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NORM ESTIMATES ON HARDY SPACES AND MULTIPLE SINGULAR INTEGRALS

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Abstract. In this article we examine certain distinctive features regarding Hardy spaces of both classical and product notions on $\mathbb{R}^N$ with our focus on their interrelations through embeddings and restrictions. Applications of our results to multiple singular integrals are included.

In the present article we primarily aim to study some of the leading roles played by Hardy spaces and a few interrelations among them. The first part, rather expository, deals with three examples showing that the theory of product Hardy spaces could refine certain established results in harmonic analysis. The second part sets forth the properties of inclusion and restriction for Hardy spaces with some applications to singular integrals.

Part I. MULTIDIMENSIONAL FOURIER ANALYSIS

1. Hardy Spaces: Classical and Product Notions

To begin with, we describe the notions of both classical and product Hardy spaces on $\mathbb{R}^N$ in the following unified approach (cf. [10] and [13]).

For some k, $1 \le k \le N$, form a partition $N = n_1 + \cdots + n_k$ and write

$$x = (x_1, \cdots, x_k) \in \mathbb{R}^N = \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k}, \quad x_j \in \mathbb{R}^{n_j}.$$

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Take any Schwartz testing function Φ on $\mathbb{R}^N$ with $\int \Phi\, dx \neq 0$ and for each k-parameters $s_1 > 0, \cdots, s_k > 0$, put

$$\Phi_{s_1,\cdots,s_k}(x) = \left(s_1^{-n_1} \cdots s_k^{-n_k}\right) \Phi\left(x_1/s_1, \cdots, x_k/s_k\right) .$$

For each $p > 0$, the Hardy space associated with this partition, denoted usually by $H^p\left(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k}\right)$, is the collection of all tempered distributions u on $\mathbb{R}^N$ such that

(1-1)
$$\|u\|_{H^p(\mathbb{R}^{n_1}\times\cdots\times\mathbb{R}^{n_k})} = \sup_{s_1,\cdots,s_k>0} \left|\left(u * \Phi_{s_1,\cdots,s_k}\right)(x)\right| \in L^p\left(\mathbb{R}^N\right) ,$$

where $*$ stands for the convolution in the distributional sense.

When $k = 1$, it reduces to the classical Hardy space of C. Fefferman and E. M. Stein [7], denoted by $H^p(\mathbb{R}^N)$. All these spaces provide inherent substitutes for $L^1(\mathbb{R}^N)$. In particular, concerning interpolations, it is known that if T is an L^2 bounded linear operator mapping $H^1(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k})$ continuously into $L^1(\mathbb{R}^N)$, then T extends to a continuous mapping from $L^p(\mathbb{R}^N)$ into itself for every p, $1 < p < 2$ (refer to A. Chang and R. Fefferman [3]).

In the next three sections, we indicate how these spaces arise naturally in Fourier analysis.

2. The Maximal Theorem of Rubio de Francia: A Sharper Version

In dealing with pointwise convergence questions of type $\lim_{t\to 0} T_t\, f(x)$ where

$$T_t\, f(x) = \int_{\mathbb{R}^N} t^{-N} K\left(\frac{x-y}{t}\right) f(y)\, dy , \tag{2-1}$$

we are led directly to the continuity question of the associated maximal operator. Making use of the Fourier transform if available, the problem is converted into the maximal multiplier theory for

$$T^* f(x) = \sup_{t>0} \left|\left(T_t\, f(x)\right)\right| , \quad (T_t)^\frown(\zeta) = m(t\zeta)\, \hat{f}(\zeta) . \tag{2-2}$$

Concerning the L^p continuity properties of T^*, J. L. Rubio de Francia established in [21] the following theorem.

THEOREM (RUBIO DE FRANCIA). *Suppose that* $m \in C^{[N/2]+2}(\mathbb{R}^N)$ *and for some fixed number* $b > 1/2$, $|D^\alpha m(\zeta)| \leq C|\zeta|^{-b}$ *for all multi-indicies* α, $|\alpha| \leq [N/2]+2$. *Then* T^* *maps* $L^p(\mathbb{R}^N)$ *continuously into itself provided* $2N/(N+2b-1) < p < (2N-2)/(N-2b)$.

In this connection, consider the following problem :

Problem A : *In* $\mathbb{R}^9$, *assume that* m *only satisfies the condition*

$$|D^\alpha m(\zeta)| \leq |\zeta|^{-5}, \quad \textit{for all} \quad \alpha \quad \textit{with} \quad \alpha_j \leq 2.$$

What can be said about the L^p *continuities of* T^* ?

Unless the function m is a product of one dimensional functions, nothing may be deduced from Rubio de Francia's theorem due to the smoothness hypothesis in different directions. Nevertheless, if we exploit the theory of product Hardy spaces instead, then we are able to conclude that T^* maps $L^p(\mathbb{R}^N)$ boundedly into itself for $9/5 < p < \infty$. The key idea is to replace T^* by the multiparameter maximal operator S^* given by

$$S^* f(x) = \sup_{t_1,\cdots,t_N>0} |(T_{t_1,\cdots,t_N} * f)(x)|, \tag{2-4}$$

where $(T_{t_1,\cdots,t_N} * f)\hat{}(\zeta) = m(t_1\zeta_1,\cdots,t_N\zeta_N)\hat{f}(\zeta)$. Clearly, $T^* \leq S^*$ and keeping similar schemes as in the proof of Rubio de Francia's theorem but considering rather $(H^1(\mathbb{R}\times\cdots\times\mathbb{R}), L^1)$ estimates, our conclusion follows (see [4] for details). Moreover, this approach enables us to obtain an unexpected property regarding almost everywhere convergence : the limit

$$\lim_{t_1\to 0,\cdots,t_N\to 0} (T_{t_1,\cdots,t_N} * f(x))$$

exists almost everywhere if $f \in L^p$, $9/5 < p < \infty$.

