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THE SPHERICAL NON-COMMUTATIVE TORI

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ABSTRACT. We define the spherical non-commutative torus $\mathbb{L}_\omega$ as the crossed product obtained by an iteration of l crossed products by actions of $\mathbb{Z}$, the first action on $C(S^{2n+1})$. Assume the fibres are isomorphic to the tensor product of a completely irrational non-commutative torus A_ρ with a matrix algebra $M_m(\mathbb{C})$ $(m > 1)$. We prove that $\mathbb{L}_\omega \otimes M_p(\mathbb{C})$ is not isomorphic to $C(\text{Prim}(\mathbb{L}_\omega)) \otimes A_\rho \otimes M_{mp}(\mathbb{C})$, and that the tensor product of $\mathbb{L}_\omega$ with a UHF-algebra M_{p^∞} of type p^∞ is isomorphic to $C(\text{Prim}(\mathbb{L}_\omega)) \otimes A_\rho \otimes M_m(\mathbb{C}) \otimes M_{p^\infty}$ if and only if the set of prime factors of m is a subset of the set of prime factors of p.

Furthermore, it is shown that the tensor product of $\mathbb{L}_\omega$ with the C^*-algebra $\mathcal{K}(\mathcal{H})$ of compact operators on a separable Hilbert space $\mathcal{H}$ is not isomorphic to $C(\text{Prim}(\mathbb{L}_\omega)) \otimes A_\rho \otimes M_m(\mathbb{C}) \otimes \mathcal{K}(\mathcal{H})$ if $\text{Prim}(\mathbb{L}_\omega)$ is homeomorphic to $L^k(n) \times \mathbb{T}^{l'}$ for k and l' non-negative integers $(k > 1)$, where $L^k(n)$ is the lens space.

1. Introduction

Given a locally compact abelian group G and a multiplier ω on G, one can associate to them the twisted group C^*-algebra $C^*(G, \omega)$, which is the universal object for unitary ω-representations of G. $C^*(\mathbb{Z}^{l+1}, \omega)$ is said to be a *non-commutative torus of rank* $l+1$ and denoted by A_ω. The multiplier ω determines a subgroup S_ω of G, called its *symmetry group*, and the multiplier ω is called *totally skew* if the

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symmetry group S_ω is trivial. And A_ω is called *completely irrational* if ω is totally skew. It was shown in [1] that if G is a locally compact abelian group and ω is a totally skew multiplier on G, then $C^*(G,\omega)$ is a simple C^*-algebra.

It is well-known (cf. [5]) that A_ω can be obtained by an iteration of l ordinary crossed products by actions of $\mathbb{Z}$, the first action on $C(S^1)$. And since A_ω is the universal object for unitary ω-representations of $\mathbb{Z}^{l+1}$, A_ω is realized as $C^*(u_0,u_1,\cdots,u_l \mid u_iu_j = e^{2\pi i\theta_{ji}}u_ju_i)$, where u_i are unitaries and θ_{ji} are real numbers for $0 \le i,j \le l$.

Consider the crossed product $\mathbb{L}_\omega$ obtained by an iteration of l ordinary crossed products by actions of $\mathbb{Z}$, the first action on $C(S^{2n+1})$, where $C(S^1)$ in the crossed product $C(S^1)\times_{\alpha_1}\mathbb{Z}\times_{\alpha_2}\cdots\times_{\alpha_l}\mathbb{Z}$ representing A_ω is replaced by $C(S^{2n+1})$, and the actions of $\mathbb{Z}$ on $C(S^{2n+1})$ are induced from the homeomorphisms

$$(z_0,z_1,\cdots,z_n)\in S^{2n+1}\mapsto(e^{2\pi i\theta_{0i}}z_0,e^{2\pi i\theta_{0i}}z_1,\cdots,e^{2\pi i\theta_{0i}}z_n)\in S^{2n+1}$$

for $i=1,2,\cdots,l$.

In this paper, using Pimsner-Voiculescu exact sequence for a crossed product, we compute the K-theory of $\mathbb{L}_\omega$ and we are going to show that the class $[1_{\mathbb{L}_\omega}]\in K_0(\mathbb{L}_\omega)$ is primitive. Using the fact that the class $[1_{\mathbb{L}_\omega}]\in K_0(\mathbb{L}_\omega)$ is primitive, we are going to show that for θ_{0i} rational numbers $(i=1,\cdots,l)$ the tensor product of $\mathbb{L}_\omega$ (with fibres $A_\rho\otimes M_m(\mathbb{C})$ for A_ρ a simple non-commutative torus and m a positive integer $(m>1)$) with a matrix algebra $M_p(\mathbb{C})$ is not isomorphic to $C(\mathrm{Prim}(\mathbb{L}_\omega))\otimes A_\rho\otimes M_{mp}(\mathbb{C})$, and that the tensor product of $\mathbb{L}_\omega$ with a UHF-algebra M_{p^∞} of type p^∞ is isomorphic to $C(\mathrm{Prim}(\mathbb{L}_\omega))\otimes A_\rho\otimes M_m(\mathbb{C})\otimes M_{p^\infty}$ if and only if the set of prime factors of m is a subset of the set of prime factors of p.

