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SPACE-LIKE SURFACES WITH 1-TYPE GENERALIZED GAUSS MAP

SOON MEEN CHOI, U-HANG KI AND YOUNG JIN SUH

ABSTRACT. Chen and Piccinni [7] have classified all compact surfaces in a Euclidean space $\mathbf{R}^{2+p}$ with 1-type generalized Gauss map. Being motivated by this result, the purpose of this paper is to consider the Lorentz version of the classification theorem and to obtain a complete classification of space-like surfaces in indefinite Euclidean space $\mathbf{R}_p^{2+p}$ with 1-type generalized Gauss map.

1. Introduction

As is well known, the theory of Gauss maps is always one of interesting topics in Riemannian geometry and it has been investigated from the various viewpoints by many geometers. The generalized Gauss maps are also taken up by Obata [14], Ruh and Vilms [17] and so on. In particular, for the Euclidean space a Gauss map is defined as follows.

Let M be a real hypersurface in an $(m+1)$-dimensional Euclidean space $\mathbf{R}^{m+1}$ and ξ a unit vector field normal to M. Then, for any point z in M, we can regard $\xi(z)$ as a point in an m-dimensional unit sphere $\mathbf{S}^m(1)$ by translating parallelly to the origin in the ambient space $\mathbf{R}^{m+1}$. The map ξ of M into $\mathbf{S}^m(1)$ is called the Gauss map of M in $\mathbf{R}^{m+1}$.

On the other hand, let $G(m,p)$ be a Grassmann manifold consisting of all oriented m-planes through the origin of $\mathbf{R}^{m+p}$. For an isometric immersion x of an m-dimensional oriented Riemannian manifold M into an n-dimensional Euclidean space $\mathbf{R}^n$, $n=m+p$, a generalized

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Gauss map is by definition a map G that assigns each point z in M to the m-plane through the origin in $\mathbf{R}^{m+p}$ obtained by translating parallelly the tangent space at z of M in $\mathbf{R}^{m+p}$. Ruh and Vilms [17] gave a condition for the generalized Gauss map G of the compact Riemannian submanifold to be harmonic. The Grassmann manifold $G(m,p)$ is also isometrically immersed in $\mathbf{R}^N$, $N = \binom{m+p}{m}$. The extension G_0 of the generalized Gauss map G to R^N is also called by the same one. It is said to be of *1-type* if it satisfies

$$\triangle G_0 = \lambda G_0$$

for a constant λ. With this relation of 1-type generalized Gauss map G_0 Chen and Piccinni [7], [8] classified the compact surfaces in a Euclidean space $\mathbf{R}^{2+p}$ in such a way that

THEOREM A. *Let M be a compact surface in $\mathbf{R}^{2+p}$. Then the generalized Gauss map $G_0 : M \to \mathbf{R}^N$, $N = (p+1)(p+2)/2$, is of 1-type if and only if M is one of the following surfaces:*
(1) *the sphere $\mathbf{S}^2 \subset \mathbf{R}^3 \subset \mathbf{R}^{2+p}$,*
(2) *the product of two plane circles $\mathbf{S}^1 \times \mathbf{S}^1 \subset \mathbf{R}^4 \subset \mathbf{R}^{2+p}$.*

On the other hand, the Gauss map on a space-like submanifold in an indefinite Euclidean space is also researched by Aiyama [2], [3] and Palmer [16]. Recently the first author [12] has proved rigidity theorems for ruled surfaces along any non-null curve in $\mathbf{R}^3_1$ by the Gauss map. Moreover, the present authors [13] have characterized a class of non-degenerate ruled surfaces along the null curve in $\mathbf{R}^3_1$, which are said to be of null scrolls, by the Gauss map.

Being motivated by Chen and Piccinni's study [7], [8] and these results, the purpose of this paper is to give the indefinite version of Theorem A as follows:

THEOREM. *Let M be a space-like surface in $\mathbf{R}^{2+p}_p$. Then the only surfaces with 1-type generalized Gauss map $G_0 : M \to \mathbf{R}^N, N = (p+1)(p+2)/2$ are locally the following spaces:*

(1) *the Euclidean space $\mathbf{R}^2$, the hyperbolic space $\mathbf{H}^2$ and the hyperbolic cylinder $\mathbf{H}^1 \times \mathbf{R}$ in $\mathbf{R}^3_1$,*

(2) *the product $\mathbf{H}^1 \times \mathbf{H}^1$ of two hyperbolic curves in $\mathbf{R}^4_2 \subset \mathbf{R}^{2+p}_p$.*

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2. Preliminaries

In this section we shall recall the theory of semi-Riemannian submanifolds in a semi-Riemannian manifold. Throughout this paper, we assume that all geometric objects are smooth and all manifolds are connected and without boundary, unless otherwise stated. Let (M', g') be an n-dimensional semi-Riemannian manifold of index q. For the semi-Riemannian manifold M' we can choose a local field $\{E_A\}$ of orthonormal frames adapted to the semi-Riemannian metric g' on M'. With respect to the frame field E_A, there exist 1-forms ω_A and ω_{AB} on M, which are usually called canonical forms and connection forms on M' respectively.

