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SEMIALGEBRAIC G CW COMPLEX STRUCTURE OF SEMIALGEBRAIC G SPACES

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ABSTRACT. Let G be a compact Lie group and M a semialgebraic G space in some orthogonal representation space of G. We prove that if G is finite then M has an equivariant semialgebraic triangulation. Moreover this triangulation is unique. When G is not finite we show that M has a semialgebraic G CW complex structure, and this structure is unique. As a consequence compact semialgebraic G space has an equivariant simple homotopy type.

1. Introduction

Let G be a compact Lie group. In this paper, we describe the semialgebraic G CW complex structures of semialgebraic G spaces. Many people considered the problem of triangulations and CW complex structures of smooth manifolds, algebraic varieties, semialgebraic sets, subanalytic sets in [7], [3], [9], [5] and [12], etc. The purpose of this paper is to study the equivariant semialgebraic triangulation and G CW complex structure of semialgebraic G spaces.

A *semialgebraic G space M* is by definition a G invariant semialgebraic set in some orthogonal real representation space Ω of G. For example, all algebraic G varieties are semialgebraic G space by the equivariant algebraic embedding theorem for real algebraic G variety in a finite dimensional orthogonal representation space of G (see, [8] or [14]).

In section 2 we review the properties of semialgebraic sets and maps, and semialgebraic triangulations of semialgebraic sets. In section 3 we

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will show that the orbit space M/G of a semialgebraic G space has a semialgebraic triangulation which is compatible with the orbit types (Theorem 3.5).

In section 4 we prove the following equivariant semialgebraic triangulation result of semialgebriac G space for finite group G.

THEOREM 1.1 (Theorem 4.3). *Let G be a finite group and M a semialgebraic G space in Ω. Then there exist a finite equivariant open simplicial complex L and an equivariant semialgebraic homeomorphism $\eta\colon |L| \to M$ such that for each open simplex* $\mathrm{int}(\Delta^n)$ *of L,* $\eta(\mathrm{int}(\Delta^n))$ *is a locally closed analytic submanifold of Ω and $\eta|_{\mathrm{int}(\Delta^n)}$ is an analytic isomorphism to its image.*

Moreover this triangulation is unique. In particular if M is closed in Ω we may take L by a 'complete' equivariant simplicial complex.

In the nonequivariant case, S. Lojasiewicz [7] obtained semialgebraic triangulations of semialgebraic sets. Later H. Hironaka [3] gave a different proof of Lojasiewicz's result (Proposition 2.5) using the projection theorem of Seidenberg-Tarski (Proposition 2.3).

In section 5 we show that any semialgebraic G space has a semialgebraic G CW complex structure as the following theorem shows.

THEOREM 1.2 (Theorem 5.2). *Let G be a compact Lie group. Any semialgebraic G space in Ω has a finite open G CW complex structure such that each equivariant-cell is semialgebraic, and each open equivariant-cell is a semialgebraic analytic G submanifold of Ω.*

Moreover each characteristic G map $f_{\Delta^n}\colon G/H_{\Delta^n} \times \Delta^n \to M$ is semialgebraic and the restriction $f_{\Delta^n}|$ to $G/H_{\Delta^n} \times (\mathrm{int}(\Delta^n))$ is an analytic isomorphism to its image.

This structure is unique and if M is closed in Ω we may take a 'complete' finite G CW complex structure.

As a consequence, any compact semialgebraic G space has an equivariant simple homotopy type.

Usually in other categories (such as subanalytic, semi-analytic, or smooth) we can find 'complete' (possibly infinite) simplicial or CW complex structures. But in the semialgebraic category we can only obtain finite open simplicial or CW complex structures as in Theorems 1.1 and 1.2. The reason is that the semialgebraic category is not stable under infinite union. For example, even if $f_\alpha\colon A_\alpha \to B$ are semialgebriac for

$\alpha \in \Lambda$, $|\Lambda| = \infty$ such that $f_\alpha = f_\beta$ on $A_\alpha \cap A_\beta$ for all α, $\beta \in \Lambda$, the map $F = \cup f_\alpha \colon \cup A_\alpha \to B$ need not be semialgebraic.

The general idea of the proofs of Theorems 1.1 and 1.2 is as follows. First we show that the union of the orbits with isotropy type (H), denoted by $M_{(H)}$, is a semialgebraic G space. Second, we find a semialgebraic set structure of the orbit space M/G such that the orbit map $\pi \colon M \to M/G$ is semialgebraic. Third, take a semialgebraic traingulation K of M/G which is compatible with the orbit types $\{\pi(M_{(H)})\}$ by the nonequivariant semialgebriac triangulation theorem. Finally, lifting each simplex Δ in the barycentric subdivision of K, we obtain our results using the fact that $\pi^{-1}(\mathrm{int}\,\Delta)$ is a semialgebraic G space which is semialgebraically G homeomorphic to $G/H \times \mathrm{int}(\Delta)$.

