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THE G-SEQUENCE OF A MAP AND ITS EXACTNESS

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ABSTRACT. In this paper, we extend the G-sequence of a CW-pair to the G-sequence of a map and show the existence of a map with nonexact G-sequence. We also give an example of a finite CW-pair with nontrivial ω-homology in high order.

1. Introduction

Gottlieb [1, 2] defined and studied the Gottlieb groups $G_n(X)$ of homotopy groups $\pi_n(X)$. The importance of Gottlieb groups is that these subgroups have many applications on topology, especially, on the fibration theory, on the fixed point theory, and on the theory of identification of spaces.

The homotopy sequence of a topological pair plays an important role in computing homotopy groups. In [6], Lee and Woo introduced the G-sequence of a CW-pair consisting of Gottlieb groups, generalized evaluation subgroups and relative evaluation subgroups and made some improvements in computing evaluation subgroups. They also introduced the ω-homology of a CW-pair and showed there is an example of a finite CW-pair whose ω-homology is not trivial only in order one. In [9], it was shown that there is an example whose ω-homology group is nontrivial for higher order but it is not a finite CW-pair. So we need to show a finite CW-pair with nontrivial ω-homology in high order.

The main object of this paper is to extend the concept of G-sequence of a CW-pair to that of a map and find a finite CW-pair with nonexact G-sequence and nontrivial ω-homology in high order.

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After the brief introduction of Gottlieb groups, we give definitions of G-sequences and ω-homologies from [6] for the reader's convenience in section 2. Our main results and a solution for an open problem are given in section 3 and 4.

2. Preliminaries

Let X be a CW-complex. The Gottlieb group $G_n(X)$ of X consists of all $\alpha \in \pi_n(X, x_0)$ such that there exists an affiliated map $A : S^n \times X \to X$ with a homotopy commutative diagram

$$\begin{array}{ccc} S^n \times X & \xrightarrow{A} & X \\ \uparrow & \nearrow \alpha \vee 1_X & \\ S^n \vee X & & \end{array}$$

This group, $G_n(X)$, is also characterized by $G_n(X) = w_\sharp(\pi_n(X^X, 1_X)) \subset \pi_n(X, x_0)$ where $\omega : X^X \to X$ is an evaluation map at $x_0 \in X$. Thus $G_n(X)$ is also called an *evaluation subgroup* of $\pi_n(X, x_0)$. Gottlieb extensively studied $G_1(X)$ in [1] and $G_n(X)$ for $n \geq 2$ in [2]. Among other things he has shown that if X is an H-space then $G_n(X) = \pi_n(X)$ for all n. He also had computed

$$G_n(S^n) = \begin{cases} 0 & \text{for } n \text{ even} \\ Z & \text{for } n = 1, 3, 7 \\ 2Z & \text{for other odd } n\text{'s} \end{cases}$$

Here we introduce the G-sequence from [6]. For convenience, from here on we assume that a space is a homotopy type of a CW-complex and a topological pair is a pointed CW-pair.

The inclusion map $i : A \to X$ induces a homotopy exact sequence for the pair (X, A)

$$\cdots \to \pi_n(A) \xrightarrow{i_\sharp} \pi_n(X) \xrightarrow{j_\sharp} \pi_n(X, A) \xrightarrow{\partial} \pi_{n-1}(A) \to \cdots$$

Let X^A and A^A be the spaces of all maps from A into X and from A into A, respectively. Then $i : A \to X$ induces the inclusion map