Furthermore, we are going to show that $\mathcal{O}_{2d}\otimes\mathbb{L}_\omega$ has the trivial bundle structure if and only if m and $2d-1$ are relatively prime, and that $\mathcal{O}_\infty\otimes\mathbb{L}_\omega$ has always a non-trivial bundle structure if $m>1$.

By comparison of the K-theory, it is shown that the tensor product of $\mathbb{L}_\omega$ with the C^*-algebra $\mathcal{K}(\mathcal{H})$ of compact operators on a separable Hilbert space $\mathcal{H}$ is not isomorphic to $C(\mathrm{Prim}(\mathbb{L}_\omega))\otimes A_\rho\otimes M_m(\mathbb{C})\otimes\mathcal{K}(\mathcal{H})$ if $\mathrm{Prim}(\mathbb{L}_\omega)$ is homeomorphic to $L^k(n)\times\mathbb{T}^{l'}$ for k and l' non-negative integers $(k>1)$, where $L^k(n)$ is the lens space.

2. The K-theory of spherical non-commutative tori

One can canonically replace $C(S^1)$ in the crossed product $C(S^1)\times_{\alpha_1} \mathbb{Z}\times_{\alpha_2}\cdots\times_{\alpha_l}\mathbb{Z}$ representing A_ω by $C(S^{2n+1})$ and the actions of $\mathbb{Z}$ on $C(S^{2n+1})$ are induced from the homeomorphisms

$$(z_0,z_1,\cdots,z_n)\in S^{2n+1}\mapsto(e^{2\pi i\theta_{0i}}z_0,e^{2\pi i\theta_{0i}}z_1,\cdots,e^{2\pi i\theta_{0i}}z_n)\in S^{2n+1}$$

for $i=1,2,\cdots,l$.

DEFINITION 2.1. The crossed product, constructed as above, obtained by an iteration of l ordinary crossed products by actions of $\mathbb{Z}$, the first action on $C(S^{2n+1})$, is said to be a *spherical non-commutative torus of rank* $l+1$, and denoted by $\mathbb{L}_\omega$, where θ_{0i} in the crossed product representing the non-commutative torus A_ω of rank $l+1$ are rational numbers for $i=1,\cdots,l$.

We are going to show that the class $[1_{\mathbb{L}_\omega}]\in K_0(\mathbb{L}_\omega)$ is primitive.

THEOREM 2.2. *Let $\mathbb{L}_\omega$ be a spherical non-commutative torus of rank $l+1$. Then $K_0(\mathbb{L}_\omega)\cong K_1(\mathbb{L}_\omega)\cong\mathbb{Z}^{2^l}$, and $[1_{\mathbb{L}_\omega}]\in K_0(\mathbb{L}_\omega)$ is primitive.*

Proof. The proof is by induction on l. If $l=1$, $\mathbb{L}_\omega$ is isomorphic to $C(S^{2n+1})\times_{\alpha_1}\mathbb{Z}$. The action α_1 is induced from the homeomorphism

$$(z_0,z_1,\cdots,z_n)\in S^{2n+1}\mapsto(e^{2\pi i\theta_{01}}z_0,e^{2\pi i\theta_{01}}z_1,\cdots,e^{2\pi i\theta_{01}}z_n)\in S^{2n+1}.$$

Note that this action is homotopic to the trivial action, since we can homotope θ_{01} to 0. Hence $\mathbb{Z}$ acts trivially on the K-theory of $C(S^{2n+1})$. The Pimsner-Voiculescu exact sequence for a crossed product gives

$$\cdots\xrightarrow{1-(\alpha_1)_*}K_0(C(S^{2n+1}))\xrightarrow{\Phi}K_0(\mathbb{L}_\omega)\to K_1(C(S^{2n+1}))\xrightarrow{1-(\alpha_1)_*}\cdots$$

and similarly for K_1, where the map Φ is induced by inclusion. Since $(\alpha_1)_*=1$ and since the K-groups of $C(S^{2n+1})$ are free abelian, this reduces a split short exact sequence

$$\{0\}\to K_0(C(S^{2n+1}))\xrightarrow{\Phi}K_0(\mathbb{L}_\omega)\to K_1(C(S^{2n+1}))\to\{0\}$$

and similarly for K_1. But $K_0(C(S^{2n+1})) \cong K_1(C(S^{2n+1})) \cong \mathbb{Z}$ (see [8, II.1.34]). So $K_j(\mathbb{L}_\omega)$ are free abelian of rank 2.