An n-dimensional semi-Riemannian manifold is called a *semi-space form of constant curvature* c *and with index* q if M' is of constant curvature c and of index q. We denote by $\mathbf{M}_q^n(c)$ an n-dimensional semi-space form of constant curvature c and with index q.

Now we give here standard models of complete semi-space forms. For an n-dimensional Euclidean space $\mathbf{R}^n$ with standard coordinate system $\{x_A\}$ a *semi- Euclidean space* $\mathbf{R}_q^n$ of index q is a semi-Riemannian manifold whose line element is given by $ds^2 = -\sum_{i=1}^q (dx_i)^2 + \sum_{j=q+1}^n (dx_j)^2$. For $q = 0$ the semi-Eucliedan space $\mathbf{R}_0^n$ is Euclidean. For $n \geq 2$, $\mathbf{R}_1^n$ is called an *n-dimensional Minkowski space.* The metric tensor g' of $\mathbf{R}_q^n$ can be written as

$$g' = \sum_A \epsilon_A dx_A \otimes dx_A,$$

where $\epsilon_A = -1$ for $1 \leq A \leq q$ and $\epsilon_A = 1$ for $q + 1 \leq A \leq n$.

For any $n(\geq 3)$ and q $(0 \leq q \leq n)$ a pseudosphere $\mathbf{S}_q^{n-1}(c)$ of constant curvature $c > 0$ is the hypersurface in an n-dimensional semi-Euclidean space $\mathbf{R}_q^n$ defined by

$$\mathbf{S}_q^{n-1}(c) = \left\{(x_A) \in \mathbf{R}_q^n \middle| -\sum_{i=1}^q (x_i)^2 + \sum_{j=q+1}^n (x_j)^2 = \frac{1}{c} = r^2, r > 0\right\}.$$

In particular, $\mathbf{S}_1^{n-1}(c)$ is called a *de Sitter space* of constant curvature c.

A pseudohyperbolic space $\mathbf{H}_{q-1}^{n-1}(c)$ of constant curvature $c < 0$ is the hypersurface in an n-diemnsional semi-Euclidean space $\mathbf{R}_q^n$ defined by

$$\mathbf{H}_{q-1}^{n-1}(c) = \{(x_A)\in\mathbf{R}_q^n | -\sum_{i=1}^{q}(x_i)^2+\sum_{j=q+1}^{n}(x_j)^2 = \frac{1}{c} = -r^2, r > 0\}.$$

In particular, $\mathbf{H}_1^{n-1}(c)$ is called an *anti-de Sitter space* of constant curvature c.

Now let $M' = \mathbf{M}_q^{m+p}(c)$ be an $(m+p)$-dimensional semi-space form of constant curvature c and of index r. The canonical forms $\{w_A\}$ and the connection forms $\{\omega_{AB}\}$ restricted to M are also denoted by the same symbols. We then have

$$\omega_\alpha = 0 \quad for \quad \alpha = m+1, \cdots, m+p \tag{2.1}$$

and the induced semi-Riemannian metric g of M is given by $g = \sum_j \epsilon_j \omega_j \otimes \omega_j$. Here and in the sequel the following convention on the range of indices is used, unless otherwise stated:

$$1 \le i, j, \cdots \le m, \ m+1 \le \alpha, \beta, \cdots \le m+p, \quad 1 \le A, B, \cdots \le m+p.$$

Thus $\{E_j\}$ is a local field of orthonormal frames with respect to this metric and $\{\omega_j\}$ is a local dual frame field due to $\{E_j\}$. They are canonical forms on M. It follows from (2.1) and the Cartan lemma that the exterior derivative of (2.1) gives rise to

$$\omega_{\alpha i} = \sum_j \epsilon_j h_{ij}^\alpha \omega_j, \quad h_{ij}^\alpha = h_{ji}^\alpha. \tag{2.2}$$

The *second fumdamental form* α on M is defined by

$$\alpha = \sum_{\alpha,i,j} \epsilon_\alpha \epsilon_i \epsilon_j h_{ij}^\alpha \omega_i \otimes \omega_j \otimes E_\alpha.$$

The *mean curvature vector field* $\mathbf{h}$ of M is defined by

$$\mathbf{h} = \frac{1}{m} \sum_{\alpha,j} \epsilon_\alpha \epsilon_j h_{jj}^\alpha E_\alpha$$

and the *mean curvature* H is defined by $H = |\mathbf{h}|$, if the codimension is equal to or greater than 2. So it satisfies

$$(2.3) \qquad H^2 = |<\mathbf{h},\mathbf{h}>| = \frac{1}{m^2}|\sum_\alpha \epsilon_\alpha(\sum_i \epsilon_i h_{ii}^\alpha)^2|.$$