In [12] the authors proved similar results to Theorems 1.1 and 1.2 for real algebraic G varieties. Since the proofs of the results here are similar to those in [12], we can abbreviate many arguments. For more details we refer the reader to the cited paper. The results here generalize the results in [12] not only to semialgebraic G spaces but also the map $\eta|_{\mathrm{int}(\Delta^n)}$ in Theorem 1.1 and the restriction $f_{\Delta^n}|$ to $G/H_{\Delta^n} \times (\mathrm{int}(\Delta^n))$ in Theorem 1.2 are analytic isomorphisms, which will be used in later research.

2. Semialgebraic sets and their semialgebraic triangulations

In this section we discuss basic concepts of semialgebraic sets, semialgebraic maps and semialgebraic triangulations of semialgebraic sets which will be used throughout this paper.

A *polyhedron* means the underlying space of some realized simplicial complex in $\mathbb{R}^n$. For n-simplex Δ^n, $\mathrm{int}(\Delta^n)$ is the interior of Δ^n. If v is a 0-simplex, let $\mathrm{int}(v) = v$. A *finite open simplicial complex* K is obtained from some finite (i.e. compact) realized simplicial complex L by omitting some open simplices $\mathrm{int}(\Delta^n)$ of L. In the other words K is the finite union of some open simplices $\mathrm{int}(\Delta^n)$ of some finite (i.e. compact) realized simplicial complex L. We denote this by $\mathrm{int}(\Delta^n) \in K$. Thus $\overline{K}$ is a compact polyhedron. Our notation of a simplicial complex differs slightly from the classical one. Classical simplicial complexes are called here "complete" simplicial complexes. We begin with the definition of semialgebraic sets.

DEFINITION 2.1. The class of *semialgebraic sets* in $\mathbb{R}^n$ is the smallest collection of subsets containing all $\{x \in \mathbb{R}^n | p(x) > 0\}$ for real polynomial $p(x) = p(x_1, \dots, x_n)$ which is stable under finite union, finite intersection and complement.

For example, compact polyhedra, finite open simplicial complexes and algebraic varieties are semialgebraic sets. It follows from the definition of semialgebraic set that a subset M of $\mathbb{R}^n$ is semialgebraic if and only if there exist polynomials $f_{ij}(x)$ and $g_{ij}(x)$ for $i = 1, \dots, p$ and $j = 1, \dots, q$, such that

$$M = \bigcup_i \{x \in \mathbb{R}^n |\ f_{ij}(x) > 0,\ g_{ij}(x) = 0 \text{ for all } j\}.$$

It is easy to see that finite unions and finite intersections of semialgebraic sets are semialgebraic and that the complement of a semialgebraic set is semialgebraic. Furthermore, the closure, and thus the interior, of a semialgebraic set are semialgebraic. In addition, every connected component of a semialgebraic set is semialgebraic and the family of the connected components of a semialgebraic set is finite (see, for example, [1] and [3]).

A finite *stratification* of a subset M of $\mathbb{R}^n$ means a finite decomposition $M = \bigcup \Gamma_j$ such that

(1) $\{\Gamma_j\}$ is a finite collection of disjoint semialgebraic sets in $\mathbb{R}^n$.
(2) Each Γ_j is a smooth connected real-analytic submanifold, locally closed in $\mathbb{R}^n$.
(3) If $\overline{\Gamma}_j \cap \Gamma_{j'} \neq \emptyset$, then $\overline{\Gamma}_j \supset \Gamma_{j'}$.

PROPOSITION 2.2 ([3, p.167]). *Given a finite system of semialgebraic sets $\{M_i\}$ in $\mathbb{R}^n$, there exists a finite semialgebraic stratifiction $\{\Gamma_j\}$ of $\bigcup M_i$ which is compatible with every M_i, i.e., each M_i is a union of some Γ_j.*

The following proposition is one of the basic properties of semialgebraic sets.

PROPOSITION 2.3 (Seidenberg-Tarski [3, p.167]). *The image of a semialgebraic set in $\mathbb{R}^n$ by a polynomial map $\mathbb{R}^n \to \mathbb{R}^m$ is a semialgebraic set in $\mathbb{R}^m$.*

In particular, the image of a semialgebraic set by the projection map $\mathbb{R}^{n+1} \to \mathbb{R}^n$ is a semialgebraic set. Proposition 2.3 implies that the cartesian product of two semialgebraic sets is a semialgebraic set.

DEFINITION 2.4. Let M and N be semialgebraic subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively. A map $f : M \to N$ is said to be *semialgebraic* if it is continuous and its graph is a semialgebraic set in $\mathbb{R}^m \times \mathbb{R}^n$.

For example, PL (piecewise linear) maps between finite (open) simplicial complexes, polynomial and rational maps are semialgebraic. Also, it is easy to show that the composition of two semialgebraic maps and also the inverse of a semialgebraic homeomorphism are semialgebraic, see [3]. Moreover, the image and the preimage of a semialgebraic set by a semialgebraic map are semialgebraic. If $f\colon A \to B$ and $g\colon A \to C$ are two semialgebraic maps, then the map $(f,g)\colon A \to B \times C$ defined by $(f,g)(a) = (f(a), g(a))$ is semialgebraic. Moreover if $f\colon A_1 \to B_1$ and $g\colon A_2 \to B_2$ are semialgebraic maps, then the map $f \times g\colon A_1 \times A_2 \to B_1 \times B_2$ defined by $(f \times g)(a,b) = (f(a), g(b))$ is semialgebraic.