Since the inclusion $C(S^{2n+1}) \to \mathbb{L}_\omega$ sends $1_{C(S^{2n+1})}$ to $1_{\mathbb{L}_\omega}$, $[1_{\mathbb{L}_\omega}]$ is the image of $[1_{C(S^{2n+1})}]$, which is primitive in $K_0(C(S^{2n+1}))$ (see [8, II.1.21]). Hence the image is primitive, since the Pimsner-Voiculescu exact sequence is a split short exact sequence of torsion-free groups.

Assume the result is true for all spherical non-commutative tori of rank $l = i-1$. Write $\mathbb{L}_i = C^*(\mathbb{L}_{i-1}, u_i)$, where $\mathbb{L}_i = C^*(C(S^{2n+1}), u_1, \ldots, u_i)$. Then the inductive hypothesis applies to $\mathbb{L}_{i-1}$. Also, we can think of $\mathbb{L}_i$ as the crossed product of $\mathbb{L}_{i-1}$ by an action α_i of $\mathbb{Z}$, where the generator of $\mathbb{Z}$ corresponds to u_i, which acts on $C^*(u_1, u_2, \cdots, u_{i-1})$ by conjugation (sending u_j to $u_i u_j u_i^{-1} = \lambda_j u_j$, $\lambda_j = \exp(2\pi i \theta_{ji})$), and acts on $C(S^{2n+1})$ by the automorphism induced from the homeomorphism given as above. Note that this action is homotopic to the trivial action, since we can homotope θ_{ji} to 0. Hence $\mathbb{Z}$ acts trivially on the K-theory of $\mathbb{L}_{i-1}$. The Pimsner-Voiculescu exact sequence for a crossed product gives

$$K_0(\mathbb{L}_{i-1}) \xrightarrow{1-(\alpha_i)_*} K_0(\mathbb{L}_{i-1}) \xrightarrow{\Phi} K_0(\mathbb{L}_i) \to K_1(\mathbb{L}_{i-1}) \xrightarrow{1-(\alpha_i)_*} K_1(\mathbb{L}_{i-1})$$

and similarly for K_1, where the map Φ is induced by inclusion. Since $(\alpha_i)_* = 1$ and since the K-groups of $\mathbb{L}_{i-1}$ are free abelian, this reduces a split short exact sequence

$$\{0\} \to K_0(\mathbb{L}_{i-1}) \xrightarrow{\Phi} K_0(\mathbb{L}_i) \to K_1(\mathbb{L}_{i-1}) \to \{0\}$$

and similarly for K_1. So $K_0(\mathbb{L}_i)$ and $K_1(\mathbb{L}_i)$ are free abelian of rank $2 \cdot 2^{i-1} = 2^i$. Furthermore, since the inclusion $\mathbb{L}_{i-1} \to \mathbb{L}_i$ sends $1_{\mathbb{L}_{i-1}}$ to $1_{\mathbb{L}_i}$, $[1_{\mathbb{L}_i}]$ is the image of $[1_{\mathbb{L}_{i-1}}]$, which is primitive in $K_0(\mathbb{L}_{i-1})$ by inductive hypothesis. Hence the image is primitive, since the Pimsner-Voiculescu exact sequence is a split short exact sequence of torsion-free groups.

Therefore, $K_0(\mathbb{L}_\omega) \cong K_1(\mathbb{L}_\omega) \cong \mathbb{Z}^{2^l}$, and $[1_{\mathbb{L}_\omega}] \in K_0(\mathbb{L}_\omega)$ is primitive. □

It is well-known (cf. [10, 11]) that the set $[M, BPU(k)]$ of homotopy classes of continuous maps of a compact Hausdorff space M into the classifying space $BPU(k)$ of the Lie group $PU(k)$ is in bijective correspondence with the set of equivalence classes of principal $PU(k)$-bundles over M, and that principal $PU(k)$-bundles over M and k-homogeneous C^*-algebras over M are in bijective correspondence, i.e., every k-homogeneous C^*-algebra A is isomorphic to the C^*-algebra of sections of a locally trivial C^*-algebra bundle with base space M, fibre $M_k(\mathbb{C})$, and structure group $\mathrm{Aut}(M_k(\mathbb{C})) \cong PU(k)$.

