \{\gamma > 0 \mid s_k(\gamma) > 0\} \end{aligned}$$

are open and $s_i(\cdot) \in C^1(I_i)$, $i = 1, 2, \cdots k$. Clearly $I_j \subset I_i$, $s_j(\gamma) < s_i(\gamma)$, for $b \in I_j$, $j > i$.

Consider the sets

$$\begin{aligned} J_1 \equiv \{\gamma > 0 \mid u'(\tau,\gamma) = 0 &\text{ for some } \tau \in (0, s^*), \\ &\text{and } u(s,\gamma) > 0 \text{ in } (\tau, s^*)\}, \\ J_2 \equiv \{\gamma > 0 \mid u'(\tau,\gamma) = 0 &\text{ for some } \tau \in (0, s_1(\gamma)), \\ &\text{and } u(s,\gamma) < 0 \text{ in } (\tau, s_1(\gamma))\}, \\ &\vdots \\ J_k \equiv \{\gamma > 0 \mid u'(\tau,\gamma) = 0 &\text{ for some } \tau \in (0, s_{k-1}(\gamma)), \\ &\text{and } (-1)^k u(s,\gamma) > 0 \text{ in } (\tau, s_{k-1}(\gamma))\}. \end{aligned}$$

If $\gamma \in J_i$ and $u'(\tau, \gamma) = 0$, then

$$f(u(\tau,\gamma)) \neq 0. \tag{2.6}$$

For, if $f(u(\tau,\gamma)) = 0$, then the initial value problem

$$v''(s) + \rho(s) f(v(s)) = 0 \quad \text{in } (\tau, s_{i-1}(\gamma)),$$
$$v(\tau) = u(\tau,\gamma), \; v'(\tau) = 0$$

has a solution $v(s) = \text{constant} = u(\tau,\gamma)$ for any $s \in (0, s^*)$. Therefore, the uniqueness of initial value problem of O. D. E. implies $u(s,\gamma) = u(\tau,\gamma)$ for any $s \in (0,s^*)$, a contradiction. By (A1) and (2.6), if $\gamma \in J_i$ and $u'(\tau_i,\gamma) = 0$ for some $\tau_i \in (0, s_{i-1}(\gamma))$, then for i odd

$$u'(s,\gamma) < 0 \quad \text{for } s \in (\tau_i, s_{i-1}(\gamma)),$$
$$u'(s,\gamma) > 0 \quad \text{for } s \in (\bar{s}(\gamma), \tau_i)$$

and if i is even

$$u'(s,\gamma) > 0 \quad \text{for } s \in (\tau_i, s_{i-1}(\gamma)),$$
$$u'(s,\gamma) < 0 \quad \text{for } s \in (\bar{s}(\gamma), \tau_i),$$

where $\bar{s}(\gamma) = s_i(\gamma)$ if $\gamma \in I_i$, $\bar{s}(\gamma) = 0$ if $\gamma \notin I_i$. Therefore, we shall denote this unique τ_i by $\tau_i(\gamma)$ which is also the maximum (or minimum) point of $u(\cdot,\gamma)$ in $(\bar{s}(\gamma), s_{i-1}(\gamma))$. It can be verified that J_i are open sets and $\tau_i(\cdot) \in C^0(J_i)$.

LEMMA 2.1. *Assume that the conditions (A0), (A1) and (A3) are satisfied. Suppose that for some $i \in \mathbb{N}$,*

$$\lim_{\gamma \to \infty} \tau_i(\gamma) = s^*,$$

$$\lim_{\gamma \to \infty} u(\tau_i(\gamma),\gamma) = \begin{cases} \infty, & \text{if } i \text{ is odd}, \\ -\infty, & \text{if } i \text{ is even} \end{cases}$$

and

$$\lim_{\gamma \to \infty} s_i(\gamma) = s^*.$$

Then

$$\lim_{\gamma \to \infty} u'(s_i(\gamma),\gamma) = \begin{cases} \infty, & \text{if } i \text{ is odd}, \\ -\infty, & \text{if } i \text{ is even}. \end{cases} \tag{2.7}$$