Now we consider the semialgebraic triangulations of semialgebraic sets in the non-equivariant case.

PROPOSITION 2.5 ([3, p.170] or [7, Theorem 3]). *Given a finite system of bounded semialgebraic sets M_i in $\mathbb{R}^n$, there is a finite complete simplicial complex K in $\mathbb{R}^n$ and a semialgebraic homeomorphism $\tau\colon |K| \to \bigcup \overline{M}_i$ such that*

(1) *each M_i is a finite union of some of the $\tau(\operatorname{int}(\Delta^n))$, where $\Delta^n \in K$,*
(2) *$\tau(\operatorname{int}(\Delta^n))$ is a locally closed smooth real-analytic submanifold of $\mathbb{R}^n$ and τ induces a real-analytic isomorphism $\operatorname{int}(\Delta^n) \to \tau(\operatorname{int}(\Delta^n))$, for every $\Delta^n \in K$.*

Notice that Hironaka's simplices are open by the definition on p. 168 of [3]. A pair (K, τ) of a finite open simplicial complex K and a semialgebraic map τ is called a *semialgebraic triangulation* of $\bigcup M_i$ if $\tau\colon |K| \to \bigcup M_i$ is a homeomorphism. Moreover, it is called *compatible with* $\{M_i\}$ if each M_i is a union of some $\tau(\operatorname{int}(\Delta^n))$ where $\operatorname{int}(\Delta^n) \in K$.

COROLLARY 2.6. *Given a finite system of semialgebraic sets $\{M_i\}$ in $\mathbb{R}^n$, there is a semialgebraic triangulation (K, τ) of $M = \bigcup M_i$ which is compatible with $\{M_i\}$ such that for each open simplex $\operatorname{int}(\Delta^n) \in K$, $\tau(\operatorname{int}(\Delta^n))$ is a locally closed smooth real-analytic submanifold of $\mathbb{R}^n$ and τ induces a real-analytic isomorphism $\operatorname{int}(\Delta^n) \to \tau(\operatorname{int}(\Delta^n))$.*

Furthermore, the above semialgebraic triangulation is unique in the sense that if there are two such semialgebraic triangulations (K_1, τ_1), (K_2, τ_2), we have a semialgebraic triangulation (K_3, τ_3) which is compatible with $\{\tau_1(\operatorname{int}\Delta^n) | \operatorname{int}\Delta^n \in K_1\}$ and $\{\tau_2(\operatorname{int}\Delta^n) | \operatorname{int}\Delta^n \in K_2\}$.

In particular if $\bigcup M_i$ *is closed in* $\mathbb{R}^n$ *we can take for* K *a complete simplical complex.*

Proof. First if $M = \bigcup M_i$ is compact then Propositon 2.5 implies this corollary with a finite complete simplicial complex K.

Now let $M = \bigcup M_i$ be a closed unbounded semialgebraic subset in $\mathbb{R}^n$. We may assume $0 \notin M$ because otherwise we can replace M by $M \times \{1\} \subset \mathbb{R}^{n+1}$. Let $\theta\colon \mathbb{R}^n - \{0\} \to \mathbb{R}^n - \{0\}$ be the inversion through the unit sphere, $\theta(x) = \frac{x}{|x|^2}$. Clearly θ is a semialgebraic, analytic diffeomorphism, and thus $\theta(M) \subset \mathbb{R}^n - \{0\}$ is a semialgebraic set. From this, we can see that $\theta(M) \cup \{0\}$ is a compact semialgebraic set in $\mathbb{R}^n$. By the compact case for $\{\theta(M_i), \{0\}\}$, there exists a finite simplicial complex K and a semialgebraic homeomorphism $\eta\colon |K| \to \theta(M) \cup \{0\}$ which is compatible with $\{\theta(M_i), \{0\}\}$. Then $v = \eta^{-1}(0)$ is a vertex of K. Define $\tau = \theta \circ \eta|\colon |K| - \{v\} \to M$. Thus we proved this for the closed case.

From the stellar subdivision (with center v, see [13] and [12, Theorem 2.12]) of this simplicial complex, we get the second part.

Now we shall prove the general case. If $M = \bigcup M_i$ is a semialgebraic set in $\mathbb{R}^n$. In this case we consider the closure $\overline{M}$ of M in $\mathbb{R}^n$. Then $\overline{M}$ is also a closed semialgebraic set in $\mathbb{R}^n$. We take the semialgebraic triangulation of $\overline{M}$ which is compatible with $\{\overline{M} - M, M_i\}$ as in closed case. This implies the existence of the corollary.

Finally we prove uniqueness. Pick two such semialgebraic triangulations $(K_i, \tau_i), i = 1, 2$ of M. Then let $\{X_\alpha\}$ be the collection of all those $\tau_i(\operatorname{int} \Delta_i)$ where $\operatorname{int}(\Delta_i) \in K_i$ $(i = 1, 2)$. It is a finite system of semialgebraic sets in M. Hence there exists a semialgebraic triangulation (K_3, τ_3) which is compatible with $\{X_\alpha\}$. This implies uniqueness. □

3. Semialgebraic triangulations of the orbit space of semialgebraic G spaces

In this section we show that the orbit space of a semialgebraic G space has a semialgebraic triangulation compatible with the orbit types where G is a compact Lie group. For general terminology and theory of compact group actions we refer the reader to [2].