From the structure equations of the ambient space the connection forms $\{\omega_{ij}\}$ of M are characterized by the structural equations:

$$(2.4) \qquad \begin{cases} d\omega_i + \sum_k \epsilon_k \omega_{ik} \wedge \omega_k = 0, & \omega_{ij} + \omega_{ji} = 0, \\ d\omega_{ij} + \sum_k \epsilon_k \omega_{ik} \wedge \omega_{kj} = \Omega_{ij}, & \\ \Omega_{ij} = -\frac{1}{2}\sum_{k,l} \epsilon_k \epsilon_l R_{ijkl} \omega_k \wedge \omega_l, & \end{cases}$$

where $\omega = (\omega_{ij})$ (resp. R_{ijkl}) denotes the curvature form (resp. the components of the Riemannian curvature tensor R) of M. For the Riemannian curvature tensor R of M it follows from (2.2) and (2.4) that we have the Gauss equation

$$(2.5) \qquad R_{ijkl} = c\epsilon_i\epsilon_j(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}) + \sum_\alpha \epsilon_\alpha(h_{il}^\alpha h_{jk}^\alpha - h_{ik}^\alpha h_{ik}^\alpha),$$

Moreover we also have the following relationships:

$$(2.6) \qquad \begin{aligned} d\omega_{\alpha\beta} + \sum_\gamma \epsilon_\gamma \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta} &= \Omega_{\alpha\beta}, \\ \Omega_{\alpha\beta} &= -\frac{1}{2}\sum_{k,l} \epsilon_k \epsilon_l R_{\alpha\beta kl} \omega_k \wedge \omega_l, \end{aligned}$$

where $\Omega_{\alpha\beta}$ (resp. $R_{\alpha\beta kl}$) is called the *normal curvature form* (resp. the components of the normal curvature tensor $R^\perp$) of M. By means of (2.2) and (2.6) we have

$$(2.7) \qquad R_{\alpha\beta kl} = \sum_j \epsilon_j(h_{jl}^\alpha h_{jk}^\beta - h_{jk}^\alpha h_{jl}^\beta).$$

Let ∇, ∇' and $\nabla^\perp$ be the Levi-Civita connections on the semi-Riemannian submanifold M, $\mathbf{R}_q^{m+p}$ and the normal connection of the normal bundle $T^\perp M$ on M, respectively. Then we get

$$\begin{aligned} &\nabla_X E_i = \sum_j \epsilon_j \omega_{ji}(X) E_j, \quad \nabla'_X E_i = \sum_A \epsilon_A \omega_{Ai}(X) E_A, \\ &\nabla_X^\perp E_\alpha = \sum_\beta \epsilon_\beta \omega_{\beta\alpha}(X) E_\beta \end{aligned}$$

for any vector field X tangent to M.

Now, the components h^{α}_{ijk} of the covariant derivative $\nabla\alpha$ of the second fundamental form α of M are given by

$$\sum_k \epsilon_k h^{\alpha}_{ijk}\omega_k = dh^{\alpha}_{ij} - \sum_k \epsilon_k(h^{\alpha}_{kj}\omega_{ki} + h^{\alpha}_{ik}\omega_{kj}) - \sum_\beta \epsilon_\beta h^{\beta}_{ij}\omega_{\beta\alpha}$$

and the components h^{α}_{ijkl} of the covariant derivative $\nabla^2\alpha$ of $\nabla\alpha$ are given by

$$\sum_l \epsilon_l h^{\alpha}_{ijkl}\omega_l = dh^{\alpha}_{ijk} - \sum_l \epsilon_l(h^{\alpha}_{ljk}\omega_{li} + h^{\alpha}_{ilk}\omega_{lj} + h^{\alpha}_{ijl}\omega_{lk}) - \sum_\beta \epsilon_\beta h^{\beta}_{ijk}\omega_{\beta\alpha}$$

The Codazzi equation and the Ricci formula for the second fundamental form are given by

$$\begin{aligned} (2.8) \qquad h^{\alpha}_{ijk} - h^{\alpha}_{ikj} &= 0, \\ h^{\alpha}_{ijkl} - h^{\alpha}_{ijlk} &= -\sum_n \epsilon_n(h^{\alpha}_{nj}R_{nikl} + h^{\alpha}_{in}R_{njkl}) \\ &\quad - \sum_\beta \epsilon_\beta h^{\beta}_{ij}R_{\beta\alpha kl}. \end{aligned}$$

On a semi-Riemannian manifold there are natural generalizations of the well known differential operators of vector calculus on $\mathbf{R}^3$: *gradient, divergence* and *Laplacian.* Let M be an m-dimensional semi-Riemannian manifold with local coordinate system $\{x_i\}$. For the components g_{ij} of the semi-Riemannian metric g on M we denote by (g^{ij}) the inverse matrix of the matrix (g_{ij}). Then the direct calculation gives us to the local representation of the Laplacian of a function f on M as follows;

$$\begin{aligned} (2.9) \qquad \triangle f &= -\frac{1}{\sqrt{|\mathfrak{g}|}}\sum_{i,j}\frac{\partial}{\partial x_i}(\sqrt{|\mathfrak{g}|}g^{ij}\frac{\partial}{\partial x_j}) \\ &= -\sum_i \epsilon_i\{E_iE_if - (\nabla_{E_i}E_i)f\}, \end{aligned}$$

where $\mathfrak{g}$ denotes the determinant of the matrix (g_{ij}). In particular, f is said to be of 1-type if it satisfies the above formula $\triangle f = \lambda f$ for a constant λ. For the submanifold of finite type we refer to Chen [5] and his survey [6].