By a change of basis for $\mathbb{Z}^l$, one can assume that $\theta_{01} = \frac{m}{k}$ is rational with $(k,m) = 1$, and that θ_{0i} are zero for $i = 2, \cdots, l$. Since the cyclic group $\mathbb{Z}/k\mathbb{Z}$ acts freely on $C(S^{2n+1})$ and $S^{2n+1}/(\mathbb{Z}/k\mathbb{Z}) = L^k(n)$, each point of $L^k(n) \times \widehat{k\mathbb{Z}} = \mathrm{Prim}(C(S^{2n+1}) \times_{\alpha_1} \mathbb{Z})$ has a contractible neighborhood and the fibres at each point of $L^k(n) \times \widehat{k\mathbb{Z}}$ are $M_k(\mathbb{C})$. So the crossed product $C(S^{2n+1}) \times_{\alpha_1} \mathbb{Z}$ is a k-homogeneous C^*-algebra over $L^k(n) \times \widehat{k\mathbb{Z}}$. Thus the crossed product $C(S^{2n+1}) \times_{\alpha_1} \mathbb{Z}$ is realized as the C^*-algebra of sections of a locally trivial C^*-algebra bundle over $L^k(n) \times \widehat{k\mathbb{Z}}$ with fibres $M_k(\mathbb{C})$ where $L^k(n)$ is the lens space. And the cyclic group $\mathbb{Z}/k\mathbb{Z}$ acts freely on the non-commutative torus $C^*(u_1, u_2, \cdots, u_l)$ of rank l, and the Mackey machine for a crossed product says that the crossed product of the cyclic group $\mathbb{Z}/k\mathbb{Z}$ on $C^*(u_1, u_2, \cdots, u_l)$ is isomorphic to $M_k(\mathbb{C}) \otimes C^*(u_1^k, u_2, \cdots, u_l)$. But $C^*(u_1^k, u_2, \cdots, u_l)$ is a non-commutative torus of rank l. $C^*(u_1^k, u_2, \cdots, u_l)$ is isomorphic to the C^*-algebra of sections of a locally trivial C^*-algebra bundle over $\mathrm{Prim}(C^*(u_1^k, u_2, \cdots, u_l))$ with fibres $A_\rho \otimes M_{k'}(\mathbb{C})$ for A_ρ a simple non-commutative torus and k' a positive integer. Hence the spherical non-commutative torus $\mathbb{L}_\omega$ of rank $l+1$ is isomorphic to the C^*-algebra of sections of a locally trivial C^*-algebra bundle over $\mathrm{Prim}(\mathbb{L}_\omega) = L^k(n) \times \mathrm{Prim}(C^*(u_1^k, u_2, \cdots, u_l))$ with fibres $A_\rho \otimes M_m(\mathbb{C})$, where $M_m(\mathbb{C}) := M_k(\mathbb{C}) \otimes M_{k'}(\mathbb{C})$. See [7] for details.

COROLLARY 2.3. *Let p be a positive integer. $\mathbb{L}_\omega \otimes M_p(\mathbb{C})$ is not isomorphic to $C(\mathrm{Prim}(\mathbb{L}_\omega)) \otimes A_\rho \otimes M_{mp}(\mathbb{C})$ if $m > 1$.*

Proof. Assume $\mathbb{L}_\omega \otimes M_p(\mathbb{C})$ is isomorphic to $C(\mathrm{Prim}(\mathbb{L}_\omega)) \otimes A_\rho \otimes M_{mp}(\mathbb{C})$. Then the unit $1_{\mathbb{L}_\omega} \otimes I_p$ maps to the unit $1_{C(\mathrm{Prim}(\mathbb{L}_\omega)) \otimes A_\rho} \otimes I_{mp}$, where I_s denotes the $s \times s$ identity matrix. So

$$[1_{\mathbb{L}_\omega} \otimes I_p] = [1_{C(\mathrm{Prim}(\mathbb{L}_\omega)) \otimes A_\rho} \otimes I_{mp}].$$

Thus there is a projection $e \in \mathbb{L}_\omega$ such that $p[1_{\mathbb{L}_\omega}] = (mp)[e]$. But $K_0(\mathbb{L}_\omega) \cong \mathbb{Z}^{2^l}$ is torsion-free, so $[1_{\mathbb{L}_\omega}] = m[e]$, a contradiction if $m > 1$.

Therefore, $\mathbb{L}_\omega \otimes M_p(\mathbb{C})$ is not isomorphic to $C(\text{Prim}(\mathbb{L}_\omega)) \otimes A_\rho \otimes M_{mp}(\mathbb{C})$ if $m > 1$. □

By comparison of the K-theory, one can show that $\mathbb{L}_\omega \otimes \mathcal{K}(\mathcal{H})$ has a non-trivial bundle structure if $\text{Prim}(\mathbb{L}_\omega)$ is homeomorphic to $L^k(n) \times \mathbb{T}^{l'}$ for k and l' non-negative integers ($k > 1$), where $L^k(n)$ is the lens space.

COROLLARY 2.4. *Let $\mathbb{L}_\omega$ be a spherical non-commutative torus of rank $l+1$ with fibres $A_\rho \otimes M_m(\mathbb{C})$ for A_ρ a simple non-commutative torus and* $\text{Prim}(\mathbb{L}_\omega) = L^k(n) \times \mathbb{T}^{l'}$ *for k and l' non-negative integers ($k > 1$). Then $\mathbb{L}_\omega \otimes \mathcal{K}(\mathcal{H})$ is not isomorphic to* $C(\text{Prim}(\mathbb{L}_\omega)) \otimes A_\rho \otimes M_m(\mathbb{C}) \otimes \mathcal{K}(\mathcal{H})$.