$\bar{i} : A^A \to X^A$ given by $\bar{i}(f) = if$. Let $\omega : A^A \to A$ and $\omega : X^A \to X$ be the corresponding evaluation maps at the base point $x_0 \in A \subset X$. Then ω induces $\omega_\sharp : \pi_n(A^A, id) \to \pi_n(A, x_0)$ and $\omega_\sharp : \pi_n(X^A, i) \to \pi_n(X, x_0)$. Denote the images of $\omega_\sharp$ by $G_n(A)$ and $G_n(X, A)$, respectively. It is well known that $G_n(X) \subset G_n(X, A)$ for all $A \subset X$, i.e., $G_n(X)$ is the lower bound for all $A \subset X$. Now let $\omega : (X^A, A^A, id) \to (X, A, x_0)$ be the relative evaluation map at $x_0 \in X$. Then ω induces $\omega_\sharp : \pi_n(X^A, A^A, id) \to \pi_n(X, A, x_0)$. Denote the image $\omega_\sharp(\pi_n(X^A, A^A, id)) \subset \pi_n(X, A, x_0)$ by $G_n^{Rel}(X, A)$. There are two equivalent definitions for each of these subgroups.

DEFINITION 2.1. $G_n(X) = \omega_\sharp(\pi_n(X^X, 1_X)) = \{[f] \in \pi_n(X) | \exists$ map $H : X \times I^n \longrightarrow X$ such that $[H|_{x_0 \times I^n}] = [f]$ and $H|_{X \times u} = 1_X$ for $u \in \partial I^n\}$ (see [2]).

DEFINITION 2.2. $G_n(X, A) = \omega_\sharp(\pi_n(X^A, i)) = \{[f] \in \pi_n(X) | \exists$ map $H : A \times I^n \longrightarrow X$ such that $[H|_{x_0 \times I^n}] = [f]$ and $H|_{A \times u} = i$ for $u \in \partial I^n\}$ (see [5]).

DEFINITION 2.3. $G_n^{Rel}(X, A) = \omega_\sharp(\pi_n(X^A, A^A, i)) = \{[f] \in \pi_n (X, A) | \exists$ map $H : (X \times I^n, A \times \partial I^n) \longrightarrow (X, A)$ such that $[H|_{x_0 \times I^n}] = [f]$ and $H|_{X \times u} = 1_X$ for $u \in J^{n-1}\}$ (see [6]).

The inclusion map i and the evaluation map ω induce the following commutative diagram

$$\begin{array}{ccccccccccc}
\cdots \to & \pi_n(A^A) & \xrightarrow{i_\sharp} & \pi_n(X^A) & \xrightarrow{j_\sharp} & \pi_n(X^A, A^A) & \xrightarrow{\partial} & \pi_{n-1}(A^A) & \to \cdots \\
& \downarrow \omega_\sharp & & \downarrow \omega_\sharp & & \downarrow \omega_\sharp & & \downarrow \omega_\sharp & \\
\cdots \to & G_n(A) & \xrightarrow{i_\sharp} & G_n(X, A) & \xrightarrow{j_\sharp} & G_n^{Rel}(X, A) & \xrightarrow{\partial} & G_{n-1}(A) & \to \cdots \\
& \downarrow \cap & & \downarrow \cap & & \downarrow \cap & & \downarrow \cap & \\
\cdots \to & \pi_n(A) & \xrightarrow{i_\sharp} & \pi_n(X) & \xrightarrow{j_\sharp} & \pi(X, A) & \xrightarrow{\partial} & \pi_{n-1}(A) & \to \cdots
\end{array}$$

where the top and the bottom rows are exact and the middle part makes a sequence. We call this middle sequence the *G-sequence* of a CW-pair (X, A). This sequence is not necessarily exact and there are a number of theorems describing under what conditions the G-sequence to be exact. Here we quote a typical theorem from [6].

THEOREM 2.1. *Let (X, A) be a connected CW-pair. If the inclusion map $i : A \to X$ is null homotopic or has a left homotopy inverse, then the G-sequence of (X, A) is exact.*

The G-sequence of a CW-pair is a half exact sequence, i.e., the composition of two adjacent homomorphisms is zero. Thus we can think of the G-sequence

$$\cdots \xrightarrow{j_\sharp{}^{n+1}} G_{n+1}^{Rel}(X,A) \xrightarrow{\partial^{n+1}} G_n(A) \xrightarrow{i_\sharp^n} G_n(X,A) \xrightarrow{j_\sharp{}^n} G_n^{Rel}(X,A) \rightarrow \cdots$$

as a chain complex.