At the current state of development, we can give a sharper form of his theorem as follows (see [4]).

THEOREM 1. *Suppose that for some* k, $1 \le k \le N$, *there exists a fixed number* $b > k/2$ *such that*

$$m \in C^{[n_1/2]+2,\cdots,[n_k/2]+2}(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k}), \quad |D^\alpha m(\zeta)| \le C|\zeta|^{-b},$$
$$\alpha = (\alpha_1, \cdots, \alpha_k),\ |\alpha_j| \le [n_j/2]+2,\ \alpha_j \in \mathbb{Z}_+^{n_j},\ j = 1, \cdots, k.$$

Then T^* *maps* $L^p(\mathbb{R}^N)$ *continuously into itself provided*

$$\max_{1\le j\le k} (2kn_j/(2b + kn_j - k)) < p < \min_{1\le j\le k} (2k(n_j - 1)/(kn_j - 2b)).$$

3. Hörmander's Multiplier Theorem Revisited

In the same spirit as in the previous discussions, we take into account the multiplier operator T_1 defined in (2-2). In [14], L. Hörmander set up the following theorem :

THEOREM (HÖRMANDER). *If* $m \in C^{[N/2]+1}(\mathbb{R}^N - 0)$ *is bounded and satisfies, away from the origin,*

$$|D^\beta m(\xi)| \le C_\beta |\xi|^{-|\beta|}, \quad \textit{for} \quad |\beta| \le [N/2]+1,$$

then T_1 *maps* $L^p(\mathbb{R}^N)$ *continuously into itself for* $1 < p < \infty$.

A typical case occurs when m corresponds to a bounded function homogeneous of degree 0, which is usual in the theory of elliptic partial differential equations. As before, it is possible to give a sharper version of the theorem with the aid of refined Hardy space methods.

Problem B : *What are the* L^p *boundedness properties of* T_1 *on* $\mathbb{R}^9$ *if*

$$m(\xi) = \frac{\xi_1}{(\xi_1^2 + \cdots + \xi_5^2)^{1/2}} \frac{\xi_7}{(\xi_6^2 + \cdots + \xi_9^2)^{1/2}}. \tag{3-1}$$

This simple example doesn't fall under the scope of the above theorem. To solve a problem of this kind, in [12], R. Fefferman and K. C. Lin made use of the theory of product Hardy spaces. To put briefly, let Δ_k denote the cross

$$\Delta_k = \{ \zeta = (\zeta_1, \cdots, \zeta_k) \in \mathbb{R}^N : \zeta_j = 0 \text{ for some } j, \ \zeta_j \in \mathbb{R}^{n_j} \}.$$

THEOREM 2. *Suppose that* $m \in C^{[n_1/2]+1, \cdots, [n_k/2]+1}(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k} - \Delta_k)$ *is bounded and, away from the cross* Δ_k, *satisfies*

$$|D^\beta m(\zeta)| \leq C_\beta |\zeta_1|^{-|\beta_1|} \cdots |\zeta_k|^{-|\beta_k|},$$
$$\beta = (\beta_1, \cdots, \beta_k), \quad |\beta_j| \leq [n_j/2] + 1, \ j = 1, \cdots, k.$$

It follows that $\|T_1 f\|_p \leq C_p \|f\|_p, \quad 1 < p < \infty$.

4. Singular Integrals

In this section we consider singular integrals of A. P. Calderón and A. Zygmund [1]. Those are, generally speaking, generalizations of the Hilbert transform on the real line. Classical extensions are the Riesz transforms.

As another extension, consider the multiple Hilbert transform associated with an index α given by

$$(4\text{-}1) \qquad \mathcal{H}_\alpha f(x) = \left(\frac{1}{x^\alpha} *_\alpha f \right)(x), \quad \alpha_j = 0 \text{ or } 1, \ j = 1, \cdots, N,$$

where $*_\alpha$ denotes the convolution in the variables $x_{\alpha_1}, \cdots, x_{\alpha_N}$, $\alpha_j = 1$ and the integral is interpreted in the principal value sense.

When $N = 1$, $\alpha = 1$, it reduces to the usual Hilbert transform and when $N = 2$, $\alpha = (1,1)$, it is referred to as the double Hilbert transform, a variant of which has been used by C. Fefferman [6] to control partial sums of the double Fourier series. When $\alpha = (1, 1, \cdots, 1)$, it has an interesting connection with the ordinary Riesz transforms $\mathcal{R}_j, j = 1, \cdots, N$:

$$(4\text{-}2) \qquad (-1)^{N+1} \mathcal{H}_\alpha^2 = \sum_{j=1}^{N} \mathcal{R}_j^2 = -I,$$

where I denotes the identity operator.

Problem C : *Describe the behavior of* $\mathcal{H}_\alpha$ *on* $L^1(\mathbb{R}^N)$.

In the case when $N = 1$, $\alpha = 1$, what is known are the weak type (1,1) inequality and the $(H^1(\mathbb{R}), L^1(\mathbb{R}))$ continuity. However, the Hardy space $H^1(\mathbb{R}^N)$ doesn't serve as a good $L^1(\mathbb{R}^N)$ substitute for the general $\mathcal{H}_\alpha$ any more. In order to see this, we consider the double Hilbert transform on $\mathbb{R}^2$, $\alpha = (1,1)$, and choose the function f defined by

$$f(x_1, x_2) = \left(\chi_{[0,1]} - \chi_{[-1,0]}\right)(x_1)\,\chi_{[-1,1]}(x_2)\,.$$

Plainly $f \in H^1\left(\mathbb{R}^2\right)$ because f is a compactly supported bounded function with $\int f\,dx_1\,dx_2 = 0$. A simple computation shows that

$$\mathcal{H}_\alpha(f)(x_1, x_2) = \frac{1}{\pi^2}\log\left(\frac{|x_1|^2}{|x_1^2 - 1|}\right)\log\left|\frac{x_2+1}{x_2-1}\right|\,, \quad |x_1| \neq 0, 1\,,\ |x_2| \neq 1\,.$$

For large $|x_1|$, $|x_2|$, Taylor's theorem implies that

$$\mathcal{H}_\alpha(f)(x_1, x_2) \approx \frac{C}{|x_1|^2\,|x_2|}\,,$$

which shows $\mathcal{H}_\alpha(f) \notin L^1\left(\mathbb{R}^2\right)$.