Now, let M be an m-dimensional semi-Riemannian submanifold of an $(m+p)$- dimensional semi-space form $\mathrm{M}_q^{m+p}(c)$ of index q. Then the Laplacian $\triangle h_{ij}^\alpha$ of the components h_{ij}^α of the second fundamental form α is given by

$$\triangle h_{ij}^\alpha = \sum_k \epsilon_k h_{ijkk}^\alpha.$$

From (2.8) we get

(2.10)

$$\begin{aligned}&\triangle h_{ij}^\alpha \\ &= \sum_k \epsilon_k \{ h_{kkij}^\alpha - \sum_n \epsilon_n (h_{kn}^\alpha R_{nijk} + h_{ni}^\alpha R_{nkjk}) \\ &\quad - \sum_\beta \epsilon_\beta h_{ki}^\beta R_{\beta\alpha jk} \}\end{aligned}$$

For an endomorphism F on the tangent bundle TM of M we denote by TrF the trace of F. That is, it is defined by

$$TrF = \sum_{i=1}^m \epsilon_i g(FE_i, E_i).$$

Let H^α be an $m \times m$ symmetric matrix (h_{ij}^α) for any index α. We put $S_{\alpha\beta} = Tr(H^\alpha H^\beta)$ and $S_\alpha = S_{\alpha\alpha}$. Then $(S_{\alpha\beta})$ is also an $m \times m$ symmetric matrix. In general, for a matrix A we define $N(A) = Tr(A^t A)$. By using the Gauss and Codazzi equations, (2.7), (2.10) and using these terminology, we get the following Simons type formula:

(2.11)

$$\begin{aligned}\frac{1}{2}\triangle < \alpha, \alpha > = &< \nabla\alpha, \nabla\alpha > + mc < \alpha, \alpha > \\ &- \sum_{\alpha\beta} \epsilon_\alpha \epsilon_\beta N(H^\alpha H^\beta - H^\beta H^\alpha) - \sum_\alpha S_\alpha^2 \\ &+ \sum_{\alpha\beta} \epsilon_\alpha \epsilon_\beta Tr H^\alpha Tr(H^\beta H^\alpha H^\beta),\end{aligned}$$

where $< \alpha, \alpha >$ denotes an inner product of the second fundamental form α given by $\sum_{i,j,\alpha} \epsilon_i \epsilon_j \epsilon_\alpha {h_{ij}^\alpha}^2$. This is obtained by Cheng and the first author (See [9] and [10]).

3. 1-type generalized Gauss map

This section is devoted to investigating the generalized Gauss map of space-like submanifold in an indefinite Eucliden space, which is closely related to the Gauss map of space-like hypersurfaces in a Minkowski space.

Let $\mathbf{R}_p^{m+p}$ be an $(m+p)$-dimensional indefinite Eucliden space with standard coordinate system $\{x_A\}$ whose line element is given by $ds^2 = \sum_{i=1}^{m}(dx_i)^2 - \sum_{\alpha=m+1}^{m+p}(dx_\alpha)^2$. Let $G(m,p)$ be the set of all m-dimensional positive definite subspace V through the origin in $\mathbf{R}_p^{m+p}$.

First we introduce the Riemannian manifold structure to $G(m,p)$. The semi-orthogonal group $O(m,p)$ consisting of all isometries $\mathbf{R}_p^{m+1} \to \mathbf{R}_p^{m+1}$ is given by

$$O(m,p) = \left\{(a_{AB}) \in GL(m+p,R) : \sum \varepsilon_c a_{AC} a_{CB} = \varepsilon_A \delta_{AB}\right\}$$

and its connected components $O^{++}(m,p)$ is the subgroup of $O(m,p)$ whose elements preserve space-like and time-like orientations:

$$O^{++}(m,p) = \left\{(a_{AB}) \in O(m,p) : det(a_{ij}) > 0, det(a_{\alpha\beta}) > 0\right\}.$$