DEFINITION 3.1. We recall that a *semialgebraic G space* M is the G invariant semialgebraic set in some orthogonal real representation space Ω of G.

It follows from the equivariant algebraic embedding theorem for (real) algebraic G variety in a finite dimensional orthogonal representation space Ω (see [8], [14] and [12, Proposition 3.2]) that an algebraic G variety is a closed semialgebraic G space in Ω. We know that every compact Lie group has a unique algebraic group structure (see [10, p.247]), and thus it has a semialgebraic group structure.

For a G space M and a point $x \in M$, let G_x denote the isotropy subgroup $G_x = \{g \in G \mid gx = x\}$. Every representation space of G has finite orbit types for G a compact Lie group (see, [11, p.45]). Thus every semialgebraic G space has only finitely many orbit types for a compact Lie group G. There exist only finitely many non-empty subsets $M_{(H)} = \{ x \in M \mid G_x = gHg^{-1} \text{ for some } g \in G \}$ of M when H is a closed subgroup of G. Moreover $\Omega_{(H)}$ is a semialgebraic G invariant subset of Ω ([12, Lemma 3.3]). Because of these and $M_{(H)} = \Omega_{(H)} \cap M$ we can prove

LEMMA 3.2. *If H is a subgroup of a compact Lie group G and M is a semialgebraic G space, then $M_{(H)}$ is a semialgebraic G invariant set.*

Now we consider equivariant stratifictions of semialgebraic G spaces.

LEMMA 3.3. *Let M be a semialgebraic G space in Ω. Given a finite system of G invariant semialgebraic subsets $\{M_i\}$ of $M = \bigcup M_i$, there exist finite disjoint G invariant subsets $\{\Gamma_j\}$ of M such that*

(1) *each Γ_j is a semialgebraic G space in Ω.*
(2) *Each Γ_j is a smooth real-analytic G submanifold, locally closed in Ω.*
(3) *Each M_i is a union of some Γ_j.*
(4) *If $\overline{\Gamma}_j \cap \Gamma_{j'} \neq \emptyset$, then $\overline{\Gamma}_j \supset \Gamma_{j'}$.*

Proof. The method is similar to the nonequivariant subanalytic case in [15]. Let $\dim \Omega = n$. We add Ω to $\{M_i\}$. We will prove this by induction. Assume that there exists a finite equivariant stratification $\{\Gamma_j\}$ of a G invariant semialgebraic subset Y_{k+1} of Ω such that for each i, j, we get $\Gamma_j \subset M_i$ or $\Gamma_j \cap M_i = \emptyset$. And such that $Z_k = \Omega - Y_{k+1}$ is a closed G invariant semialgebraic set with $\dim Z_k \leq k$ and $\dim \Gamma_j > k$. The case $k = n$ is trivial with $Y_{n+1} = \emptyset$.

Put $M_i' = M_i \cap Z_k$. For each M_i' we have semialgebraic partition N_i and S_i of M_i' such that N_i, S_i are G invariant, M_i' is the disjoint union of N_i and S_i, and N_i is an analytic manifold of dimension k and $\dim S_i < k$

in a way similar to the algebraic variety V case with $Nonsing(V)$ and $Sing(V)$.

Let Z_{k-1} be the union of all $\overline{N}_i - N_i$, $\overline{S}_i$ and the connected components of dimension $< k$ the sets in the form $N_i \cap N_{i'}$. Then Z_{k-1} is a closed G invariant semialgebraic set and of dimension $< k$. It follows $Z_k - Z_{k-1} = \bigcup_i \{N_i - Z_{k-1}\}$ because $\Omega \in \{M_i\}$, and it is also G invariant semialgebraic analytic manifold.

Let $\{\gamma'_j\}$ be all connected components, note there are finitely many connected components, of $Z_k - Z_{k-1}$. Put Γ'_j is the union of some γ'_j which is the smallest G invariant set containing γ'_j. Then each Γ'_j contains γ'_j, but γ'_j contained in some N_i and thus Γ'_j contained in N_i since N_i is G invariant. Moreover $\Gamma'_j \cap N_i \neq \emptyset$ implies $\Gamma'_j \subset N_i$.

Hence $\{\Gamma_j, \Gamma'_j\}$ is a finite equivariant stratification of $Y_k = \Omega - Z_{k-1}$ compatible with $\{M_i\}$ whose strata are of dimension $\geq k$. □

Let G be a compact Lie group and Ω an orthogonal representation space of G. By the theorem of Hilbert and Hurwitz ([16, Ch.VIII]), the graded algebra $\mathbb{R}[\Omega]^G$ of G invariant polynomials on Ω is finitely generated, say, by homogeneous elements $p_1, \ldots, p_d$. Let $p = (p_1, \ldots, p_d) \colon \Omega \to \mathbb{R}^d$, and let I be the ideal of relations of the p_i's in $\mathbb{R}[y_1, \ldots, y_d]$. Let Z be the corresponding algebraic variety in $\mathbb{R}^d$. Then the image $Im(p)$ of p is a semialgebraic subset of Z. Let $\overline{p} \colon \mathbb{R}^n/G \to Im(p)$ denote the mapping induced by p. Then the map p and $\overline{p}$ are proper, hence $Im(p)$ is closed in $Z \subset \mathbb{R}^d$, and the map $\overline{p} \colon \Omega/G \to Im(p)$ is a homeomorphism, see [14, p.136]. Thus we have