In summary, in higher dimensions, it is necessary to refine our considerations to get full comprehensions over multidimensional harmonic analysis and for this purpose the product Hardy spaces play vital roles.

Part II. NORM INEQUALITIES ON HARDY SPACES

1. Inclusion Relationships and Preliminaries

As to various Hardy spaces, it is interesting to find out their interrelations. To begin with, there are simple inclusion relationships.

THEOREM I. *For each* k, $1 \le k \le N$, *and* $p > 0$, *we have continuous inclusions*

$$H^p(\mathbb{R} \times \cdots \times \mathbb{R}) \hookrightarrow H^p(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k}) \hookrightarrow H^p(\mathbb{R}^N).$$

More precisely, we have $\|f\|_{H^p(\mathbb{R}^N)} \leq A_p \|f\|_{H^p(\mathbb{R}^{n_1}\times\cdots\times\mathbb{R}^{n_k})} \leq B_p \|f\|_{H^p(\mathbb{R}\times\cdots\times\mathbb{R})}$. *Furthermore, if the partition* $N = m_1 + \cdots + m_l$ *is a refinement of* $N = n_1 + \cdots + n_k$, *then*

$$H^p(\mathbb{R}^{m_1} \times \cdots \mathbb{R}^{m_l}) \hookrightarrow H^p(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k}).$$

It should be noted that the dimension N is fixed in the statement. Proofs are elementary so omitted. Although trivial, it shows that product Hardy spaces are much finer subspaces of the classical Hardy space and they have most of properties that the classical Hardy space possesses.

For any Schwartz functions ϕ, ψ in $\mathbb{R}^n$, $\mathbb{R}^m$ with nonvanishing integrals, respectively, and for each $u \in H^p(\mathbb{R}^n \times \mathbb{R}^m)$, write

$$u_x^+(x,y) = \sup_{t>0} |(\psi_t *_2 u)(x,y)|, \quad \psi_t(y) = t^{-m}\psi(y/t),\ t > 0,$$

(1-1)

$$u_y^+(x,y) = \sup_{s>0} |(\phi_s *_1 u)(x,y)|, \quad \phi_s(x) = s^{-n}\phi(x/s),\ s > 0.$$

The above definitions may not make sense for general distribution elements in Hardy spaces because such an expression

$$(\psi_t *_2 u)(x,y) = < u(x,y'),\ \psi_t(y-y') >, \quad x \in \mathbb{R}^n$$

may not be a function. However, it follows from the atomic decomposition theorem of A. Chang and R. Fefferman that the set all finite linear combinations of $H^p(\mathbb{R}^n \times \mathbb{R}^m)$ atoms is dense in $H^p(\mathbb{R}^n \times \mathbb{R}^m)$ and this subspace consists of all bounded functions with compact support that satisfy certain moment conditions (see [2] and [3]). Thus we always restrict our attention to this dense subspace in considering (1-1). The definition (1-1) will be extended inductively to general $H^p(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k})$.

In what follows, we shall rely on the theory of atomic decompositions associated with product Hardy spaces. Owing to the fact that atoms are generally supported on open sets of finite volume, in order to utilize this theory in such a way that proceeds along similar lines with the classical case, we need certain covering lemmas which make different flavours according to the number of product factors. For the sake of completeness, we record those lemmas.

For a bounded and open set $\Omega \subset \mathbb{R}^N = \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k}$, let $\mathcal{M}_i(\Omega)$, $i = 1, \cdots, k$ denote the collection of all dyadic rectangles contained in Ω that are maximal in the x_i-direction. If $M_s^{(k)}$ stands for the strong maxiaml function associated with this product structure, put $\tilde{\Omega} = \{M_s^{(k)}(\chi_\Omega) > 2^{-N}\}$, a subset of $\mathbb{R}^N$ containing Ω. By the strong maximal theorem, we note that $|\tilde{\Omega}| \leq C|\Omega|$. J. L. Journé's covering lemma [16] in the case $k = 2$ states that

LEMMA (JOURNÉ). *For $R = I \times J \in \mathcal{M}_2(\Omega)$, if $\hat{I}$ denotes the largest dyadic cube containing I with $\hat{I} \times J \subset \tilde{\Omega}$, then for any $\delta > 0$,*

$$\sum_{R \in \mathcal{M}_2(\Omega)} |R| \left(\frac{|I|}{|\hat{I}|}\right)^\delta \leq C_\delta |\Omega|.$$

The other direction regarding $\mathcal{M}_1(\Omega)$ can be stated similarly.

For the case when $k = 3$, we set the following notations. If I is a dyadic cube in $\mathbb{R}^{n_1}$, then for each integer $j \geq 1$, I_j denotes the 2^j-fold concentric dyadic cube containing I and further

$$A_{I,j} = \bigcup \left\{R : I \times R \in \mathcal{M}_3(\Omega),\ \hat{I}(R) = I_{j-1}\right\}.$$

J. Pipher's extension [20] runs as follows :

LEMMA (PIPHER). *For any bounded and open $\Omega \subset \mathbb{R}^N$ and $\delta > 0$,*

(1)
$$\sum_I \sum_j 2^{-j\delta} |I|\, |A_{I,j}| \leq C|\Omega|,$$

(2) *for each $P = I \times J \times Q \in \mathcal{M}_3(\Omega)$, if $\hat{I}$ denotes the largest dyadic cube containing I such that $\hat{I} \times J \times Q \subset \tilde{\Omega}$ and $\hat{J}$ the largest dyadic cube containing J with $\hat{I} \times \hat{J} \times Q \subset \tilde{\tilde{\Omega}}$, then we have*

$$\sum_{P \in \mathcal{M}_3(\Omega)} |P| \left(\frac{|I|}{|\hat{I}|}\right)^\delta \left(\frac{|J|}{|\hat{J}|}\right)^\delta \leq C_\delta |\Omega|.$$

Of course there are symmetric forms regarding other $\mathcal{M}_i(\Omega)$ for the above two lemmas and when $k \geq 4$, Pipher's lemma can be extended inductively (for the covering lemmas of weighted version, refer to the article of D. Krug and A. Torchinsky [18]).