The group $O(m,p)$(respectively $O^{++}(m,p)$) acts transtively on $G(m,p)$. If V_0 is a subspace of $\mathbf{R}_p^{m+p}$ spanned by the first m vectors in the canonical basis, then $V_0 \in G(m,p)$ and the isotropy group of V_0 is $O(m) \times O(p)$ (resp. $SO(m) \times SO(p)$) since both V_0 and the perpendicular subspace $V_0^\perp$ are definite. Then we get the manifold structure on $G(m,p)$ as a quotient space:

$$G(m,p) = O(m,p)/O(m) \times O(p) = O^{++}(m,p)/SO(m) \times SO(p).$$

The manifold $G(m,p)$ is called a Grassmann manifold. For the Grassmann manifold the following property is known by O'Neill [15]:

THEOREM 3.1. *The Grassmann manifold $G(m,p)$ consisting of all space-like subspaces in $\mathbf{R}_p^{m+p}$ is a Riemannian symmetric space of the non-compact type.*

In fact, $G(m,p)$ is a Hadamard manifold, that is, it is a complete, simply connected Riemannian manifold of non-positive curvature. In particular, in the case where $p = 1, G(m,1)$ is an m-dimensional hyperbolic space as the set of time-like lines through the origin in $\mathbf{R}_1^{m+1}$. Now, let V be an m-dimensional oriented space-like subspaces in $\mathbf{R}_p^{m+p}$. We denote by $e_1, ..., e_m$ an orthonomal basis for V. Then $e_1\wedge\cdots\wedge e_m$ is a m-vector with norm 1 and gives the orientation on V.

Conversely, for any m-vector of norm 1, it determnes a unique m-dimensional oriented space-like subspace in $\mathbf{R}_p^{m+p}$. Consequently, an element in $G(m,p)$ can be identified naturally with the m-vectors of norm 1 in the N-dimensional Euclidean space $\Lambda^m\mathbf{R}^{m+p} = \mathbf{R}^N$, $N = \binom{m+p}{m}$. Let $\mathbf{S}^{N-1}$ be the unit sphere in $\mathbf{R}^N$ centered at origin. Then $G(m,p)$ is an mp-dimensional submanifold which is isometrically immersed in $\mathbf{S}^{N-1}\subset\mathbf{R}^N$.

Let $x : M\to\mathbf{R}_p^{m+p}$ be an isometric immersion of an m-dimensional Riemannian manifold into $\mathbf{R}_p^{m+p}$. For each vector u tangent to M, we identify u with its image under the differential dx of the isometric immersion x. Then the generalized Gauss map G of M is by definition a map which assigns each point z in M to the space-like m-plane through the origin in $\mathbf{R}_p^{m+p}$ obtained by translating parallelly the tangent space at z of M in $\mathbf{R}_p^{m+p}$. Hence we see that

$$G : M\to G(m,p)\subset\mathbf{S}^{N-1}\subset\mathbf{R}_p^N.$$

Let $E_1, \cdots, E_m$ be a local field of orthonormal frames on M. So the Gauss map G is given by $G(z) = (E_1\wedge\cdots\wedge E_m)(z)$.

Firstly, we obtain the following fundamental formula for the generalized Gauss map on M:

LEMMA 3.2. *Let $x : M\to\mathbf{R}_p^{m+p}$ be an isometric immersion of an oriented m-dimensional Riemannian manifold M into $\mathbf{R}_p^{m+p}$. Then the Laplacian of the generalized Gauss map $G : M\to G(m,p)\subset\mathbf{R}_p^N$, $N = \binom{m+p}{m}$ is given by*

(3.1)

$$\begin{aligned}\triangle G =&\sum_{\alpha,i,j} E_1\wedge\cdots\wedge \overset{j-th}{h^{\alpha}_{iij}E_\alpha}\wedge\cdots\wedge E_m\\ &+\sum_{\alpha<\beta}\sum_{j,k} R_{\alpha\beta jk}E_1\wedge\cdots\wedge \overset{k-th}{E_\beta}\wedge\cdots\wedge \overset{j-th}{E_\alpha}\wedge\cdots\wedge E_m\\ &- SG,\end{aligned}$$

where S denotes the square norm of the second fundamental form α given by $S = \sum_{i,j,\alpha} {h_{ij}^{\alpha}}^2$.

Proof. Let ∇' and $\nabla^{\perp}$ be the Levi-Civita connections on $\mathbf{R}_p^{m+p}$ and the normal connection on M respectively. Then we get

$$\nabla'_X E_i = \sum_A \epsilon_A \omega_{Ai}(X) E_A$$

and

$$\nabla^{\perp}_X E_\alpha = \sum_\beta \epsilon_\beta \omega_{\beta\alpha}(X) E_\beta$$

for any vector field X tangent to M. Since we can regard the generalized Gauss map G of M into $G(m,p)$ as in $\mathbf{R}^N$-valued function on M, we have

$$E_i G = \sum_j E_1 \wedge \cdots \wedge \nabla'_{E_i} E_j \wedge \cdots \wedge E_m.$$

Then (2.2) and the Gauss formula imply

$$E_i G = -\sum_{j,\alpha} h_{ij}^{\alpha} E_1 \wedge \cdots \wedge \overset{j-th}{E_\alpha} \wedge \cdots \cdots \wedge E_m$$