DEFINITION 2.4. If (X,A) is a CW-pair, the ω-homology

$$H_*^\omega(X,A) = \{H_{n+1}^{g\omega}(X,A), H_{n+1}^{r\omega}(X,A), H_n^{a\omega}(X,A)\}_{n\geq 0}$$

of (X,A) is defined to be

$$\begin{aligned} H_{n+1}^{g\omega}(X,A) &= \text{ker of } j_\sharp^{n+1}/\text{Image of } i_\sharp^{n+1} \\ H_{n+1}^{r\omega}(X,A) &= \text{ker of } \partial^{n+1}/\text{Image of } j_\sharp^{n+1} \\ H_n^{a\omega}(X,A) &= \text{ker of } i_\sharp^n/\text{Image of } \partial^n \end{aligned}$$

for $n \geq 0$ (see [6]).

In [5], Kim and Woo have generalized the Gottlieb group $G_n(X)$ to a subgroup $G_n^f(X,A) = \{[\alpha] \in \pi_n(X) | \exists H : A \times S^n \longrightarrow X$ such that $H|_{a_0 \times S^n} = \alpha$ and $H|_{A \times s_0} = f\}$ for any map $f : (A, a_0) \rightarrow (X, x_0)$, where A need not be a subspace of X. It is shown that $G_n^f(X,A)$ is the image of $\omega_\sharp : \pi_n(X^A, f) \rightarrow \pi_n(X, x_0)$. In this reason, these groups are called the *generalized evaluation subgroups* for the map f. Especially, if A is a subcomplex of X and f is the inclusion from A to X, then $G_n^f(X,A)$ is equal to $G_n(X,A)$.

Obviously, if the G-sequence of (X,A) is exact, then the ω-homology $H_*^\omega(X,A)$ of (X,A) is trivial. Before we show an example of a finite CW-pair (X,A) whose ω-homology is not trivial in high order, we extend the G-sequence of a CW-pair to the G-sequence of a map.

3. The G-sequence of a map

In this section, we extend the generalized evaluation subgroups, the relative evaluation subgroups and the G-sequence of a CW-pair to the generalized evaluation subgroups for a map, the evaluation subgroups for a map and the G-sequence of a map, respectively. We begin with the following lemma to define the evaluation subgroups for a map.

LEMMA 3.1. *Let (X, A) be a CW-pair. Then $[\alpha] \in G_n^{Rel}(X, A)$ if and ony if there exists a map $H : (A \times I^n, A \times \partial I^n) \to (X, A)$ such that $H|_{x_0 \times I^n} = \alpha$ and $H|_{A \times J^{n-1}} = i$.*

Proof. Let $H : (A \times I^n, A \times \partial I^n) \to (X, A)$ be a map such that $H|_{x_0 \times I^n} = \alpha$ and $H|_{A \times J^{n-1}} = i$. Define $G : X \times I^{n-1} \times 1 \cup (A \times I^{n-1} \cup X \times \partial I^{n-1}) \times I \to X$ by

$$G(x, v, s) = \begin{cases} x, & (\text{if } x \in X, v \in I^{n-1}, s = 1) \\ H(x, v, s), & (\text{if } x \in A, v \in I^{n-1}, s \in I) \\ x, & (\text{if } x \in X, v \in \partial I^{n-1}, s \in I) \end{cases}$$

Then there exists an extension $\bar{G} : X \times I^n \longrightarrow X$ of G by the absolute homotopy extension property such that $\bar{G}(A \times \partial I^n) \subset A, \bar{G}(x_0, v) = \alpha(v)$ and $\bar{G}(x, u) = x$ for all $x \in X, v \in I^n$ and $u \in J^{n-1}$. Thus $\bar{G} : (X \times I^n, A \times \partial I^n) \to (X, A)$ is a homotopy such that $\bar{G}|_{x_0 \times I^n} = \alpha$ and $\bar{G}|_{A \times J^{n-1}} = i$. Therefore $[\alpha] \in G_n^{Rel}(X, A)$. □