2. Restriction Inequalities : Bi-product Domains

We are now ready to state one of our main theorems for the case of $k = 2$.

THEOREM II. *Whenever* $f \in H^p(\mathbb{R}^n \times \mathbb{R}^m)$, $p > 0$, *there exist* $A_p, B_p > 0$ *depending only on* p *and the dimension* $N = n + m$ *such that*

$$\text{(2-1)}\qquad \begin{aligned} \Big\| \, \|f(\cdot,y)\|_{H^p_x(\mathbb{R}^n)} \Big\|_{L^p(\mathbb{R}^m,dy)} &\leq A_p \, \|f\|_{H^p(\mathbb{R}^n\times\mathbb{R}^m)} \\ \Big\| \, \|f(x,\cdot)\|_{H^p_y(\mathbb{R}^m)} \Big\|_{L^p(\mathbb{R}^n,dx)} &\leq B_p \, \|f\|_{H^p(\mathbb{R}^n\times\mathbb{R}^m)} \, . \end{aligned}$$

On the other hand, if u *is a function in* $\mathbb{R}^n \times \mathbb{R}^m$ *such that* $u_x^+ \in L^p_{dy}(H^p_x)$ *or* $u_y^+ \in L^p_{dx}(H^p_y)$ *for some Schwartz functions, then* u *belongs to* $H^p(\mathbb{R}^n \times \mathbb{R}^m)$ *with the inequalities*

$$\text{(2-2)}\qquad \begin{aligned} \|u\|_{H^p(\mathbb{R}^n\times\mathbb{R}^m)} &\leq A_p \, \Big\| \, \|u_x^+\|_{H^p_x(\mathbb{R}^n)} \Big\|_{L^p(\mathbb{R}^m,dy)} \\ \|u\|_{H^p(\mathbb{R}^n\times\mathbb{R}^m)} &\leq B_p \, \Big\| \, \|u_y^+\|_{H^p_y(\mathbb{R}^m)} \Big\|_{L^p(\mathbb{R}^n,dx)} \, . \end{aligned}$$

In particular, if $f(x,y) \in H^1(\mathbb{R}^n \times \mathbb{R}^m)$, *then* $f(\cdot,y) \in H^1(\mathbb{R}^n)$ *for* a.e. $y \in \mathbb{R}^m$ and $f(x,\cdot) \in H^1(\mathbb{R}^m)$ *for* a.e. $x \in \mathbb{R}^n$.

The statements in regards to (2-2) are elementary so we only prove the first part.

Proof. Since these results are trivial for $p > 1$, we only need to deal with the cases when $0 < p \leq 1$. Take a function $\phi \in C_0^\infty(\mathbb{R}^n)$, $\int \phi \, dx = 1$, supported in the unit ball. By the atomic decomposition theory related with $H^p(\mathbb{R}^n \times \mathbb{R}^m)$, it suffices to show that $\|T(a_\Omega)\|_{L^p(\mathbb{R}^N)} \leq C_p$, where

$$Ta_\Omega(x,y) = (a_\Omega)_y^+(x,y) \, ,$$

and a_Ω denotes an $H^p(\mathbb{R}^n \times \mathbb{R}^m)$ atom associated with a bounded and open set Ω. For such an atom, write $a_\Omega = \sum_{R \in \mathcal{M}(\Omega)} \alpha_R$ according to the property of atomic decomposition, and for each $R = I \times J \in \mathcal{M}(\Omega)$, let $\hat{I}$ denote the largest dyadic cube containing I so that $\hat{I} \times J \subset \tilde{\Omega}$ and then $\hat{J}$ the largest dyadic cube containing J so that $\hat{I} \times \hat{J} \subset \tilde{\tilde{\Omega}}$. Repeat this procedure once more to obtain a larger rectangle $\overline{R} = \hat{\hat{I}} \times \hat{\hat{J}}$ and set $\overline{\Omega} = \cup \overline{R}$. By the strong maximal theorem we notice that $|\overline{\Omega}| \leq C|\Omega|$. If we denote the Hardy-Littlewood maximal operator acting only on $\mathbb{R}^n$ by M_1, then $|T(a_\Omega)(x,y)| \leq M_1(a_\Omega(\cdot,y))(x)$ so that the size conditions of atoms and Hölder's inequality imply

$$\begin{aligned}\int_{\overline{\Omega}} |T(a_\Omega)|^p \, dxdy &\leq \int_{\overline{\Omega}} |M_1(a_\Omega)|^p \, dx\, dy \\ &\leq |\overline{\Omega}|^{1-p/2} \left(\int_{\mathbb{R}^N} |M_1(a_\Omega)|^2 \, dx\, dy \right)^{p/2} \\ &\leq |\overline{\Omega}|^{1-p/2} \|a_\Omega\|_{L^2}^p \leq C_p.\end{aligned}$$