Proof. By Theorem 2.2 $K_0(\mathbb{L}_\omega \otimes \mathcal{K}(\mathcal{H})) \cong K_0(\mathbb{L}_\omega) \cong \mathbb{Z}^{2^l}$, torsion-free. On the other hand, $K_0(C(\text{Prim}(\mathbb{L}_\omega)) \otimes A_\rho \otimes M_m(\mathbb{C}) \otimes \mathcal{K}(\mathcal{H})) \cong K_0(C(\text{Prim}(\mathbb{L}_\omega)) \otimes A_\rho) \cong K_0(C(L^k(n)) \otimes C(\mathbb{T}^{l'}) \otimes A_\rho) \cong K_0(C(L^k(n))) \otimes K_0(C(\mathbb{T}^{l'}) \otimes A_\rho) \oplus K_1(C(L^k(n))) \otimes K_1(C(\mathbb{T}^{l'}) \otimes A_\rho)$ by Künneth Theorem (see [2, Theorem 23.1.3]). But $K_0(C(L^k(n))) \otimes K_0(C(\mathbb{T}^{l'}) \otimes A_\rho) \cong (\mathbb{Z}/k^n\mathbb{Z} \oplus \mathbb{Z}) \otimes \mathbb{Z}^{2^{l-1}} \cong (\mathbb{Z}/k^n\mathbb{Z})^{2^{l-1}} \oplus \mathbb{Z}^{2^{l-1}}$ (see [8, IV.2.11]). So $K_0(C(\text{Prim}(\mathbb{L}_\omega)) \otimes A_\rho \otimes M_m(\mathbb{C}) \otimes \mathcal{K}(\mathcal{H}))$ is not torsion-free. Hence $\mathbb{L}_\omega \otimes \mathcal{K}(\mathcal{H})$ is not isomorphic to $C(\text{Prim}(\mathbb{L}_\omega)) \otimes A_\rho \otimes M_m(\mathbb{C}) \otimes \mathcal{K}(\mathcal{H})$.□

We have obtained that $[1_{\mathbb{L}_\omega}] \in K_0(\mathbb{L}_\omega)$ is primitive. This result is very useful to investigate the bundle structure of the tensor products of the spherical non-commutative tori with UHF-algebras and Cuntz algebras.

3. The tensor products of spherical non-commutative tori with UHF-algebras

In this section, we investigate the bundle structure of the tensor products of the spherical non-commutative tori $\mathbb{L}_\omega$ with UHF-algebras M_{p^∞} of type p^∞ and Cuntz algebras.

The following is useful.

THEOREM 3.1 [6, Theorem 7.1]. *Suppose there exists an intertwining of the sequence of C^*-algebra homomorphisms $A_1 \to A_2 \to \cdots$*

and $B_1 \to B_2 \to \cdots$. *Then the inductive limit* C^**-algebras* $\lim A_i$ *and* $\lim B_i$ *are isomorphic.*

THEOREM 3.2. *Let* $\mathbb{L}_\omega$ *be a spherical non-commutative torus of rank* $l+1$ *with fibres* $A_\rho \otimes M_m(\mathbb{C})$ *for* A_ρ *a simple non-commutative torus and* m *a positive integer. Let* M_{p^∞} *be a* UHF*-algebra of type* p^∞. *Then* $\mathbb{L}_\omega \otimes M_{p^\infty}$ *is isomorphic to* $C(\mathrm{Prim}(\mathbb{L}_\omega)) \otimes A_\rho \otimes M_m(\mathbb{C}) \otimes M_{p^\infty}$ *if and only if the set of prime factors of* m *is a subset of the set of prime factors of* p.

Proof. Assume the set of prime factors of m is a subset of the set of prime factors of p. To show that $\mathbb{L}_\omega \otimes M_{p^\infty}$ is isomorphic to $C(\mathrm{Prim}(\mathbb{L}_\omega)) \otimes A_\rho \otimes M_m(\mathbb{C}) \otimes M_{p^\infty}$, it is enough to show that $\mathbb{L}_\omega \otimes M_{m^\infty}$ is isomorphic to $C \otimes M_{m^\infty}$ where $C := C(\mathrm{Prim}(\mathbb{L}_\omega)) \otimes A_\rho$. But there exist the canonical C^*-algebra homomorphisms:

$$\mathbb{L}_\omega \hookrightarrow C \otimes M_m(\mathbb{C}) \hookrightarrow \mathbb{L}_\omega \otimes M_m(\mathbb{C}) \hookrightarrow C \otimes M_{m^2}(\mathbb{C}) \hookrightarrow \cdots .$$

The inductive limit of the odd terms

$$\cdots \to \mathbb{L}_\omega \otimes M_{m^d}(\mathbb{C}) \to \mathbb{L}_\omega \otimes M_{m^{d+1}}(\mathbb{C}) \to \cdots$$

is $\mathbb{L}_\omega \otimes M_{m^\infty}$, and the inductive limit of the even terms

$$\cdots \to C \otimes M_{m^d}(\mathbb{C}) \to C \otimes M_{m^{d+1}}(\mathbb{C}) \to \cdots$$

is $C \otimes M_{m^\infty}$. Thus by Theorem 3.1, $\mathbb{L}_\omega \otimes M_{m^\infty}$ is isomorphic to $C(\mathrm{Prim}(\mathbb{L}_\omega)) \otimes A_\rho \otimes M_{m^\infty}$.