Let h_{ijk}^{α} be the components of the covariant derivative $\nabla\alpha$ of the second fundamental form α of M. Since the Laplacian of G is given by

$$\triangle G = -\sum_i \{E_i E_i G - (\nabla_{E_i} E_i) G\}.$$

By (2.9), we have

$$\begin{aligned}
\triangle G =& \sum_{\alpha,i,j} \{dh_{ij}^{\alpha} - \sum_\beta h_{ij}^{\beta} \omega_{\alpha\beta} + \sum_k (h_{kj}^{\alpha} \omega_{ik} + h_{ik}^{\alpha} \omega_{jk})\}(E_i) \\
& E_1 \wedge \cdots \wedge E_\alpha \wedge \cdots E_m \\
& - \sum_{\alpha,\beta,i,j,k} h_{ij}^{\alpha} h_{ik}^{\beta} E_1 \wedge \cdots \wedge \overset{k-th}{E_\beta} \wedge \cdots \wedge \overset{j-th}{E_\alpha} \wedge \cdots \wedge E_m \\
& - S E_1 \wedge \cdots \wedge E_m \\
=& \sum_{\alpha,i,j} h_{iji}^{\alpha} E_1 \wedge \cdots \wedge \overset{j-th}{E_\alpha} \wedge \cdots \wedge E_m \\
& - \sum_{\alpha,\beta,i,j,k} h_{ij}^{\alpha} h_{ik}^{\beta} E_1 \wedge \cdots \wedge \overset{k-th}{E_\beta} \wedge \cdots \wedge \overset{j-th}{E_\alpha} \wedge \cdots \wedge E_m - SG
\end{aligned}$$

where we have used the definitions of the connections and the covariant derivative of the second fundamental form α. It completes the proof.□

Let x_0 be the isometric immersion of the Grassmann manifold $G(m,p)$ into $\mathbf{R}^N$. For the generalized Gauss map G a map $G_0 : M \rightarrow \mathbf{R}^N$ is defined by $G_0 := x_0 \circ G$. This is also called a *generalized Gauss map.*

THEOREM 3.3. *Let M be a space-like submanifold in $\mathbf{R}^{m+p}_p$. Then the generalized Gauss map G_0 is of 1-type if and only if the mean curvature vector field of M is parallel in the normal bundle, the normal curvature is flat and the scalar curvature is constant.*

Proof. The generalized Gauss map G_0 is of 1-type if it satisfies

$$\triangle G_0 = \lambda G_0$$

for a constant λ. By Theorem 3.1 we can regard $G(m,p)$ as an mp-dimensional complete simply connected Riemannian manifold of non-positive curvature. We consider a curve γ through the point $E_1 \wedge \cdots \wedge E_m$ in $G(m,p)$ defined by

$$\gamma(s) = E_1 \wedge \cdots \wedge (\cosh sE_j + \sinh sE_\alpha) \wedge \cdots \wedge E_m.$$

Then the tangent vector $\gamma'(0)$ of the curve γ at $s = 0$ is given as

$$E_1 \wedge \cdots \wedge \overset{j-th}{E_\alpha} \wedge \cdots \wedge E_m.$$

So we see that

$$\{E_1 \wedge \cdots \wedge \overset{j-th}{E_\alpha} \wedge \cdots \wedge E_m : j, \alpha\}$$

is an orthonormal basis for the tangent space at the point. When we regard $G(m,p)$ as an mp-dimensional Riemannian submanifold in the unit sphere $\mathbf{S}^{N-1}$ and the position vector G is normal to $\mathbf{S}^{N-1}$ in $\mathbf{R}^N$,

$$E_1 \wedge \cdots \wedge \overset{k-th}{E_\beta} \wedge \cdots \wedge \overset{j-th}{E_\alpha} \wedge \cdots \wedge E_m$$

is normal to $G(m,p)$ in $\mathbf{S}^{N-1}$. Thus we have

$$\sum h^\alpha_{iij} = 0, \ R_{\alpha\beta jk} = 0, \ S = -\lambda,$$

which means that the mean curvature vector field is parallel and the normal connection is flat, and moreover the squared norm S of the second fundamental tensor in time-like normal space is constant. The last statement is equivalent to the fact that the scalar curvature is constant. It completes the proof of our theorem. □

4. Proof of Main Theorem

In this section let us prove our theorem, which is an indefinite version of Chen and Piccini's result [7]. Let M be a space-like surface in $\mathbf{R}_p^{2+p}$. Then by Theorem 3.3 and a theorem due to Aiyama (Theorem 10 in [3]), we have

PROPOSITION 4.1. *Let M be a space-like surface in $\mathbf{R}_p^{2+p}$. If the generalized Gauss map $G_0 : M \to \mathbf{R}^N, N = (p+1)(p+2)/2$, is of 1-type, then M is locally one of the following surfaces:*

(1) *maximal space-like surfaces of $\mathbf{R}_p^{2+p}$,*

(2) *maximal space-like surfaces of a totally umbilical hypersurface $\bar{M} = \mathbf{M}_{p-1}^{p+1}(c)$ in $\mathbf{R}_p^{2+p}$,*

(3) *space-like surfaces with constant mean curvature of a totally umbilic 3-dimensional submanifold $M' = \mathbf{M}_1^3(c)$ in $\mathbf{R}_p^{2+p}$.*

Now we are in a position to prove the theorem in the introduction.