Since we may assume that each α_R is supported in R, we note

$$\begin{aligned}\int_{\overline{\Omega}^c} |T(a_\Omega)|^p \, dxdy &= \int_{\overline{\Omega}^c} \left| T\left(\sum_{R \in \mathcal{M}(\Omega)} \alpha_R \right) \right|^p dx\, dy \\ &\leq \int_{\overline{\Omega}^c} \sum_{R \in \mathcal{M}(\Omega)} |T(\alpha_R)|^p \, dx\, dy \\ &\leq \sum_{R \in \mathcal{M}(\Omega)} \int_{(\hat{I})^c \times J} |T(\alpha_R)|^p \, dx\, dy . \qquad \text{(2-3)}\end{aligned}$$

Utilizing the cancellation properties of α_R, we have the identity

$$(\phi_t *_1 \alpha_R)(x,y) = \int_I (\phi_t(x-z) - \Delta_{x,t}(z)) \, \alpha_R(z,y) \, dz ,$$

where $\Delta_{x,t} = \sum_{|\delta| \leq d} C_{\delta,t} (z - \bar{x})^\delta$, $d = [n(1/p - 1)]$, the Taylor polynomial of the function $z \to \phi_t(x-z)$ expanded about the center $\bar{x}$ of

I. Observe that when $x \in \left(\hat{I}\right)^c$, $z \in I$, $|x - z| \le t$,

$$|\phi_t(x-z) - \Delta_{x,t}(z)| \le C \frac{|z-\bar{x}|^{d+1}}{t^{n+d+1}} \le C \frac{|z-\bar{x}|^{d+1}}{|x-\bar{x}|^{n+d+1}}$$

and therefore we obtain the estimate

$$|T(\alpha_R)(x,y)| \le C \frac{|I|^{(d+1)/n}}{|x-\bar{x}|^{n+d+1}} \int_I |\alpha_R(z,y)|\, dz .$$

Integrating and applying Hölder's inequality twice, we are led to

$$\begin{aligned}
&\int_{\left(\hat{I}\right)^c \times J} |T(\alpha_R)(x,y)|^p\, dxdy \\
&\le C\, |I|^{\frac{p}{n}(d+1)}\, |\hat{I}|^{-\frac{p}{n}(n+d+1)+1} \int_J \left(\int_I |a(z,y)|\, dz \right)^p dy \\
&\le C\, |I|^{-p+1} \left(\frac{|I|}{|\hat{I}|} \right)^{\delta} |J|^{1-p/2}\, |I|^{p/2}\, \|\alpha_R\|_{L^2}^p \\
&\le C\, |R|^{1-p/2} \left(\frac{|I|}{|\hat{I}|} \right)^{\delta} \|\alpha_R\|_{L^2}^p ,
\end{aligned} \tag{2-4}$$

where $\delta = \left(p(n+d+1) - n\right)/n > 0$. We conclude by Hölder's inequality and Journé's lemma that

$$\begin{aligned}
&\sum_{R \in \mathcal{M}(\Omega)} \int_{\left(\hat{I}\right)^c \times J} |T(\alpha_R)|^p\, dxdy \\
&\le C \sum_{R \in \mathcal{M}(\Omega)} |R|^{1-p/2} \left(\frac{|I|}{|\hat{I}|} \right)^{\delta} \|\alpha_R\|_{L^2}^p \\
&\le C \left(\sum_{R \in \mathcal{M}(\Omega)} |R| \left(\frac{|I|}{|\hat{I}|} \right)^{\delta'} \right)^{1-p/2} \left(\sum_{R \in \mathcal{M}(\Omega)} \|\alpha_R\|_{L^2}^2 \right)^{p/2} \\
&\le C\, |\Omega|^{p/2-1} \left(\sum_{R \in \mathcal{M}_2(\Omega)} |R| \left(\frac{|I|}{|\hat{I}|} \right)^{\delta'} \right)^{1-p/2} \le C_p .
\end{aligned} \tag{2-5}$$

The proof is now complete. □

3. The Cases of Higher Number of Parameters

We now state restriction results of the kind (2-1) for $k = 3$.

THEOREM III. *For $f \in H^p\left(\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^l\right)$, there exists $C_p > 0$ depending only on p and the dimension $N = n + m + l$ such that*

$$\left\| \|f(\cdot,\cdot,z)\|_{H^p_{(x,y)}(\mathbb{R}^n\times\mathbb{R}^m)} \right\|_{L^p(\mathbb{R}^l,dz)} \le C_p \, \|f\|_{H^p(\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^l)} . \tag{3-1}$$

In particular, whenever $f \in H^1\left(\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^l\right)$, we have

$$f(\cdot,\cdot,z) \in H^1\left(\mathbb{R}^n \times \mathbb{R}^m\right) \quad \textit{for a.e.} \quad z \in \mathbb{R}^l$$

There are obvious symmetric results concerning other variables, which we omit the statements for simplicity. It is now evident how Theorem III can be extended inductively to higher number of product factors.

Proof. Choose arbitrary $\phi \in C_0^\infty(\mathbb{R}^n)$, $\psi \in C_0^\infty(\mathbb{R}^m)$, supported in the unit ball in each space, such that $\int \phi\, dx = 1$, $\int \psi\, dy = 1$. In view of the atomic decomposition theory, it suffices to handle

$$T(a_\Omega)(x,y,z) = (a_\Omega)^+_z(x,y,z) = \sup_{s,t>0} \left|\left(\phi_s\psi_t \, * a_\Omega(\cdot,\cdot,z)\right)(x,y)\right| ,$$

where $*$ denotes the convolution with respect to (x,y) and a_Ω an atom for the product Hardy space $H^p\left(\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^l\right)$, $0 < p \le 1$, associated with an open and bounded set Ω. Let $M_s^{(3)}$ denote the strong maximal operator relevant to this product structure and let $M_s^{(1,2)}$ denote the partial maximal operator acting only on the variable (x,y). It is plain to observe that

$$T(a_\Omega)(x,y,z) \le \left(M_s^{(1,2)}(a_\Omega)(\cdot,\cdot,z)\right)(x,y) .$$

As before we put $a_\Omega = \sum_{P\in\mathcal{M}(\Omega)} \alpha_P$ and for each $P = I \times J \times Q \in \mathcal{M}(\Omega)$ we form $\overline{P} = \hat{\hat{I}} \times \hat{\hat{J}} \times \hat{\hat{Q}}$, $\overline{\Omega} = \cup\, \overline{P}$ so that Pipher's lemma may be applicable.