Conversely, assume $\mathbb{L}_\omega \otimes M_{p^\infty}$ is isomorphic to $C(\mathrm{Prim}(\mathbb{L}_\omega)) \otimes A_\rho \otimes M_m(\mathbb{C}) \otimes M_{p^\infty}$. Let $C := C(\mathrm{Prim}(\mathbb{L}_\omega)) \otimes A_\rho$. Then the unit $1_{\mathbb{L}_\omega} \otimes 1_{M_{p^\infty}}$ maps to the unit $1_C \otimes 1_{M_{p^\infty}} \otimes I_m$. So

$$[1_{\mathbb{L}_\omega} \otimes 1_{M_{p^\infty}}] = [1_C \otimes 1_{M_{p^\infty}} \otimes I_m].$$

And $[1_{\mathbb{L}_\omega} \otimes 1_{M_{p^\infty}}] = [1_{\mathbb{L}_\omega}] \otimes [1_{M_{p^\infty}}]$ and $[1_C \otimes 1_{M_{p^\infty}} \otimes I_m] = m([1_C] \otimes [1_{M_{p^\infty}}])$. But $K_0(\mathbb{L}_\omega \otimes M_{p^\infty}) \cong [\frac{1}{p}](K_0(\mathbb{L}_\omega))$ and $K_0(C \otimes M_{p^\infty} \otimes M_m(\mathbb{C})) \cong m[\frac{1}{p}](K_0(C))$. If there is a prime factor q of m such that $q \nmid p$, then $[1_{M_{p^\infty}}] \neq q[e_\infty]$ for e_∞ a projection in M_{p^∞} under the assumption that the unit $1_{\mathbb{L}_\omega} \otimes 1_{M_{p^\infty}}$ maps to the unit $1_C \otimes 1_{M_{p^\infty}} \otimes I_m$.

So there is a projection $e \in \mathbb{L}_\omega$ such that $[1_{\mathbb{L}_\omega}] = q[e]$. This contradicts Theorem 2.2. Thus the set of prime factors of m is a subset of the set of prime factors of p.

Therefore, $\mathbb{L}_\omega \otimes M_{p^\infty}$ is isomorphic to $C(\mathrm{Prim}(\mathbb{L}_\omega)) \otimes A_\rho \otimes M_m(\mathbb{C}) \otimes M_{p^\infty}$ if and only if the set of prime factors of m is a subset of the set of prime factors of p. □

We have obtained that $\mathbb{L}_\omega \otimes M_{p^\infty}$ has the trivial bundle structure if and only if the set of prime factors of m is a subset of the set of prime factors of p.

Let us apply the previous results to the tensor products of $\mathbb{L}_\omega$ with Cuntz algebras.

The Cuntz algebra $\mathcal{O}_d, 2 \leq d < \infty$, is the universal C^*-algebra generated by d isometries $s_1, \ldots, s_d$, i.e., $s_j^* s_j = 1$ for all j, with the relation $s_1 s_1^* + \cdots + s_d s_d^* = 1$. Cuntz [3, 4] proved that $\mathcal{O}_d$ is simple and the K-theory of $\mathcal{O}_d$ is $K_0(\mathcal{O}_d) = \mathbb{Z}/(d-1)\mathbb{Z}$ and $K_1(\mathcal{O}_d) = 0$. He proved that $K_0(\mathcal{O}_d)$ is generated by the class of the unit.

PROPOSITION 3.3. *Let $\mathbb{L}_\omega$ be a spherical non-commutative torus of rank $l+1$ with fibres $A_\rho \otimes M_m(\mathbb{C})$ for A_ρ a simple non-commutative torus and m a positive integer ($m > 1$). Let d be a positive integer such that m and $d-1$ are not relatively prime. Then $\mathcal{O}_d \otimes \mathbb{L}_\omega$ is not isomorphic to $\mathcal{O}_d \otimes C(\mathrm{Prim}(\mathbb{L}_\omega)) \otimes A_\rho \otimes M_m(\mathbb{C})$.*