To define the relative evaluation subgroups for a map, we introduce *the category of pairs* whose objects are maps from a pointed space to a pointed space and whose morphism from f to g is a pair of maps (α_1, α_2) such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\alpha_1} & B \\ \downarrow f & & \downarrow g \\ X & \xrightarrow{\alpha_2} & Y \end{array}$$

commutes. A homotopy of (α_1, α_2) is just *a pair of homotopies* $(\alpha_{1t}, \alpha_{2t})$ such that $g\alpha_{1t} = \alpha_{2t} f$. This category reduces to the category of ordinary pairs of spaces if we restrict ourselves to maps which are inclusions.

Let $f : (A, a_0) \to (X, x_0)$ be a map. In [3], the *homotopy group of the map f* is defined by $\pi_n(f) = \{[(\alpha_1, \alpha_2)] | (\alpha_1, \alpha_2) : i_n \to f\}$, where i_n is the canonical inclusion from S^{n-1} to CS^{n-1}, the cone over S^{n-1}, and [ ] denotes the homotopy class. Especially, if $A \subset X$ and f is the inclusion map from A to X, then $\pi_n(f)$ is $\pi_n(X, A)$. Let

$f:(A,a_0)\to(X,x_0)$ be a map. Then there is the homotopy sequence of f

$$\cdots\xrightarrow{J}\pi_{n+1}(f)\xrightarrow{\partial}\pi_n(A)\xrightarrow{f_\sharp}\pi_n(X)\to\cdots\xrightarrow{J}\pi_1(f)\xrightarrow{\partial}\pi_0(A)\xrightarrow{f_\sharp}\pi_0(X)$$

Using Lemma 3.1, we generalize the relative evaluation subgroups for a CW-pair to the relative evaluation subgroups for a map as follows.

Let $[(\alpha_1,\alpha_2)]$ be an element of $\pi_n(f)$. If there exist $F_1:A\times S^{n-1}\to A$ and $F_2:A\times CS^{n-1}\to X$ such that $F_1|_{a_0\times S^{n-1}}=\alpha_1$, $F_1|_{A\times s_0}=id$, $F_2|_{a_0\times CS^{n-1}}=\alpha_2$, $F_2|_{A\times s_0}=f$ and the following diagram commutes

$$\begin{array}{ccc} A\times S^{n-1} & \xrightarrow{F_1} & A \\ \downarrow 1\times i_n & & \downarrow f \\ A\times CS^{n-1} & \xrightarrow{F_2} & X, \end{array}$$

then (F_1,F_2) is called an *affiliated map* of (α_1,α_2), where a_0 is the base point of A and s_0 is the common base point of S^{n-1} and CS^{n-1}.

DEFINITION 3.1. The *relative evaluation subgroup* $G_n^R(f)$ is defined by $G_n^R(f)=\{[(\alpha_1,\alpha_2)]\in\pi_n(f)|\exists$ affiliated map (F_1,F_2) of $(\alpha_1,\alpha_2)\}$.

REMARK. If A is a subspace of X and f is the inclusion map, then $G_n^R(f)$ is just the relative evaluation subgroup $G_n^{Rel}(X,A)$.

Let $\bar f:(A^A,id)\to(X^A,f)$ be a map given by $\bar f(g)=fg$. If we consider the evaluation maps $\omega:A^A\to A$ and $\omega:X^A\to X$, then the map $(\omega,\omega):\bar f\to f$ is called the *evaluation map* in the homotopy category of pairs.

LEMMA 3.2. *Let f and $\bar f$ be as above. Then we have* $G_n^R(f)=(\omega,\omega)_\sharp(\pi_n(\bar f))$.

Proof. Let $[(\alpha_1,\alpha_2)]$ be an element of $\pi_n(\bar f)$. If we define $F_1:A\times S^{n-1}\to A$ and $F_2:A\times CS^{n-1}\to X$ by $F_i(a,s)=\alpha_i(s)(a)(i=1,2)$, then (F_1,F_2) is an affiliated map of $[(\omega\alpha_1,\omega\alpha_2)]$ and hence $(\omega,\omega)_\sharp$ $([(\alpha_1,\alpha_2)])$ belongs to $G_n^R(f)$.