LEMMA 3.4. *Let G be a compact Lie group and let M be a semialgebraic G space in Ω. Then there exists a G invariant algebraic map $f \colon M \to \mathbb{R}^d$ for some d such that the induced map $\overline{f} \colon M/G \to f(M)$ is a homeomorphism. Moreover if M is a closed subset of Ω, then $f(M)$ is a closed semialgebraic set in $\mathbb{R}^d$.*

Notation. From the above lemma, the orbit space M/G of a semialgebraic G space has a semialgebraic structure. Thus we identify the orbit space M/G by $f(M)$ and the orbit map $\pi \colon M \to M/G$ by the polynomial map f.

A *semialgebraic triangulation* (K, τ) of the orbit space M/G is a pair of a finite open simplicial complex K and a semialgebraic homeomorphism $\tau \colon |K| \to M/G$. Let $\{M_{(H)} \mid H \text{ is a subgroup of } G\}$ be the finite collection of semialgebraic subsets of M, we say that the triangulation is *compatible with the orbit type* of M if each $\pi(M_{(H)})$ is a union of some $\tau(\operatorname{int}\Delta^n)$, where $\operatorname{int}(\Delta^n) \in K$.

With the same notations as above, there exists a semialgebraic triangulation $\tau\colon |K| \to M/G \equiv f(M)$ which is compatible with $\{\pi(M_{(H)}) \equiv f(M_{(H)}) \mid H$ is a subgroup of $G\}$ by Corollaries 2.6. This implies

THEOREM 3.5. *Let G be a compact Lie group and M a semialgebraic G space in Ω. Then there is a semialgebraic triangulation (K, τ) of M/G compatible with the orbit types of M such that for each open simplex $\operatorname{int}(\Delta^n)$ of K, $\tau(\operatorname{int}(\Delta^n))$ is a locally closed smooth real-analytic submanifold of some $\mathbb{R}^d$ and $\tau|_{\operatorname{int}(\Delta^n)}$ is a real-analytic isomorphism to its image. Furthermore, this triangulation is unique in the same sense as in Corollary 2.6.*

In particular if M is closed in Ω we may take K by a complete simplicial complex.

4. Equivariant semialgebraic triangulations of semialgebraic G spaces for G finite

Throughout this section, we let G be a finite group. In this section, we will define an equivariant semialgebraic triangulation and show that every semialgebraic G space has an equivariant semialgebraic triangulation which induces a semialgebraic triangulation of the orbit space compatible with the orbit types.

DEFINITION 4.1. A *complete simplicial G complex* consists of a complete simplicial complex K together with a G action $\varphi\colon G \times K \to K$ such that the map $\varphi_g = g\colon K \to K$ is a simplicial homeomorphism for every $g \in G$. Let K and L be simplicial G complexes. An *equivariant simplicial* (*respectively PL*) *map* $f\colon K \to L$ is an equivariant map (i.e., G map) which is a simplicial (respectively PL) map as a map between ordinary simplicial complexes K and L.

We say that a simplicial G complex K is an *equivariant complete simplicial complex* if the following conditions are satisfied.

(1) For every subgroup H of G, we have that if $\Delta^n =< v_0, \dots, v_n >$ and $\Delta^{n'} =< h_0v_0, \dots, h_nv_n >$ are simplices of K for $h_i \in H$, then there exists $h \in H$ such that $hv_i = h_iv_i$ for all i.

(2) For any simplex Δ^n of K, the vertices $v_0, \dots, v_n$ of Δ^n can be ordered in such a way that we have $G_{v_n} \subset \cdots \subset G_{v_0}$.

An abstract simplicial G complex satisfying condition (1) above is called *regular* [2, p.116]. Moreover, it is easy to see that in the presence of condition (2), condition (1) can be replaced by the following weaker condition(see, [5]);

(1') If $\Delta^n =< v_0, \dots, v_n >$ and $\Delta^{n'} =< g_0v_0, \dots, g_nv_n >$ are simplices for $g_i \in G, i = 0, \dots, n$, then there exists $g \in G$ such that $gv_i = g_iv_i$ for all i.

A *finite equivariant open simplicial complex* K is defined as a G invariant subset of some realized finite (i.e. compact) equivariant simplicial complex L by omitting some open simplices of L.

We know that the second barycentric subdivision of any simplicial G complex is an equivariant simplicial complex(see, [5]). The orbit space K/G of an equivariant simplicial complex K has the structure of an ordinary simplicial complex such that the orbit map $\pi\colon K \to K/G$ is simplicial and π maps each simplex of K PL homeomorphically onto the corresponding image simplex of K/G. Let K and L be equivariant simplicial complexes and let $f\colon |K| \to |L|$ be an equivariant map. Then it is easy to see that f is an equivariant simplicial map or an equivariant PL map if and only if the induced map $\overline{f}\colon K/G \to L/G$ is simplicial or PL, respectively.