It is clear from the strong maximal theorem that $\left|\overline{\Omega}\right| \leq C\,|\Omega|$ and from Hölder's inequality that

$$\begin{aligned}\int_{\overline{\Omega}} |T(a_\Omega)|^p \,dx\,dy\,dz &\leq \int_{\overline{\Omega}} \left| M_s^{(1,2)}(a_\Omega)(x,y,z)\right|^p dx\,dy\,dz \\ &\leq \left|\overline{\Omega}\right|^{1-p/2} \left\| M_s^{(1,2)} a_\Omega \right\|_{L^2}^p \\ &\leq C_p\,|\Omega|^{1-p/2}\,\|a_\Omega\|_{L^2}^p \leq C_p\,,\end{aligned}$$

where we exploited the L^2 continuity of $M_s^{(1,2)}$.

As for $\int_{\overline{\Omega}^c} |T(a_\Omega)|^p \,dx\,dy\,dz$, we first note that when $(x,y,z) \in \overline{\Omega}^c$, we have, in view of the support conditions,

$$\begin{aligned} T\,(a_\Omega)\,(x,y,z) &= T\,\Big(\sum_{P\in\mathcal{M}(\Omega)} \alpha_P\Big)\,(x,y,z) \\ &\leq \sum_{P\in\mathcal{M}(\Omega)} T(\alpha_P)(x,y,z)\left(\chi_{(\hat{I})^c} + \chi_{(\hat{J})^c} + \chi_{(\hat{I})^c\times(\hat{J})^c}\right). \end{aligned} \tag{3-2}$$

Writing $P = I\times J\times Q = I\times R$, $(y,z) = \zeta$, we note regarding the first term

$$\begin{aligned} &\int \left| \sum_{P\in\mathcal{M}(\Omega)} T(\alpha_P)\,(x,y,z)\,\chi_{(\hat{I})^c}\right|^p dx\,dy\,dz \\ &\leq \sum_I \sum_k \int \left| T\,\Big(\sum_{\substack{R:I\times R\in\mathcal{M}_3(\Omega)\\ \hat{I}(R)=I_{k-1}}} \alpha_{I\times R}\Big)\,(x,y,z)\,\chi_{I_{k-1}^c}\right|^p dx\,dy\,dz \\ &= \sum_I \sum_k \int_{x\in I_{k-1}^c} \iint \left| T\,\Big(\sum_{R\in\mathcal{M}_2(A_{I,k})} \alpha_{I\times R}\Big)\,(x,\zeta)\right|^p d\zeta\,dx. \end{aligned} \tag{3-3}$$

For each $R = J\times Q \in \mathcal{M}_2\,(A_{I,k})$, form its enlargement and put $\tilde{R} = \tilde{J}\times\tilde{Q}$, $S_k = \cup\,\tilde{R}$ so that Journé's lemma may be applicable in this occasion. We now separate (3-3) into two parts

(3-4)

$$\sum_I \sum_k \int_{x\in I_{k-1}^c} \left(\int_{\zeta\in S_k} + \int_{\zeta\in S_k^c}\right) \left| T\,\Big(\sum_{R\in\mathcal{M}_2(A_{I,k})} \alpha_{I\times R}\Big)\right|^p d\zeta\,dx.$$

To estimate the first part, observe when $x \in I^c_{k-1}$, $\zeta \in S_k$,

$$\begin{aligned}&\Big(\phi_s\,\psi_t * \sum_{R\in\mathcal{M}_2(A_{I,k})} \alpha_{I\times R}\Big)(\cdot,\cdot,z)\\&= \int_I \left[\phi_s(x-u) - \Delta_{x,s}(u)\right] \Big(\psi_t *_2 \sum_{R\in\mathcal{M}_2(A_{I,k})} \alpha_{I\times R}\Big)(u,y,z)\,du\,,\end{aligned}$$

where $\Delta_{x,s}(u)$ represents the Taylor polynomial of degree $d_1 = [n\,(1/p-1)]$ expanded about the center $\bar{x}$ of I. Adopting the same argument as before, if we denote the Hardy-Littlewood maximal operator acting on the second variable by M_2, then

$$\begin{aligned}&T\Big(\sum_{R\in\mathcal{M}_2(A_{I,k})} \alpha_{I\times R}\Big)(x,\zeta) = \sup_{s,t>0}\Big|\phi_s\,\psi_t * \sum_{R\in\mathcal{M}_2(A_{I,k})} \alpha_{I\times R}(x,\zeta)\Big|\\&\le C\,\frac{|I|^{(d_1+1)/n}}{|x-\bar{x}|^{n+d_1+1}} \int_I M_2\Big(\sum_{R\in\mathcal{M}_2(A_{I,k})} \alpha_{I\times R}\Big)(u,\cdot,z)\,du\end{aligned}$$

and consequently the integral portion of the first term in (3-4) is dominated by the constant $|I|^{\frac{p}{n}(1+d_1)}$ times

$$\begin{aligned}&\int_{x\in I^c_{k-1}} |x-\bar{x}|^{-p(n+d_1+1)}\\&\qquad dx \int_{\zeta\in S_k}\Big(\int_{u\in I} M_2\Big(\sum_{R\in\mathcal{M}_2(A_{I,k})} \alpha_{I\times R}\Big)(u,\zeta)\,du\Big)^p d\zeta\\&\le C_p\,|I|^{-p+1}\,(2^k)^{-p(n+d_1+1)+n}\,|S_k|^{1-p/2}\times\\&\qquad\Big\{\int_{S_k}\Big(\int_I M_2\Big(\sum_{R\in\mathcal{M}_2(A_{I,k})} \alpha_{I\times R}\Big)(u,\zeta)\,du\Big)^2 d\zeta\Big\}^{p/2}\\&\le C_p\,|I|^{1-p/2}\,(2^k)^{-p(n+d_1+1)+n}\,|\,A_{I,k}\,|^{1-p/2}\,\Big\|\sum_{R\in\mathcal{M}_2(A_{I,k})} \alpha_{I\times R}\Big\|^p_{L^2}.\end{aligned}$$