Conversely, let $[(\alpha_1,\alpha_2)]$ be an element of $G_n^R(f)$ and (F_1,F_2) an affiliated map of $[(\alpha_1,\alpha_2)]$. If we define $\beta_i(s)(a)=F_i(a,s)(i=1,2)$, then $[(\beta_1,\beta_2)]\in\pi_n(\bar f)$ and $(\omega,\omega)_\sharp([(\beta_1,\beta_2)])=[(\alpha_1,\alpha_2)]$. □

If we use the above lemma, then f and the evaluation maps induce the following commutative diagram

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & \pi_n(A^A, id_A) & \xrightarrow{\bar{f}_\sharp} & \pi_n(X^A, f) & \xrightarrow{J} & \pi_n(\bar{f}) & \xrightarrow{\partial} & \pi_{n-1}(A^A, id_A) & \longrightarrow \cdots \\
& \downarrow \omega_\sharp & & \downarrow \omega_\sharp & & \downarrow (\omega,\omega)_\sharp & & \downarrow \omega_\sharp & \\
\cdots \longrightarrow & G_n(A) & \xrightarrow{f_\sharp} & G_n^f(X,A) & \xrightarrow{J} & G_n^R(f) & \xrightarrow{\partial} & G_{n-1}(A) & \longrightarrow \cdots \\
& \downarrow \cap & & \downarrow \cap & & \downarrow \cap & & \downarrow \cap & \\
\cdots \longrightarrow & \pi_n(A) & \xrightarrow{f_\sharp} & \pi_n(X) & \xrightarrow{J} & \pi_n(f) & \xrightarrow{\partial} & \pi_{n-1}(A) & \longrightarrow \cdots
\end{array}$$

where the top and the bottom rows are also exact and the middle sequence forms a chain complex. We call this middle sequence, the *G-sequence of f*. In this paper, we will also show the sequence is not necessarily exact.

THEOREM 3.3. *Let $f : A \to X$ be a map and Z_f be the mapping cylinder of f. Then $G_n^f(X,A)$ is isomorphic to $G_n(Z_f, A)$ and $G_n^R(f)$ is also isomorphic to $G_n^{Rel}(Z_f, A)$.*

Proof. Let $r : Z_f \to X$ be a map given by $r(x) = x$ and $r(a,t) = f(a)$ and let $\bar{r} : {Z_f}^A \to X^A$ be a map given by $\bar{r}(h) = rh$. Then r and $\bar{r}$ are homotopy equivalences. If we consider the following commutative diagram

$$\begin{array}{ccc}
\pi_n({Z_f}^A, i) & \xrightarrow{\bar{r}_\sharp} & \pi_n(X^A, f) \\
\downarrow \omega_\sharp & & \downarrow \omega_\sharp \\
\pi_n(Z_f) & \xrightarrow{r_\sharp} & \pi_n(X),
\end{array}$$

then it is easy to show that $G_n^f(X,A)$ is isomorphic to $G_n(Z_f, A)$. Let $i : A \to Z_f$ be the inclusion. If we consider the following commutative diagram

$$\begin{array}{ccccccccc}
\xrightarrow{\partial} & \pi_n(A) & \longrightarrow & \pi_n(Z_f) & \longrightarrow & \pi_n(i) \cdots \longrightarrow & \pi_1(A) & \longrightarrow & \pi_1(Z_p) \\
& \downarrow = & & \downarrow \cong & & \downarrow (id, r)_\sharp & \downarrow = & & \downarrow \cong \\
\xrightarrow{\partial} & \pi_n(A) & \xrightarrow{f_\sharp} & \pi_n(X) & \xrightarrow{J} & \pi_n(f) \cdots \longrightarrow & \pi_1(A) & \longrightarrow & \pi_1(X)
\end{array}$$