DEFINITION 4.2. Let M be a semialgebraic G space. An *equivariant semialgebraic triangulation* (L, η) of M consists of a finite equivariant open simplicial complex L and an equivariant semialgebraic homeomorphism $\eta\colon |L| \to M$.

When G is a finite group, by lifting each simplex in the barycentric subdivision of a semialgebraic triangulation of the orbit space which is obtained by Theorem 3.5, we get the following main theorem using the fact that the isotropy subgroups are constant on the image of a (semialgebraic) cross section of the orbit map which is defined on each open simplex of the barycentric subdivision of a triangulation of the orbit space compatible with the orbit types. And thus the orbit of a simplex should not clash on M (see [12, Lemma 4.3]). We also need the Covering Homotopy Theorem (see [2, p.97]) for the uniqueness of these triangulations. We are now ready to prove the equivariant semialgebraic triangulation theorem for semialgebraic G spaces. We briefly prove of this by the same arguments as in [12]. Thus see to [12] for further detailed proofs.

THEOREM 4.3 (Theorem 1.1). *Let G be a finite group and M be a semialgebraic G space in Ω. Then there is an equivariant semialgebraic triangulation (L, η) of M which satisfy:*

(1) *For each open simplex $\mathrm{int}(\Delta^n)$ of L, $\eta(\mathrm{int}(\Delta^n))$ is a locally closed smooth analytic submanifold of Ω and $\eta|_{\mathrm{int}(\Delta^n)}$ is analytic isomorphism to its image.*

(2) *This induces a semialgebraic triangulation of the orbit space M/G compatible with the orbit types as in* Theorem 3.5.

Furthermore, this triangulation is unique in the sense of Corollary 2.6.

In particular if M is closed in Ω, we can take L to be a complete equivariant simplicial complex.

Proof. (i) First we can prove the case when M is compact in Ω. By Lemma 3.3, there exists a finite equivariant stratification $\{\Gamma_j\}$ which is compatible with $\{M_{(H)}\}$. Take a finite semialgebraic triangulation (K, τ) of the orbit space which is compatible with the $\{\Gamma_j\}$ as in Theorem 3.5. We may replace K by its barycentric subdivision, so that for each simplex Δ^n of K the space $\pi^{-1}(\mathrm{int}(\Delta^n))$ is an analytic, smooth G manifold. Moreover $\pi^{-1}(\tau(\Delta^n - \Delta^{n-1}))$ is G homeomorphic to $G/H \times (\Delta^n - \Delta^{n-1})$ and $\pi^{-1}(\tau(\mathrm{int}(\Delta^n))$ is analytically G diffeomorphic to $G/H \times (\mathrm{int}(\Delta^n))$. Then there is a semialgebraic section $s_{\Delta^n} \colon \tau(\Delta^n) \to M$ such that $s_{\Delta^n}(\tau(\mathrm{int}(\Delta^n)))$ is a locally closed smooth real-analytic submanifold of Ω and $s_{\Delta^n}|_{\tau(\mathrm{int}(\Delta^n))}$ is smooth, analytic isomorphism to its image. We note that $s_{\Delta^n}(\tau(\Delta^n)) = \pi^{-1}(\tau(\Delta^n))^H$ and $s_{\Delta^n}(\mathrm{int}(\tau(\Delta^n))) = \pi^{-1}(\tau(\mathrm{int}(\Delta^n)))^H$ where H is the orbit type of $\tau(\mathrm{int}(\Delta^n))$.

Let L' be the finite abstract complex with its vertex set being the collection $\{gs(\tau(v))\}$ where $g \in G$ and v vertices in K and s sections as above, and with its simplices $\{gs_{\Delta^n}(\tau(v_1)), \dots, gs_{\Delta^n}(\tau(v_n))\}$ for simplices $\Delta^n =< v_1, \dots, v_n >$ in K. Then L' is an abstract G-complex with the following action: For a simplex $\{hs(v_0), \dots, hs(v_n)\} \in L'$ and $g \in G$, then $\varphi_g(\{hs(v_0), \dots, hs(v_n)\}) = \{ghs(v_0), \dots, ghs(v_n)\}$. Now let L be a realized finite simplicial complex of L' and let $\ll gs(v_0), \dots, gs(v_n) \gg$, or briefly $\ll gs(\Delta^n) \gg$, denote the corresponding simplex to $\{gs(\tau(v_0)), \dots, gs(\tau(v_n))\}$ for $\Delta^n =< v_0, \dots, v_n >\in K$. Then L is an equivariant simplicial complex with the action which is induced by the action of L'.

For each $g \in G$, $\Delta^n =< v_0, \dots, v_n >\in K$ and section s_{Δ^n} on Δ^n, we define a linear homeomorphism $\Pi_{\ll gs(\Delta^n) \gg} \colon \ll gs(v_0), \dots, gs(v_n) \gg \to \Delta^n$ by $\Pi_{\ll gs(\Delta^n) \gg}(gs(v_i)) = v_i$. Put $\Pi = \bigcup \Pi_{\ll gs(\Delta^n) \gg} \colon |L| \to |K|$, then we can show that this is a well-defined map (see, Proof of Theorem 4.5

in [12]). Clearly Π is a simplicial map. In particular Π is PL and therefore semialgebraic. Notice that $\Pi\colon L \to K$ is the orbit map and hence $L/G = K$.