THEOREM 4.2. *Let M be a space-like surface in $\mathbf{R}_p^{2+p}$. Then the only surfaces with 1-type generalized Gauss map $G_0 : M \to \mathbf{R}^N$, $N = (p+1)(p+2)/2$, are locally one of the following spaces:*

(1) *a Euclidean space $\mathbf{R}^2$, a hyperbolic space $\mathbf{H}^2$ or a hyperbolic cylinder $\mathbf{H}^1 \times \mathbf{R}$ in $\mathbf{R}_1^3$,*

(2) *the product $\mathbf{H}^1 \times \mathbf{H}^1$ of two hyperbolic curves in $\mathbf{R}_2^4 \subset \mathbf{R}_p^{2+p}$.*

Proof. By Theorem 3.3 the space-like surface M in $\mathbf{R}_p^{2+p}$ has 1-type generalized Gauss map if and only if the mean curvature vector field $\mathbf{h}$ of M is parallel in the normal bundle, the normal curvature $R^\perp$ is flat and the scalar curvature is constant. Furthermore, because of $m = 2$, the parallelism of $\mathbf{h}$ is equivalent to the fact that the surface M has the three situations in Proposition 4.1.

Now we consider the first case (1) in Proposition 4.1. Since it is maximal in $\mathbf{R}_p^{2+p}$, the normal curvature $R^\perp$ of M is flat, that is, the formula (2.7) vanishes. So the Simons type formula (2.11) in this situation satisfies

$$\frac{1}{2}\triangle S = |\nabla\alpha|^2 + \sum\nolimits_\alpha (S_\alpha)^2$$

where

$$S = \sum_{i,j,\alpha} (h_{ij}^\alpha)^2 = -<\alpha, \alpha>, \ |\nabla\alpha|^2 = \sum_{i,j,k,\alpha} (h_{ijk}^\alpha)^2 = -<\nabla\alpha, \nabla\alpha>,$$

and $<,>$ denotes the inner product in time-like normal space. This yields $S_\alpha = 0$ for any index α, because the squared norm S of the second fundamental form is constant. Hence the surface M is totally geodesic and there is a 2-dimensional Euclidean space $\mathbf{R}^2$ in $\mathbf{R}_p^{2+p}$ in which M is contained.

Next we investigate the second case (2). Then the surface M is maximal space-like surface of a totally umbilical hypersurface $\bar{M} = \mathbf{M}_{p-1}^{p+1}(c)$ in $\mathbf{R}_p^{2+p}$. Let λ be a principal curvature of a totally umbilical hypersurface $\bar{M} = \mathbf{M}_{p-1}^{p+1}(c)$ in $\mathbf{R}_p^{2+p}$. Then the Gauss equation (2.5) implies $c = -\lambda^2 \leq 0$.

In the case of $c = 0$, we can apply the previous result in case (1) and we see that there is a 2-dimensional Euclidean space $\mathbf{R}^2$ in $\mathbf{R}_{p-1}^{p+1}$ in which M is contained. Suppose that $c < 0$. We choose a field of orthonormal frames $\{E_i, E_a, E_{2+p}\}$, $3 \leq a \leq 1 + p$, adapted to the Riemannian metric on M such that E_1, E_2 are tangent to M and hence $\{E_3, ...E_{1+p}\}$ are normal to M in $\bar{M}$ and E_{2+p} is normal to $\bar{M}$ in $\mathbf{R}_p^{2+p}$. Then $\{E_i, E_a\}$ is a local field of orthonormal frames of $\bar{M}$. When M can be regarded as the surface of $\bar{M}$, we have $\omega_\alpha = 0$ on M, which implies

$$\omega_{\alpha i} = \sum\nolimits_j h_{ij}^\alpha \omega_j$$

and hence the mean curvature vector field $\mathbf{h}_1$ of M in $\bar{M} = \mathbf{M}_{p-1}^{p+1}(c)$ is given by $\mathbf{h}_1 = -\sum_{a,i} h_{ii}^a E_a/2$.

Since the mean curvature vector field $\mathbf{h}$ of M in $\mathbf{R}_p^{2+p}$ is parallel in the normal bundle of M, we see $\sum_i h_{iij}^\alpha = 0$ for any indeces j and α. That is, we get $\sum_i h_{iij}^a = 0$ for any indices j and a. This means that the mean curvature vector field $\mathbf{h}_1$ of M in $\bar{M}$ is parallel in the normal bundle $T^\perp M$.