Now summing over all I, k, we note, with $\delta = p(n+d_1+1)-n$, the first term in (3-4) is less than

$$\Big(\sum_I \sum_k 2^{-k\delta'} \, |I| \, |A_{I,k}|\Big)^{1-p/2} \Big(\sum_I \sum_k \Big\| \sum_{R\in\mathcal{M}_2(A_{I,k})} \alpha_{I\times R} \Big\|_{L^2}^2 \Big)^{p/2}$$
$$\leq C_p \, |\Omega|^{1-p/2} \Bigg(\sum_{P\in\mathcal{M}(\Omega)} \|\alpha_P\|_{L^2}^2 \Bigg)^{p/2}$$
$$\leq C_p \, |\Omega|^{1-p/2} \, |\Omega|^{-1+p/2} \; = \; C_p \, ,$$

by the Hölder inequality and the first part of Pipher's lemma.

It is much easier to deal with the second portion of (3-4) for we can adopt similar reasonings as in the proof of Theorem II. We first note the second term in (3-4) is bounded above by

$$\sum_{R\in\mathcal{M}_2(A_{I,k})} \int_{x\in I_{k-1}^c} \int_{\zeta\in S_k^c} \big| T(\alpha_{I\times R}) \big|^p \, d\zeta \, dx \, .$$

If $x \in I_{k-1}^c$, $\zeta \in S_k^c$, then the cancelling properties of atoms enable us to have

$$(\phi_s \, \psi_t * \alpha_{I\times R}) \, (x, y, z) = \iint_{I\times J} [\, \phi_s(x-u) - \Delta_{x,s}(u) \,] \, \times$$
$$[\, \psi_t(y-v) - \Delta_{y,t}(v) \,] \; \alpha_{I\times R}(u, v, z) \, du \, dv$$
$$\leq C \, \frac{|I|^{(d_1+1)/n}}{|x-\bar{x}|^{n+d_1+1}} \, \frac{|J|^{(d_2+1)/m}}{|y-\bar{y}|^{m+d_2+1}} \iint_{I\times J} |\, \alpha_{I\times J\times Q}(u, v, z) \,| \; du \, dv \, ,$$

in which $\Delta_{y,t}(v)$ denotes the Taylor polynomial of degree $d_2 = [m\,(1/p-1)]$ associated with ψ_t. By integrating and using previous steps, we see that the quantity

$$\int_{x\in I_{k-1}^c} \int_{\zeta\in S_k^c} |\, T(\alpha_{I\times R}) \,|^p \; d\zeta \, dx$$

is dominated by, for some positive δ_1, δ_2,

$$C_p\, 2^{-k\delta_1}\, |I|^{-p+1}\, |J|^{-p+1} \left(\frac{|J|}{|\tilde{J}|}\right)^{\delta_2} \int_Q \left(\iint_{I\times J} |\,\alpha_{I\times J\times Q}(u,v,z)\,|\, du\, dv\right)^p dz$$
$$\leq C_p\, 2^{-k\delta_1}\, |I|^{1-p/2}\, |R|^{1-p/2} \left(\frac{|J|}{|\tilde{J}|}\right)^{\delta_2} \|\,\alpha_{I\times R}\,\|_{L^2}^p\,.$$

In view of Journé's lemma relevant to $A_{I,k}$ and Hölder's inequality,

$$\sum_{R\in\mathcal{M}_2(A_{I,k})} \left(\frac{|J|}{|\tilde{J}|}\right)^{\delta_2} |R|^{1-p/2}\, \|\alpha_{I\times R}\|_{L^2}^p$$
$$\leq \left(\sum_{R\in\mathcal{M}_2(A_{I,k})} \left(\frac{|J|}{|\tilde{J}|}\right)^{\delta_2'} |R|\right)^{1-p/2} \left(\sum_{R\in\mathcal{M}_2(A_{I,k})} \|\alpha_{I\times R}\|_{L^2}^2\right)^{p/2}$$
$$\leq |A_{I,k}|^{1-p/2} \left(\sum_{R\in\mathcal{M}_2(A_{I,k})} \|\alpha_{I\times R}\|_{L^2}^2\right)^{p/2}$$

so that the second portion of (3-4) is bounded above by

$$\sum_I \sum_k 2^{-k\delta_1}\, |I|^{1-p/2}\, |A_{I,k}|^{1-p/2} \left(\sum_{R\in\mathcal{M}_2(A_{I,k})} \|\alpha_{I\times R}\|_{L^2}^2\right)^{p/2}$$
$$\leq \left(\sum_I \sum_k 2^{-k\delta_1'}\, |I|\, |A_{I,k}|\right)^{1-p/2} \left(\sum_I \sum_k \sum_R \|\alpha_{I\times R}\|_{L^2}^2\right)^{p/2}$$
$$\leq C_p\, |\Omega|^{1-p/2}\, |\Omega|^{p/2-1} \quad \leq C_p\,.$$

The estimates of (3-2) involving $\chi_{(\hat{J})^c}$ can be performed in the exactly same fashion as above so we omit them.