Now we define a map $\eta\colon |L| \to M$ by

$$\eta|_{\ll gs(\Delta^n)\gg} = gs \circ \tau \circ \Pi|_{\ll gs(\Delta^n)\gg}\colon \ll gs(\Delta^n) \gg \to gs(\tau(\Delta^n))$$

for $\Delta^n \in K$, $g \in G$. Then η is a well-defined semialgebraic G homeomorphism (see, Proof of Theorem 4.5 in [12]). Notice that $\eta|_{\ll gs(\Delta^n)\gg}$ are semialgebraic homeomorphisms for all $\ll gs(\Delta^n) \gg \in L$. Moreover for each simplex $\Delta = \ll gs(\Delta^n) \gg$ of L, we have $\eta(\mathrm{int}(\Delta)) = gs(\tau(\mathrm{int}(\Delta^n)))$. Thus it is a locally closed smooth, real-analytic submanifold of Ω. Moreover $\eta|_{\ll gs(\Delta^n)\gg}$ is a smooth, analytic isomorphism from $\ll gs(\Delta^n) \gg$ to its image, since it is a composition of linear isomorpism $\Pi|_{\ll gs(\Delta^n)\gg}$ and analytic isomorphisms τ and gs_{Δ^n}. Thus (L, η) is a desired triangulation.

(ii) Now we consider the case when M is a closed (noncompact real) semialgebraic G space in Ω. We may assume $0 \notin M$, otherwise we can replace M by $M \times \{1\} \subset \Omega \times \mathbb{R}$. Then the map $\theta\colon \Omega - \{0\} \to \Omega - \{0\}$, defined by $\theta(x) = \frac{x}{|x|^2}$, is a semialgebraic anaytic G diffeomorphism since Ω is an orthogonal representation space of G. Put $M^* = \theta(M) \cup \{0\}$, then M^* is a compact semialgebraic G space. And M^* is the one point compactification of M. Let $\{\Gamma_j^*\}$ be the equivariant stratification of M^* which is compatible with orbit types.

We take a finite semialgebraic triangulation (K, τ^*) of the orbit space M^*/G which is compatible with a point $\{0\}$ and $\{\Gamma_j^*\}$. Then $\tau^{*-1}(\pi^*(0))$ is a vertex of K. We let v^* denote this vertex where π^* is the orbit map of M^*. By the compact case (i), there is a finite equivariant semialgebraic triangulation (L, η^*) of M^* which induces (K, τ^*). Let $\pi_L\colon L \to K$ be the simplicial orbit map. Then ${\pi_L}^{-1}(v^*)$ is a vertex of L since 0 is a fixed point of G, and we let v denote this vertex. We thus have the following commutative diagram:

$$\begin{array}{ccccc}
|L| - \{v\} & \xrightarrow{\eta^*|} & \theta(M) & \xrightarrow{\theta^{-1}} & M \\
\downarrow{\scriptstyle \pi_L} & & \downarrow{\scriptstyle \pi^*} & & \downarrow{\scriptstyle \pi} \\
|K| - \{v^*\} & \xrightarrow[\tau^*|]{} & \theta(M)/G & \xrightarrow[(\bar{\theta})^{-1}]{} & M/G
\end{array}$$

Set $\eta = \theta^{-1} \circ \eta^*|\colon |L| - \{v\} \to M$. Then η is a semialgebraic G-homeomorphism which induces a homeomorphism $\tau = \overline{\theta}^{-1} \circ \tau^*|\colon |K| - \{v^*\} \to M/G$ where $\overline{\theta}$ is the induced homeomorphism by θ. Then (K, τ) is compatible with the orbit types because $\theta(M_{(H)}) = \theta(M)_{(H)}$. Since π_L and $\pi \circ \eta$ are surjective semialgebraic maps, $Gr(\tau) = (\pi_L, \pi \circ \eta)(|L| - \{v\})$. Thus τ is a semialgebraic map.

For each open simplex $\mathrm{int}(\Delta^n)$ of $|K| - \{v^*\}$ there is an analytic section s on $\tau^*(\mathrm{int}(\Delta^n))$, thus the restriction map $\overline{\theta}^{-1}|_{\tau^*(\mathrm{int}(\Delta^n))} = \pi \circ \theta^{-1} \circ s$ is an analytic isomorphism to its image. Therefore so is $\tau^*|_{\mathrm{int}(\Delta^n)}$.

Using the stellar subdivision of L/G (with center v^*) and the same argument as in the proof of Theorem 1.2 in [12], we have the second part.

(iii) In the general case, if M is a semialgebraic G space in Ω. Let $\overline{M}$ be the closure of M in Ω, then it is also a closed semialgebraic G space in Ω. We take a semialgebraic triangulation of $\overline{M}/G$ which is compatible with $\{\overline{M}_{(H)}, \overline{M} - M, M\}$. These together with the closed case (ii) imply this theorem.