then $(id, r)_\sharp$ is an isomorphism. Similarly, $(\bar{id}, \bar{r})_\sharp : \pi_n(\bar{i}) \to \pi_n(\bar{f})$ is also an isomorphism. Using the following commutative diagram

$$\begin{array}{ccc} \pi_n(\bar{i}) & \xrightarrow{(\bar{id},\bar{r})_\sharp} & \pi_n(\bar{f}) \\ \downarrow (\omega,\omega)_\sharp & & \downarrow (\omega,\omega)_\sharp \\ \pi_n(i) & \xrightarrow{(id,r)_\sharp} & \pi_n(f) \end{array}$$

we can prove the other part. □

COROLLARY 3.4. *Let $f : A \to X$ be a map and Z_f be the mapping cylinder of f. Then the G-sequence of the pair (Z_f, A) is isomorphic to the G-sequence of f as follows*

$$\begin{array}{ccccccccc} \longrightarrow & G_n(A) & \xrightarrow{i_\sharp} & G_n(Z_f, A) & \xrightarrow{j_\sharp} & G_n^{Rel}(Z_f, A) & \xrightarrow{\partial} & G_{n-1}(A)\cdots \\ & \downarrow = & & \downarrow \cong & & \downarrow \cong & & \downarrow = \\ \longrightarrow & G_n(A) & \xrightarrow{f_\sharp} & G_n^f(X, A) & \xrightarrow{J} & G_n^R(f) & \xrightarrow{\partial} & G_{n-1}(A)\cdots \end{array}$$

If we use Theorem 2.1 and the above corollary, then we obtain the following theorem.

THEOREM 3.5. *If $f : A \to X$ has a left homotopy inverse or is homotopic to the constant map, then the G-sequence of f is exact.*

COROLLARY 3.6. *If $f : A \to X$ has a left homotopy inverse, then we have $G_n^f(X, A) = G_n(A) \oplus G_n^R(f)$.*

Proof. If f has a left homotopy inverse r, then the induced homomorphism $f_\sharp : \pi_n(A) \to \pi_n(X)$ is a monomorphism and $\partial = 0$. Therefore, it is sufficient to show $r_\sharp(G_n^f(X, A)) \subset G_n(A)$. Let $[\alpha]$ be an element of $G_n^f(X, A)$ and F be an affiliated map of $[\alpha]$. Then $G = rF$ is an affiliated map of $[r\alpha]$. □

4. The existence of a nonexact G-sequence

The purpose of this section is to solve the open problem proposed in [6], i.e. we show a finite CW-pair with nontrivial ω-homology groups (equivalently, nonexact G-sequence) in high order. It was shown in [9] for the existence of an infinite CW-pair with nontrivial ω-homology groups in high order. To show the existence of a finite CW-pair with nonexact G-sequence, we need the following theorem.

THEOREM 4.1. *Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a principal fibration. Then we have the following commutative diagram*

$$(*)\qquad \begin{array}{ccccc} \coprod F_f^E & \xrightarrow{j} & E_{id}^E & \xrightarrow{\bar{p}} & B_p^E \\ \downarrow{\scriptstyle\omega} & & \downarrow{\scriptstyle\omega} & & \downarrow{\scriptstyle\omega} \\ F & \xrightarrow{i} & E & \xrightarrow{p} & B \end{array}$$

where X_f^Y is the component of the function space X^Y containing f, $\coprod$ is the disjoint union over a set of maps containing the constant map, the first sequence is a fibration sequence up to homotopy and ω is the evaluation map at the base point e_0.