Proof. Let p be a prime such that $p \mid m$ and $p \mid d-1$. Suppose that $\mathcal{O}_d \otimes \mathbb{L}_\omega$ is isomorphic to $\mathcal{O}_d \otimes C(\mathrm{Prim}(\mathbb{L}_\omega)) \otimes A_\rho \otimes M_m(\mathbb{C})$. The the unit $1_{\mathcal{O}_d \otimes \mathbb{L}_\omega}$ maps to the unit $1_{\mathcal{O}_d \otimes C(\mathrm{Prim}(\mathbb{L}_\omega)) \otimes A_\rho} \otimes I_m$, i.e.,

$$[1_{\mathcal{O}_d \otimes \mathbb{L}_\omega}] = [1_{\mathcal{O}_d \otimes C(\mathrm{Prim}(\mathbb{L}_\omega)) \otimes A_\rho} \otimes I_m].$$

So there is a projection e in $\mathcal{O}_d \otimes \mathbb{L}_\omega$ such that $[1_{\mathcal{O}_d \otimes \mathbb{L}_\omega}] = m[e]$. But $[1_{\mathcal{O}_d \otimes \mathbb{L}_\omega}] = [1_{\mathcal{O}_d}] \otimes [1_{\mathbb{L}_\omega}]$ and $[1_{\mathcal{O}_d}]$ is a generator of $K_0(\mathcal{O}_d) \cong \mathbb{Z}/(d-1)\mathbb{Z}$ (see [4]). By assumption $p \mid d-1$. $[1_{\mathcal{O}_d}] \neq p[e_1]$ for e_1 a projection in $\mathcal{O}_d$. So $[1_{\mathbb{L}_\omega}] = m'[e_2]$ for a projection $e_2 \in \mathbb{L}_\omega$ and an integer m' with $p \mid m'$. This is a contradiction. Hence m and $d-1$ are relatively prime.

Therefore, $\mathcal{O}_d \otimes \mathbb{L}_\omega$ is not isomorphic to $\mathcal{O}_d \otimes C(\mathrm{Prim}(\mathbb{L}_\omega)) \otimes A_\rho \otimes M_m(\mathbb{C})$ if m and $d-1$ are not relatively prime. □

More generally, it follows from a theorem of Rørdam that m and $d-1$ are relatively prime if and only if $\mathcal{O}_d \otimes \mathbb{L}_\omega$ is isomorphic to $\mathcal{O}_d \otimes C(\mathrm{Prim}(\mathbb{L}_\omega)) \otimes A_\rho \otimes M_m(\mathbb{C})$.

The following result is useful to understand the bundle structure of $\mathcal{O}_d \otimes \mathbb{L}_\omega$.

PROPOSITION 3.4 [9, Theorem 7.2]. *Let A and B be unital simple inductive limits of even Cuntz algebras. If $\alpha : K_0(A) \to K_0(B)$ is an isomorphism of abelian groups satisfying $\alpha([1_A]) = [1_B]$, then there is an isomorphism $\phi : A \to B$ which induces α.*

COROLLARY 3.5.

(1) *Let p be an odd integer such that p and $2d-1$ are relatively prime. Then $\mathcal{O}_{2d}$ is isomorphic to $\mathcal{O}_{(2d-1)p+1} \otimes M_{p^\infty}$. That is, $\mathcal{O}_{2d}$ is isomorphic to $\mathcal{O}_{2d} \otimes M_{p^\infty}$.*
(2) *$\mathcal{O}_{2d}$ is isomorphic to $\mathcal{O}_{2d} \otimes M_{(2d)^\infty}$.*

THEOREM 3.6. *Let $\mathbb{L}_\omega$ be a spherical non-commutative torus given as above. Then $\mathcal{O}_{2d} \otimes \mathbb{L}_\omega$ is isomorphic to $\mathcal{O}_{2d} \otimes C(\mathrm{Prim}(\mathbb{L}_\omega)) \otimes A_\rho \otimes M_m(\mathbb{C})$ if and only if m and $2d-1$ are relatively prime.*

Proof. Assume that m and $2d-1$ are relatively prime. Let $m = p2^c$ for some odd integer p. Then p and $2d-1$ are relatively prime. Then by Corollary 3.5 $\mathcal{O}_{2d}$ is isomorphic to $\mathcal{O}_{2d} \otimes M_{p^\infty}$, and $\mathcal{O}_{2d}$ is isomorphic to $\mathcal{O}_{2d} \otimes M_{(2d)^\infty} \cong \mathcal{O}_{2d} \otimes M_{(2d)^\infty} \otimes M_{(2^c)^\infty} \cong \mathcal{O}_{2d} \otimes M_{(2^c)^\infty}$. So $\mathcal{O}_{2d}$ is isomorphic to $\mathcal{O}_{2d} \otimes M_{p^\infty} \otimes M_{(2^c)^\infty} \cong \mathcal{O}_{2d} \otimes M_{m^\infty}$. Thus by Theorem 3.2 $\mathcal{O}_{2d} \otimes \mathbb{L}_\omega$ is isomorphic to $\mathcal{O}_{2d} \otimes M_{m^\infty} \otimes \mathbb{L}_\omega$, which in turn is isomorphic to $\mathcal{O}_{2d} \otimes M_{m^\infty} \otimes C(\mathrm{Prim}(\mathbb{L}_\omega)) \otimes A_\rho \otimes M_m(\mathbb{C})$. Thus $\mathcal{O}_{2d} \otimes \mathbb{L}_\omega$ is isomorphic to $\mathcal{O}_{2d} \otimes C(\mathrm{Prim}(\mathbb{L}_\omega)) \otimes A_\rho \otimes M_m(\mathbb{C})$.