Proof. If not, there exist a constant $M > 0$ and a sequence $\gamma_k \to \infty$ such that $(-1)^{i+1}u'(s_i(\gamma_k), \gamma_k) \le M$. By (2.6)

$$\begin{aligned}
0 < (-1)^{i+1}u(\tau_i(\gamma_k)) &= (-1)^{i+1}\{u(s_i(\gamma_k)) + u'(s_i(\gamma_k))(\tau_i(\gamma_k) - s_i(\gamma_k)) \\
&\qquad + \int_{s_i(\gamma_k)}^{\tau_i(\gamma_k)} (t - \tau_i(\gamma_k))\rho(t) f(u(t))dt\} \\
&= (-1)^{i+1}u'(s_i(\gamma_k))(\tau_i(\gamma_k) - s_i(\gamma_k)) \\
&\qquad - (-1)^{i+1} \int_{s_i(\gamma_k)}^{\tau_i(\gamma_k)} (\tau_i(\gamma_k) - t)\rho(t) f(u(t))dt \\
&\le M(\tau_i(\gamma_k) - s_i(\gamma_k)),
\end{aligned}$$

since $(-1)^{i+1} \int_{s_i(\gamma_k)}^{\tau_i(\gamma_k)} (\tau_i(\gamma_k) - t)\rho(t) f(u(t))dt \ge 0$. By the assumption, $(\tau_i(\gamma_k) - s_i(\gamma_k)) \to 0$ as $k \to \infty$, which is impossible. □

LEMMA 2.2. *Assume that the conditions (A0), (A1) and (A2) are satisfied. Then $\tau_i(\gamma), i = 1, \cdots$ and $s_i(\gamma), i = 1, \cdots$ are well defined when γ is sufficiently large and*

$$\lim_{\gamma \to \infty} s_i(\gamma) = s^* \quad i = 1, \cdots, \tag{2.8}$$

$$\lim_{\gamma \to \infty} \tau_i(\gamma) = s^* \quad i = 1, \cdots, \tag{2.9}$$

$$\lim_{\gamma \to \infty} u(\tau_i(\gamma), \gamma) = \begin{cases} \infty, & \text{if } i \text{ is odd}, \\ -\infty, & \text{if } i \text{ is even}. \end{cases} \tag{2.10}$$

Proof. By induction, $\lim_{\gamma \to \infty} \tau_1(\gamma) = s^*, \lim_{\gamma \to \infty} u(\tau_1(\gamma), \gamma) = \infty$ and $\lim_{\gamma \to \infty} s_1(\gamma) = s^*$, by lemma 2.1 and lemma 2.2 of [6].

Assume that

$$\lim_{\gamma \to \infty} \tau_i(\gamma) = s^*,$$
$$\lim_{\gamma \to \infty} s_i(\gamma) = s^*,$$

$$\lim_{\gamma \to \infty} u(\tau_i(\gamma), \gamma) = \begin{cases} \infty, & \text{if } i \text{ is odd}, \\ -\infty, & \text{if } i \text{ is even}. \end{cases}$$

We must show that

$$\text{(2.11)} \qquad \text{(i)} \quad \lim_{\gamma\to\infty} \tau_{i+1}(\gamma) = s^*,$$

$$\text{(2.12)} \qquad \text{(ii)} \quad \lim_{\gamma\to\infty} s_{i+1}(\gamma) = s^*,$$

$$\text{(2.13)} \qquad \text{(iii)} \quad \lim_{\gamma\to\infty} u(\tau_{i+1}(\gamma),\gamma) = \begin{cases} -\infty, & \text{if } i \text{ is odd },\\ \infty, & \text{if } i \text{ is even }. \end{cases}$$

(i) If not, there exist $\tau_0 \in (0, s^*)$ and a sequence $\gamma_k \to \infty$ such that $\tau_{i+1}(\gamma_k) \le \tau_0$ for all k.