Proof. From [8, 11], $\bar{p} : E_{id}^E \to B_p^E$ is a fibration and the fiber is $\mathbf{F} = \{s \mid s \text{ is a section of } \pi,\ q \circ s \sim id\}$, where $\pi : Q \longrightarrow E$ is the fibration induced from p along p such that

$$\begin{array}{ccc} Q & \xrightarrow{q} & E \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle p} \\ E & \xrightarrow{p} & B \end{array}$$

commutes. Note that $\pi : Q \longrightarrow E$ is a principal fibration with a cross-section. Hence it must be a trivial fibration. It is easy to see that $\mathbf{F}$ is a union of components of section space of the fibration π and there is a commutative diagram

$$\begin{array}{ccccc} \mathbf{F} & \xrightarrow{i_1} & E_{id}^E & \xrightarrow{\bar{p}} & B_p^E \\ \downarrow{\scriptstyle\omega_0} & & \downarrow{\scriptstyle\omega} & & \downarrow{\scriptstyle\omega} \\ F & \xrightarrow{i} & E & \xrightarrow{p} & B \end{array}$$

where ω is the evaluation map at e_0 and i_1 is defined by $i_1(s) = q \circ s$. Thus we have a fibre homotopy equivalence

$$\begin{array}{ccc} F & \xrightarrow{id} & F \\ \downarrow & & \downarrow \\ E \times F & \xrightarrow{k} & Q \\ {\scriptstyle p_1}\downarrow & & {\scriptstyle \pi}\downarrow \\ E & \xrightarrow{id} & E \end{array}$$

It follows immediately that there is a homotopy equivalence $\bar{k} : \coprod F_f^E \to \mathbf{F}$ such that $\omega \sim \omega_0 \circ \bar{k}$. If we define the map j in diagram $(*)$ directly by sending a map $f : E \to F$ into a map $ff : E \to E$ over the identity by $ff(e) = e \circ f(e)$, where $\circ$ is the multiplication of $E \times F \to E$ which comes from the definition of principal fibration, then we should get a commutative diagram and the first sequence in the diagram $(*)$ is a fibration sequence up to homotopy. □

In [5], it is easily shown the following.

THEOREM 4.2. *Let F be an H-space, E a space and $f : E \to F$ a map. Then the evaluation homomorphism $\omega_\sharp : \pi_n(F^E, f) \to \pi_n(F)$ is onto for each n .*

Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration. Then there are excision homomorphisms $\varepsilon_1 : \pi_n(i) \to \pi_n(B)$, $\varepsilon_2 : \pi_n(F) \to \pi_{n+1}(p)$ which are isomorphisms.

THEOREM 4.3. *Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a principal fibration. Then the following diagram is commutative and vertical homomorphisms are isomorphisms:*

$$\begin{array}{ccccccccc} \cdots \longrightarrow G_n(E) & \xrightarrow{p_\sharp} & G_n^p(B,E) & \xrightarrow{J} & G_n^R(p) & \xrightarrow{\partial} & G_{(n-1)}(E) & \xrightarrow{p_\sharp} & \cdots \\ \downarrow id & & \downarrow id & & \downarrow \varepsilon_2^{-1} & & \downarrow id & & \\ \cdots \longrightarrow G_n(E) & \xrightarrow{p_\sharp} & G_n^p(B,E) & \xrightarrow{\delta} & \pi_{n-1}(F) & \xrightarrow{\bar{i}} & G_{(n-1)}(E) & \xrightarrow{p_\sharp} & \cdots \end{array}$$

Proof. By Theorem 4.1, we have the following commutative diagrams

$$\begin{array}{ccccc} \coprod F_f^E & \xrightarrow{j} & E_{id}^E & \xrightarrow{\bar{p}} & B_p^E \\ {\scriptstyle\omega}\downarrow & & {\scriptstyle\omega}\downarrow & & {\scriptstyle\omega}\downarrow \\ F & \xrightarrow{i} & E & \xrightarrow{p} & B \end{array}$$

By this diagram, we have the following commutative diagram

$$\begin{array}{ccc} \pi_n(B_p^E, p) & \xrightarrow{\delta} & \pi_{n-1}(F_c^E, c) \\ \searrow J & & \swarrow \varepsilon_2 \\ & \pi_n(\bar{p}) & \\ \downarrow \omega_\sharp & \downarrow \omega_\sharp & \downarrow \omega_\sharp \\ & \pi_n(p) & \\ \nearrow J & & \nwarrow \varepsilon_2 \\ \pi_n(B) & \xrightarrow{\delta} & \pi_{n-1}(F) \end{array}$$

where c is the constant map from E to F. Thus we can show easily ${\varepsilon_2}^{-1}J = \delta id$.