(iv) The uniqueness part proved by the same way as the proof of Corollary 2.6. □

5. Semialgebraic G CW complex structure of a semialgebraic G space for G a compact Lie group

In this section G denotes a compact Lie group. We will show that any semialgebraic G space M has a G CW complex structure. When M is compact in Ω, we also show that M has an equivariant simple homotopy type.

DEFINITION 5.1. A G CW *complex structure* of a G space M is a pair (X, ξ) consisting of a G CW complex X and an equivariant homeomorphism $\xi\colon X \to M$. The G-CW complex structure (X, ξ) is said to *induce a triangulation* of M/G if X/G is a simplicial complex and each characteristic G map of an equivariant n-cell $f_{\Delta^n}\colon G/H_{\Delta^n} \times \Delta^n \to X$ induces a linear characteristic map $(G/H_{\Delta^n} \times \Delta^n)/G = \Delta^n \to X/G$ of some simplex in X/G. Moreover, if M is a semialgebraic G space and for the induced homeomorphism $\overline{\xi}\colon X/G \to M/G$ the composition $(\overline{\xi})^{-1} \circ \pi$ is semialgebraic, then we say that (X, ξ) *induces a semialgebraic triangulation* of the orbit space M/G. A *finite open* G CW *complex* *structure* Y is defined as a G invariant subset of some finite (i.e. compact) G CW complex structure X by removing some open equivariant n-cells $f_{\Delta^n}(G/H_{\Delta^n} \times \text{int}(\Delta^n))$ of X. See [4] or [9] for any unexplained terminology.

We can easily check that if a G CW complex structure (X, ξ) of a G space M induces a triangulation of M/G then $(X/G, \bar{\xi})$ is compatible with the orbit types.

Let G be a compact Lie group. For each closed subgroup H of G, G/H has a natural semialgebraic and analytic structure from the orbit map $G \to G/H$.

THEOREM 5.2 (Theorem 1.2). *Let G be a compact Lie group. Any semialgebraic G space M in Ω has a finite open G CW complex structure X such that*

(1) *which induces a semialgebraic triangulation of M/G compatible with the orbit types as in* Theorem 3.5.
(2) *each equivariant-cell is semialgebraic and each open equivariant-cell is a semialgebraic smooth analytic G submanifold of Ω.*
(3) *each characteristic G map $f_{\Delta^n}\colon G/H_{\Delta^n} \times \Delta^n \to M$ is semialgebraic and the restriction $f_{\Delta^n}|$ to $G/H_{\Delta^n} \times (\text{int}(\Delta^n))$ is a smooth analytic isomorphism to its image.*

In particular if M is closed in Ω, we can replace X by a complete finite G CW complex structure.

Proof. (i) If M is compact. Let (K, τ) be the barycentric subdivision of a semialgebraic triangulation of the orbit space which is compatible with the equivariant stratification of M. Then for each simplex Δ^n of K, note that $G/H_{\Delta^n} \times (\text{int}\,\Delta^n)$ is analytic G diffeomorphic to $\pi^{-1}(\tau(\text{int}\,\Delta^n))$. And thus $\tau(\Delta^n)$ has a semialgebraic section s_{Δ^n} with $s_{\Delta^n}(\tau(\Delta^n)) = (\pi^{-1}(\tau(\Delta^n)))^{H_{\Delta^n}}$ and $s_{\Delta^n}|_{\tau(\text{int}\,\Delta^n)}$ is analytic. Define the characteristic G map $f_{\Delta^n}\colon G/H_{\Delta^n} \times \Delta^n \to M$ by $f_{\Delta^n}(gH_{\Delta^n}, x) = gs_{\Delta^n}(x)$. Then f_{Δ^n} is semialgebraic and the restriction $f_{\Delta^n}|$ on $G/H_{\Delta^n} \times (\Delta^n - \Delta^{n-1})$ is a homeomorphism since there is constant orbit type and $\Delta^n - \Delta^{n-1}$ is contractible.

Collecting equivariant cells $\{\pi^{-1}(\tau(\Delta^n)) = Gs_{\Delta^n}(\tau(\Delta^n))\}$ for all simplices Δ^n of K, we get a desired G CW complex structure such that the underlying space is M.

(ii) If M is closed in Ω, using the inversion map θ and one point compactification of M, we have a complete finite G CW complex structure as in compact case and a fixed equivariant 0-cell which is a point. And using the stellar subdivision with center this 0-cell as in the proof of Theorem 4.3, then we have the second part.

(iii) In general case, we obtain this theorem from the closure of M in Ω and the case (ii). □

In fact the closed case of Theorem 5.2, without semialgebraic and analytic sentences, can also be obtained from Theorem 3.5 and Corollary 6.3 of [6]. However, with our approach, we get the uniqueness of G CW complex structure for compact M. In fact, let M be a compact semialgebraic G space. Since two such sections are concordant (see [12, Lemma 5.2]), we get the uniqueness theorem for such G CW complex structure as in Theorem 5.2 (see [12, Lemma 5.3]). This defines the equivariant simple homotopy type of M. Thus we have

THEOREM 5.3. *Any compact semialgebraic G space has an equivariant simple homotopy type where G is a compact Lie group.*

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