Let $\overline{s} = \frac{\tau_0 + s^*}{2}$. Then, by induction hypothesis, for every sufficiently large k, $s_i(\gamma_k) > \tau_0$ and

$$\text{(2.14)} \qquad \begin{aligned} &(-1)^i u(s,\gamma_k) > 0\\ &(-1)^i u'(s,\gamma_k) \le 0 \quad \text{for } s \in (\tau_0, \overline{s}). \end{aligned}$$

We now claim that

$$\text{(2.15)} \qquad \limsup_{k\to\infty} (-1)^i u(\overline{s},\gamma_k) = \infty.$$

Suppose that this is not the case. Then there exists a constant $M > 0$ such that $(-1)^i u(\overline{s},\gamma_k) \le M$ for all k. Now, by (2.4) and lemma 2.1,

$$\begin{aligned} M &\ge (-1)^i u(\overline{s},\gamma_k)\\ &= (-1)^i \{u(s_i(\gamma_k),\gamma_k) + u'(s_i(\gamma_k),\gamma_k)(\overline{s} - s_i(\gamma_k))\\ &\qquad\qquad + \int_{s_i(\gamma_k)}^{\overline{s}} (t-\overline{s})\rho(t) f(u(t))dt\}\\ &= (-1)^{i-1} u'(s_i(\gamma_k),\gamma_k)(s_i(\gamma_k) - \overline{s})\\ &\qquad\qquad - (-1)^i \int_{\overline{s}}^{s_i(\gamma_k)} (t-\overline{s})\rho(t) f(u(t))dt \end{aligned}$$

(2.16)

$$\ge (-1)^{i-1} u'(s_i(\gamma_k),\gamma_k)\frac{(s^* - \overline{s})}{2} - C \qquad \text{for some constant C.}$$

By Lemma 2.1, the right side of (2.16) goes to ∞. It is a contradiction. Thus (2.15) holds. By choosing a subsequence of γ_k if necessary, we may assume,

$$\lim_{k\to\infty}(-1)^i u(\bar{s},\gamma_k)=\infty. \tag{2.17}$$

By (A0), there is a subinterval $(\alpha,\beta)\subset(\tau_0,\bar{s})$ such that $\rho(s)\ge\rho_0>0$ in (α,β). Set

$$M_k=\inf\{\frac{f(u(s,\gamma_k))}{u(s,\gamma_k)}\mid s\in(\alpha,\beta)\}.$$

Then

$$M_k\ge\inf\{\frac{f(u)}{u}\mid |u|\ge(-1)^i u(\bar{s},\gamma_k)\}.$$

By (2.17) and (A2),

$$\lim_{k\to\infty}M_k=\infty. \tag{2.18}$$

By (2.2), $u_k\equiv u(\cdot,\gamma_k)$ satisfies

$$u_k''(s)+\rho(s)\frac{f(u_k(s))}{u_k(s)}u_k(s)=0\quad\text{in }(\alpha,\beta),$$

where

$$\rho(s)\frac{f(u(s,\gamma_k))}{u(s,\gamma_k)}\ge\rho_0 M_k\quad\text{in }(\alpha,\beta). \tag{2.19}$$

Now, let v_k be a solution of

$$v''+\rho_0 M_k v=0\quad\text{in }(\alpha,\beta).$$

By (2.18) , v_k has at least two zeros in (α,β) when k is sufficiently large. By (2.19) and Sturm Comparison Theorem, u_k has at least one zero in (α,β). Because of (2.14), this is impossible. Hence (i) holds.

Now, let's show (ii). If it is false, there exist a point $s_0\in(0,s^*)$ and a sequence γ_k such that $\gamma_k\to\infty$ with $s_{i+1}(\gamma_k)\le s_0$ for all k, and

(2.20)
$$(-1)^i u(s,\gamma_k) > 0 \quad \text{and} \quad (-1)^i u'(s,\gamma_k) \geq 0 \quad \text{in } (s_0, \tau_{i+1}(\gamma_k)).$$