The commutativities of the remaining parts are similar. □

Let us consider the Hopf bundle $S^3 \xrightarrow{i} S^7 \xrightarrow{p} S^4$ as a principal fibration. Then we have the following theorem.

THEOREM 4.4. *Let $S^3 \xrightarrow{i} S^7 \xrightarrow{p} S^4$ be the Hopf bundle. Then*

$$\cdots \to G_4(S^7) \xrightarrow{p_\sharp} G_4^p(S^4, S^7) \xrightarrow{J} G_4^R(p) \xrightarrow{\partial} G_3(S^7) \to \cdots$$

is not exact at $G_4^R(p)$.

A map $f : A \to B$ is *cyclic* [4] if there exists an affiliated map F with a homotopy commutative diagram

$$\begin{array}{ccc} A \times B & \xrightarrow{F} & B \\ \uparrow & & \nearrow f \vee 1_X \\ A \vee B. & & \end{array}$$

We will use the following lemmas for the proof of Theorem 4.4.

LEMMA 4.5. *$f : \Sigma A \to \Sigma B$ is cyclic if and only if $[f, id_{\Sigma B}] = 0$, where $[,]$ is the Whitehead product* (*see* [4]).

LEMMA 4.6. *Let $\iota_4 \in \pi_4(S^4)$ be the identity element. Then we have $[\iota_4, [\iota_4, \iota_4]] \neq 0$* (*see* [7,13]).

Proof of Theorem 4.4. Suppose the theorem is not true. Then we have the isomorphisms $J : G_4^p(S^4, S^7) \xrightarrow{\cong} G_4^R(p)$ and $\delta : \pi_4(S^4) \xrightarrow{\cong} \pi_3(S^3)$ and by Theorem 4.2, we have $G_4^R(p) \cong \pi_3(S^3)$.

By Theorem 4.3, we have the following commutative diagram

$$\begin{array}{ccc} G_4^p(S^4, S^7) & \xrightarrow{\cong} & G_4^R(p) \\ \downarrow & & \downarrow\cong \\ \pi_4(S^4) & \xrightarrow{\cong} & \pi_3(S^3) \end{array}$$

Since three homomorphisms in the above diagram are isomorphisms, the last one is also isomorphism. This is equivalent to surjectivity of the evaluation homomorphism $\omega_\sharp : \pi_4(S^{4^{S^7}}, p) \to \pi_4(S^4)$. Thus $\iota_4 \in Im\omega_\sharp$ which is , in turn, equivalent to the fact that p is cyclic. By Lemma 4.5, this implies $[\iota_4, p] = [\iota_4, [\iota_4, \iota_4]] = 0$ which contradicts Lemma 4.6. $\square$

In [9], it was shown that ω-homology group $H_n^{g\omega}(K(Z,n), S^n)$ is nontrivial for all n except $n = 1, 3, 7$. But $(K(Z,n), S^n)$ is not a finite CW-pair. The following corollary show an existence of finite CW-pair with nontrivial ω-homology in high order.

COROLLARY 4.7. *There is an example of a finite CW-pair with nonexact G-sequence and nontrivial ω-homology in high order.*

Proof. Let $S^3 \xrightarrow{i} S^7 \xrightarrow{p} S^4$ be the Hopf bundle and Z_p be the mapping cylinder of p. From Corollary 3.4, we have the G-sequence

$$\cdots \to G_4(S^7) \longrightarrow G_4(Z_p, S^7) \longrightarrow G_4^{Rel}(Z_p, S^7) \longrightarrow G_3(S^7) \to \cdots$$

of $i : S^7 \to Z_p$ which is isomorphic to the G-sequence of p. Thus the G-sequence of (Z_p, S^7) is not exact at $G_4^{Rel}(Z_p, S^7)$ and hence $H_4^{r\omega}(Z_p, S^7)$ is nontrivial. □

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