For the last term in (3-2), we can modify the above reasonings to conclude a similar conclusion. To be precise, if $x \in \left(\hat{\hat{I}}\right)^c$, $y \in \left(\hat{\hat{J}}\right)^c$, then we obtain

$$\begin{aligned}
&\int \Big| \sum_{P\in\mathcal{M}(\Omega)} T(\alpha_P)\,\chi_{\left(\hat{I}\right)^c \times \left(\hat{J}\right)^c} \Big|^p dx\,dy\,dz \\
&\leq \sum_{P\in\mathcal{M}(\Omega)} \int_{\mathbf{R}^l} \iint_{\left(\hat{I}\right)^c \times \left(\hat{I}\right)^c} |T(\alpha_P)|^p \,dx\,dy\,dz \\
&\leq \sum_{P\in\mathcal{M}(\Omega)} \left(\frac{|I|}{|\hat{I}|}\right)^{\delta_1} \left(\frac{|J|}{|\hat{J}|}\right)^{\delta_2} |P|^{1-p/2}\, \|\alpha_P\|_{L^2}^p \\
&\leq \left(\sum_{P\in\mathcal{M}(\Omega)} |P| \left(\frac{|I|}{|\hat{I}|}\right)^{\delta} \left(\frac{|J|}{|\hat{J}|}\right)^{\delta} \right)^{1-p/2} \left(\sum_{P\in\mathcal{M}(\Omega)} \|\alpha_P\|_{L^2}^2 \right)^{p/2} \\
&\leq C_p |\Omega|^{1-p/2}\, |\Omega|^{p/2-1} \quad \leq\; C_p
\end{aligned}$$

by the second part of Pipher's lemma and the size restriction of an atom.

Putting all of our estimates together, we finally finish the proof. □

4. Applications to Multiple Riesz Transforms

In this section, we demonstrate that Theorem II and Theorem III provide a way of iterations in setting up boundedness properties of some typical class of singular integrals.

Generalizing a bit the multiple Hilbert transforms of section 4, Part I, consider the extended Riesz transforms $\mathcal{R}_{l,j}$, $l = 1, \cdots, n_j$, $j = 1, \cdots, k$ defined by, for testing functions,

$$(\mathcal{R}_{l,j} f)\hat{}\,(\xi) = -i\, \frac{\xi_{l,j}}{|\xi_j|}\, \hat{f}(\xi)\,, \tag{4-1}$$

where $\xi = (\xi_1, \cdots, \xi_k)$, $\xi_j = (\xi_{1,j}, \cdots, \xi_{n_j,j})$, $j = 1, \cdots, k$. These singular integrals arise in the theory of partial differential equations. For conciseness, we consider only the case when $k = 2$. Writing the variables by $(x,y) \in \mathbb{R}^n \times \mathbb{R}^m$, $x = (x_1, \cdots, x_n)$, $y = (y_1, \cdots, y_m)$, take a fourth order equation

$$\Delta_x \Delta_y u(x,y) = f(x,y) \tag{4-2}$$

or the ultrahyperbolic differential equation

$$\Delta_x u(x,y) = \Delta_y u(x,y) \tag{4-3}$$

into consideration. Concerning the regularity questions, it is meaningful to control all the second order derivatives of the type

$$\frac{\partial^2 u(x,y)}{\partial x_i \partial x_j}, \ \frac{\partial^2 u(x,y)}{\partial y_l \partial y_k}, \ 1 \le i,j \le n, \ 1 \le l,k \le m$$

by the corresponding Laplacians Δ_x, Δ_y rather than the whole Laplacian Δ.

Using (4-1), we simply note that

$$\frac{\partial^2 u(x,y)}{\partial x_i \partial x_j} = \mathcal{R}_{x,i} \mathcal{R}_{x,j} \Delta_x u(x,y), \ 1 \le i,j \le n \tag{4-4}$$

with the similar expression for the y-variables. Exploiting the well-known boundedness results of the Riesz transforms on each space, we immediately see from Theorem II that

$$\begin{aligned}\left\| \frac{\partial^2 u(x,y)}{\partial x_i \partial x_j} \right\|_{L^1} &= \| \, \|\mathcal{R}_{x,i} \ (\mathcal{R}_{x,j} \Delta_x u) \|_{L^1(\mathbb{R}^n,dx)} \|_{L^1(\mathbb{R}^m,dy)} \\ &\le C \| \, \|\mathcal{R}_{x,j} \Delta_x u\|_{H^1(\mathbb{R}^n,dx)} \|_{L^1(\mathbb{R}^m,dy)} \\ &\le C \| \, \|\Delta_x u\|_{H^1(\mathbb{R}^n,dx)} \|_{L^1(\mathbb{R}^m,dy)} \\ &\le C \|\Delta_x f\|_{H^1(\mathbb{R}^n \times \mathbb{R}^m)} . \end{aligned} \tag{4-5}$$

In summary,

THEOREM IV. *For Schwartz testing functions, with* $N = n + m$,

$$\left\| \frac{\partial^2 u(x,y)}{\partial x_i \partial x_j} \right\|_{L^1(\mathbb{R}^N)} \le C \|\Delta_x u\|_{H^1(\mathbb{R}^n \times \mathbb{R}^m)},$$

$$\left\| \frac{\partial^2 u(x,y)}{\partial y_l \partial y_k} \right\|_{L^1(\mathbb{R}^N)} \le C \|\Delta_y u\|_{H^1(\mathbb{R}^n \times \mathbb{R}^m)},$$

$$\left\| \frac{\partial^2 u(x,y)}{\partial x_j \partial y_k} \right\|_{L^1(\mathbb{R}^N)} \le C \|\Delta_x \Delta_y u\|_{H^1(\mathbb{R}^n \times \mathbb{R}^m)} .$$

Similar reasonings will show that

THEOREM V. *For any product structure* $N = n_1 + \cdots + n_j + \cdots + n_k$, *with* n_j *fixed, we have*

$$\| \mathcal{R}_{i,j} f \|_{L^1(\mathbb{R}^N)} \leq C \| f \|_{H^1(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k})}, \quad i = 1, \cdots, n_j.$$

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