Set $\overline{s} = \frac{s_0+s^*}{2}$. By (i), we may assume $\overline{s} < \tau_{i+1}(\gamma_k)$ for sufficiently large k. We claim that

$$\limsup_{k\to\infty}(-1)^i u(\overline{s},\gamma_k) < \infty. \tag{2.21}$$

Otherwise, there exists a subsequence $\gamma_k \to \infty$ such that

$$\lim_{\gamma\to\infty}(-1)^i u(\overline{s},\gamma_k) = \infty.$$

Similar to (i), by Sturm Comparison Theorem, $u_k \equiv u(\cdot,\gamma_k)$ has zeros in $(\overline{s}, \tau_{i+1}(\gamma_k))$ when k is sufficiently large, which is impossible by (2.20). Hence (2.21) holds. By (2.4)

$$\begin{aligned}
&(-1)^i u(\overline{s},\gamma_k)\\
&\quad = (-1)^i \{u(s_i(\gamma_k),\gamma_k) + u'(s_i(\gamma_k),\gamma_k)(\overline{s} - s_i(\gamma_k))\\
&\qquad + \int_{s_i(\gamma_k)}^{\overline{s}} (t-\overline{s})\rho(t) f(u(t))dt\}\\
&\quad = (-1)^i u'(s_i(\gamma_k),\gamma_k)(\overline{s} - s_i(\gamma_k))\\
&\qquad - (-1)^i \int_{\overline{s}}^{s_i(\gamma_k)} (t-\overline{s})\rho(t) f(u(t))dt\\
&\quad \geq (-1)^{i+1} u'(s_i(\gamma_k),\gamma_k)(s_i(\gamma_k) - \overline{s}) - C.
\end{aligned}$$

By Lemma 2.1. and (2.21), it is impossible. Hence (ii) is true.

(iii) Suppose it does not hold. Then there exist a constant $M > 0$ and a sequence $\gamma_k \to \infty$ such that

$$(-1)^{i+1} u(\tau_{i+1}(\gamma_k),\gamma_k) \leq M \quad \text{for all } k. \tag{2.22}$$

Set
$$F(u) = \int_0^u f(s)ds$$

and define

$$V(s) \equiv V(s,\gamma) \equiv \frac{1}{2}[u'(s)]^2 + \rho(s)F(u(s)).$$

Since

$$\begin{aligned} V'(s) &= u'(s)u''(s) + \rho(s)f(u(s))u'(s) + \rho'(s)F(u(s)) \\ &= \rho'(s)F(u(s)), \end{aligned}$$

$$V(s_i(\gamma_k)) = V(\tau_{i+1}(\gamma_k)) + \int_{\tau_{i+1}(\gamma_k)}^{s_i(\gamma_k)} \rho'(t)F(u(t))dt.$$

Therefore, we have

$$\begin{aligned} &\frac{1}{2}[u'(s_i(\gamma_k))]^2 + \rho(s_i(\gamma_k))F(u(s_i(\gamma_k)) \\ &\qquad = \frac{1}{2}[u'(\tau_{i+1}(\gamma_k))]^2 + \rho(\tau_{i+1}(\gamma_k))F(u(\tau_{i+1}(\gamma_k)) \\ &\qquad\qquad + \int_{\tau_{i+1}(\gamma_k)}^{s_i(\gamma_k)} \rho'(t)F(u(t))dt, \end{aligned}$$

$$\begin{aligned} \frac{1}{2}[u'(s_i(\gamma_k))]^2 &= \rho(\tau_{i+1}(\gamma_k))F(u(\tau_{i+1}(\gamma_k)) \\ &\quad + \int_{\tau_{i+1}(\gamma_k)}^{s_i(\gamma_k)} \rho'(t)F(u(t))dt. \end{aligned} \tag{2.23}$$

(2.22) implies that the right side of (2.23) is bounded. By lemma 2.1., it is impossible. Therefore (iii) holds. □

LEMMA 2.3. *Assume that the conditions* $(A0)$, $(A1)$ *and* $(A3)$ *are satisfied. Then we have*

$$\text{(i) if } (0,\gamma_1) \subset J_1 \quad \text{for some } \gamma_1 > 0, \text{ then } \lim_{\gamma\to 0} \tau_1(\gamma) = 0. \tag{2.24}$$

$$\text{(ii) if } (0,\gamma_1) \subset I_1 \quad \text{for some } \gamma_1 > 0, \text{ then } \lim_{\gamma\to 0} s_1(\gamma) = 0. \tag{2.25}$$

Proof. See [6]. □

LEMMA 2.4. *Assume that the conditions* $(A0)$, $(A1)$, $(A2)$ *and* $(A3)$ *are satisfied. Then we have*