Let $\mathbf{h}_1$ (resp. $\mathbf{h}_2$) be the mean curvature vector of M in $\bar{M} = \mathbf{M}_{p-1}^{p+1}(c)$ (resp. $\bar{M}$ in $\mathbf{R}_p^{2+p}$). Since $\bar{M}$ is totally umbilic in $\mathbf{R}_p^{2+p}$, we have

$$\begin{aligned} \nabla'_X \mathbf{h}_1 &= \bar{\nabla}_X \mathbf{h}_1 = -A_{1\mathbf{h}_1} X + \nabla^\perp_{1X} \mathbf{h}_1 \\ \nabla'_X \mathbf{h}_2 &= \lambda H_2 X \end{aligned}$$

where $\bar{\nabla}$ and ∇' denote the Levi-Civita connections on semi-Riemannian space form $\bar{M}$ and $\mathbf{R}_p^{2+p}$ respectively, $\nabla_1^\perp$ and A_1 denote the normal connection and the shape operator of M into $\bar{M}$ and H_2 the mean curvature of $\bar{M}$ in $\mathbf{R}_p^{2+p}$. This gives us to

$$\nabla_X^{\perp}\mathbf{h} = \nabla_{1X}^{\perp}\mathbf{h}_1,$$

which means that $\mathbf{h}_1$ is parallel in the normal bundle of M in $\bar{M}$, because so is $\mathbf{h}$.

On the other hand, since the normal connection is flat, the normal curvature tensor satisfies $R^{\perp} = 0$. So it satisfies $\Omega_{\alpha\beta} = 0$. Suppose that the hypersurface $\bar{M}$ is defined by $w_{2+p} = 0$. For any indices $\alpha, \beta \leq 1+p$ the structure equations for $\bar{M}$ satisfy

$$d\omega_{\alpha\beta} + \sum_{\gamma\leq 1+p} \epsilon_\gamma \omega_{\alpha\gamma}\wedge\omega_{\gamma\beta} + \epsilon_{2+p}\omega_{\alpha 2+p}\wedge\omega_{2+p\beta} = 0,$$

where we have used (2.6). Again, since $\bar{M}$ is a hypersurface, by the above equation we get

$$d\omega_{\alpha\beta} + \sum_{\gamma\leq 1+p} \epsilon_\gamma \omega_{\alpha\gamma}\wedge\omega_{\gamma\beta} = 0,$$

which means that the normal curvature tensor of M in $\bar{M}$ is zero. Consequently, for the maximal space-like surface M in $\bar{M}$, it has parallel mean curvature vector field $\mathbf{h}_1$ and the flat normal curvature $R_1^{\perp}$ of M in $\bar{M}$. By the argument of Aiyama [3] Chen's result (Lemma 2.5 in [5, p.108], Remark 2.1 in [5, p.114]) holds in this situation and there is a 3-dimensional anti-de Sitter space $\mathbf{H}_1^3(c)$ which is totally umbilic in $\mathbf{R}_2^4$ such that $M \subset \mathbf{H}_1^3(c)$. Accordingly, we can apply the above discussion to $M \to \mathbf{H}_1^3(c) \to \mathbf{R}_2^4$ and we see that the mean curvature vector field of M in $\mathbf{H}_1^3(c)$ is parallel, which is equivalent to the fact that the mean curvature of M in $\mathbf{H}_1^3(c)$ is constant on M. Since the scalar curvature of M is constant, so are the principal curvatures. By a congruent theorem due to Abe, Koike and Yamaguchi [1], it is contained in the 2-dimensional hyperbolic space $\mathbf{H}^2$ or the product space of the hyperbolic curves $\mathbf{H}^1 \times \mathbf{H}^1$ in $\mathbf{R}_2^4$.

Finally, we consider the case (3) where M is a space-like surface with constant mean curvature of a totally umbilic submanifold $\mathbf{H}_1^3(c)$ in $\mathbf{R}_p^{2+p}$. Since the scalar curvature of M is constant, two principal curvatures of M are both constant on M. Again, by a theorem due to Abe,Koike and Yamaguchi [1] M is contained in the 2-dimensional Euclidean space $\mathbf{R}^2, \mathbf{H}^2$, the hyperbolic cylinder $\mathbf{R}\times\mathbf{H}^1$ in $\mathbf{R}_1^3$ and the product $\mathbf{H}^1\times\mathbf{H}^1$ in $\mathbf{R}_2^4$.

Conversely, it is easy to see that these surfaces satisfy the three conditions that the mean curvature vector field is parallel in the normal bundle, the normal connection is flat and the scalar curvature is constant. It shows that the map G is of 1-type. This completes the proof of our Theorem. $\square$

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Soon Meen Choi, U-Hang Ki
Topology and Geometry Research Center
Kyungpook National University
Taegu 702-701, Korea

Young Jin Suh
Department of Mathematics
Kyungpook National University
Taegu 702-701, Korea
E-mail: yjsuh@bh.kyungpook.ac.kr