The converse is proved in Proposition 3.3.

Therefore, $\mathcal{O}_{2d} \otimes \mathbb{L}_\omega$ is isomorphic to $\mathcal{O}_{2d} \otimes C(\mathrm{Prim}(\mathbb{L}_\omega)) \otimes A_\rho \otimes M_m(\mathbb{C})$ if and only if m and $2d-1$ are relatively prime. □

Cuntz [4] computed the K-theory of the generalized Cuntz algebra $\mathcal{O}_\infty$, generated by a sequence of isometries with mutually orthogonal ranges, $K_0(\mathcal{O}_\infty) = \mathbb{Z}$ and $K_1(\mathcal{O}_\infty) = 0$. He proved that $K_0(\mathcal{O}_\infty)$ is generated by the class of the unit.

PROPOSITION 3.7. *Let $\mathbb{L}_\omega$ be a spherical non-commutative torus given as above. $\mathcal{O}_\infty \otimes \mathbb{L}_\omega$ is not isomorphic to $\mathcal{O}_\infty \otimes C(\mathrm{Prim}(\mathbb{L}_\omega)) \otimes A_\rho \otimes M_m(\mathbb{C})$.*

Proof. Suppose $\mathcal{O}_\infty \otimes \mathbb{L}_\omega$ is isomorphic to $\mathcal{O}_\infty \otimes C(\mathrm{Prim}(\mathbb{L}_\omega)) \otimes A_\rho \otimes M_m(\mathbb{C})$. Then the unit $1_{\mathcal{O}_\infty \otimes \mathbb{L}_\omega}$ maps to the unit $1_{\mathcal{O}_\infty \otimes C(\mathrm{Prim}(\mathbb{L}_\omega)) \otimes A_\rho} \otimes I_m$. By the same trick as in the proof of Proposition 3.3, one can show that $[1_{\mathcal{O}_\infty \otimes \mathbb{L}_\omega}] = m[e]$ for a projection $e \in \mathcal{O}_\infty \otimes \mathbb{L}_\omega$. $[1_{\mathcal{O}_\infty \otimes \mathbb{L}_\omega}] = [1_{\mathcal{O}_\infty}] \otimes [1_{\mathbb{L}_\omega}]$ and $[1_{\mathcal{O}_\infty}]$ is a primitive element of $K_0(\mathcal{O}_\infty) \cong \mathbb{Z}$ (see [4]). So $[1_{\mathbb{L}_\omega}] = m[e_1]$ for a projection $e_1 \in \mathbb{L}_\omega$. This contradicts Theorem 2.2.

Hence $\mathcal{O}_\infty \otimes \mathbb{L}_\omega$ is not isomorphic to $\mathcal{O}_\infty \otimes C(\mathrm{Prim}(\mathbb{L}_\omega)) \otimes A_\rho \otimes M_m(\mathbb{C})$. □

References

[1] L. Baggett and A. Kleppner, *Multiplier representations of abelian groups*, J. Funct. Anal. **14** (1973), 299–324.

[2] B. Blackadar, *K-Theory for Operator Algebras*, Springer-Verlag, New York, Berlin and Heidelberg, 1986.

[3] J. Cuntz, *Simple C^*-algebras generated by isometries*, Comm. Math. Phys. **57** (1977), 173–185.

[4] ———, *K-theory for certain C^*-algebras*, Ann. Math. **113** (1981), 181–197.

[5] G. A. Elliott, *On the K-theory of the C^*-algebra generated by a projective representation of a torsion-free discrete abelian group*, Operator Algebras and Group Representations (G. Arsene et al., ed.), vol. 1, Pitman, London, 1984, pp. 157–184.

[6] ———, *On the classification of C^*-algebras of real rank zero*, J. Reine Angew. Math. **443** (1993), 179–219.

[7] P. Green, *The local structure of twisted covariance algebras*, Acta. Math. **140** (1978), 191–250.

[8] M. Karoubi, *K-Theory*, Springer-Verlag, Berlin, Heidelberg and New York, 1978.

[9] M. Rørdam, *Classification of inductive limits of Cuntz algebras*, J. Reine Angew. Math. **XL** (1988), no. 2, 257–338.

[10] M. Takesaki and J. Tomiyama, *Applications of fibre bundles to the certain class of C^*-algebras*, Tohoku Math. J. **13** (1961), 498–522.

[11] K. Thomsen, *Inductive limits of homogeneous C^*-algebras*, preprint.

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