(i) *for any connected component* $(\overline{\gamma}_{i1}, \overline{\gamma}_{i2})$ *of* I_i,

$$\lim_{\gamma\to\overline{\gamma}_{i1}^+} s_i(\gamma) = 0 \qquad i = 1, 2, \cdots . \tag{2.26}$$

(ii) *for any connected component* $(\overline{\gamma}_{i1}, \overline{\gamma}_{i2})$ *of* J_i,

$$\lim_{\gamma\to\overline{\gamma}_{i1}^+} \tau_i(\gamma) = 0 \qquad i = 1, 2, \cdots . \tag{2.27}$$

Proof. (i) Let $(\overline{\gamma}_{i1}, \overline{\gamma}_{i2})$ be a connected component of I_i $(i = 1, 2, \cdots)$. Either $\overline{\gamma}_{i1} = 0$ or $\overline{\gamma}_{i1} > 0$. If $\overline{\gamma}_{i1} = 0$, by lemma 2.3. (ii), $\lim_{\gamma\to\overline{\gamma}_{i1}^+} s_1(\gamma) = 0$. Since $I_i \subset I_1$ and $s_1(\gamma) > s_i(\gamma)$ for each $i = 2, 3, \cdots$, (2.26) follows from (2.25).

If $\overline{\gamma}_{i1} > 0$ and suppose that this is not the case, there exist a point $s_i > 0$ and a sequence $\gamma_{ik} \to \overline{\gamma}_{i1}$ with $s_i(\gamma_{ik}) \to s_i$, $i = 1, 2, \cdots$. Then

$$u(s_i, \overline{\gamma}_{i1}) = \lim_{\gamma_{ik}\to\overline{\gamma}_{i1}} u(s_i(\gamma_{ik}), \gamma_{ik}) = 0$$

i.e. $\overline{\gamma}_{i1} \in I$, which is a contradiction to the fact that $(\overline{\gamma}_{i1}, \overline{\gamma}_{i2})$ is a connected component of I_i.

(ii) Next, let $(\overline{\gamma}_{i1}, \overline{\gamma}_{i2})$ be a connected component of J_i, $(i = 1, 2, \cdots)$. If $\overline{\gamma}_{i1} = 0$, the result follows from lemma 2.3. If $\overline{\gamma}_{i1} > 0$ and $\lim_{\gamma\to\overline{\gamma}_{i1}^+} \tau_i(\gamma) \neq 0$, then there exist a $\tau_i > 0$ and a sequence $\gamma_{ik} \to \overline{\gamma}_{i1}$ with $\tau_i(\gamma_{ik}) \to \tau_i$ as $k \to \infty$. Since

$$u'(\tau_i, \overline{\gamma}_{i1}) = \lim_{k\to\infty} u'(\tau_i(\gamma_{ik}), \gamma_{ik}) = 0$$

i.e. $\overline{\gamma}_{i1} \in I_i$, which is a contradiction. □

Now, we prove the main Theorem.

THOREM 2.5. *Assume that the conditions* $(A0)$, $(A1)$, $(A2)$ *and* $(A3)$ *are satisfied. Then for each integer* $k \in \mathbb{N}$, (1.1) *has at least one nodal solution which has exactly* k *nodes for all* a, b *such that* $0 < a < b < \infty$.

Proof. Fix $k \in \mathbb{N}$. By lemma 2.2, $I_{k+1} \neq \phi$ and there exists $\overline{\gamma} \geq 0$ such that $(\overline{\gamma}, \infty) \subset I_{k+1}$. lemma 2.4 implies

$$\lim_{\gamma \to \overline{\gamma}^+} s_{k+1}(\gamma) = 0.$$

Hence for any $s_* < s^*$, there exist $\gamma^* > \overline{\gamma}$ such that $s_{k+1}(\gamma^*) = s_*$. Since $s_i(\gamma) > s_{k+1}(\gamma)$ for $0 \leq i < k$, there exists a nodal solution which have nodes

$$s_* < s_k(\gamma^*) < s_{k-1}(\gamma^*) < \cdots < s_1(\gamma^*) < s^*. \qquad \square$$

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Department of Mathematics
Yonsei University
Seoul 120-749, Korea