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PROPERTIES OF A SEQUENCE SPACE $l(s,t)$

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ABSTRACT. Elementary properties of the sequence space $l(s,t)$ are studied with applications to Hardy space theory.

1. Introduction

Let N be the set of nonnegative integers. Corresponding to a sequence $a = \{a_k\}_{k\in N}$ and $0 < s,t < \infty$, we define

$$\|a\|_{(s,t)} = \left\{ \sum_{n\in N} \left(\sum_{k\in I_n} |a_k|^s \right)^{t/s} \right\}^{1/t},$$

where

$$I_n = \begin{cases} \{k \in N;\ 2^{n-1} \le k < 2^n\}, & n > 0 \\ \{0\}, & n = 0 \end{cases}.$$

When s or t is ∞, $\|a\|_{(s,t)}$ is defined also by use of corresponding suprema.

We say that $a = \{a_k\}_{k\in N}$ belongs to the set $l(s,t)$ if $\|a\|_{(s,t)} < \infty$, Of course, $l(s,s) = l^s$. $l(s,t)$ is a Banach space equipped with the norm $\|\cdot\|_{(s,t)}$ if $1 \le s,\ t \le \infty$. When $u = min(s,t) < 1$, $l(s,t)$ is a Fréchet space equipped with the translation invariant metric $\|a-b\|^u_{(s,t)}$ for $a, b \in l(s,t)$. See [5] for $l(s,t)$ spaces.

Let $H(D)$ denote the set of all holomorphic functions defined on the open unit disc $D = \{z : |z| < 1\}$ of the complex plane. If $f(z) = \sum_{k\in N} a_k z^k \in H(D)$, then its sequence of Taylor coefficients $\{a_k\}_{k\in N}$,

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denoted in this paper by $f^\wedge$, is uniquely determined and vice versa, so that we can identify f with $f^\wedge$.

Let A and B be two sets of complex sequences and $a = \{a_k\}_{k\in N} \in A$, $b = \{b_k\}_{k\in N} \in B$. Then we denote $\{a_k b_k\}_{k\in N}$ by $a * b$ and $\{a * b : a \in A, b \in B\}$ by $A * B$. We use the following conventions throughout this paper: For β real and $a = \{a_k\}_{k\in N}$, we denote

$$(I^\beta)^\wedge = \{(k+1)^{-\beta}\}_{k\in N}, \ \ I^\beta a = (I^\beta)^\wedge * a,$$

and

$$I^\beta A = \{I^\beta a : a \in A\}.$$

Of course, $I^\beta f^\wedge = (I^\beta f)^\wedge$ if $f \in H(D)$, where $I^\beta f$ denotes the fractional integral (or derivative) of f of order β. See [4]. We denote, $r^\wedge = \{r^k\}_{k\in N}$ and $f_r(z) = f(rz)$, for $0 \le r \le 1$ and $z \in D$, so that for example, $f_r^\wedge = f^\wedge * r^\wedge$. Also throughout, $C(p, q, ..)$ denotes a positive constant depending only on $p, q, ..$ not necessarily the same at each occurrence, and p' denotes the conjugate exponent of $p : p + p' = pp'$ if $1 \le p \le \infty$.

The purpose of this note is to study basic properties of the sequence space $l(s,t)$ that may be applicable to the researches of the related fields. Our main result is the following, which may seem a little bit striking at the first glance.

THEOREM 1. *For $0 < p < \infty$, $0 < s \le \infty$, $0 < t \le \infty$ and $-1 < \alpha < \infty$, there is a positive constant $C = C(p, s, t, \alpha)$ such that for any function $f(z) = \sum_{k\in N} a_n z^n \in H(D)$ the following holds :*

$$\frac{1}{C}\int_0^1 (1-r)^\alpha \|f_r^\wedge\|_{(s,s)}^p \, dr \le \int_0^1 (1-r)^\alpha \|f_r^\wedge\|_{(s,t)}^p \, dr \le C\int_0^1 (1-r)^\alpha \|f_r^\wedge\|_{(s,s)}^p \, dr.$$

The point of Theorem 1 is that the quantaties are independent of the index t. After describing some elementary properties of $l(s,t)$, Theorem 1 will be proven in Section 3 with more equivalent quantaties. There are some consequences of Theorem 1 in Section 4.

2. Elementary properties of $l(s,t)$

THEOREM 2. *For a sequence $\{a_n\}$, the followings hold.*

(1) *Let* $0 < \gamma < \infty$. *Then*

$$\sum_{1}^{M} n^{\gamma}|a_n|^{\gamma} = O\left(M^{\gamma}\right)$$

if and only if $\{a_n\} \ \varepsilon \ l(\gamma,\infty)$.
(2) *Let* $0 < \alpha < \infty,\ -\infty < \beta < \infty$ *and* $0 < s,\ t \leq \infty$. *Then*

$$\{|a_n|^{\alpha}\} \in I^{\beta}\ l(s,t)$$

if and only if

$$\{a_n\} \in I^{\frac{\beta}{\alpha}}\ l(\alpha s, \alpha t).$$

(3) *Let* $0 \leq \alpha < \infty,\ 0 < \beta, \gamma < \infty$. *Then*

$$\sum_{n=1}^{M} n^{\alpha}|a_n|^{\beta} = O\left(M^{\gamma}\right)$$

if and only if

$$\{a_n\} \in I^{(\alpha-\gamma)/\beta}\ l(\beta,\infty).$$

(4) *Let* $0 < \alpha \leq \infty$ *and* $0 < s,t \leq \infty$. *Then*

$$I^{\alpha}l(s,t) \subset l\left(\frac{s}{\alpha s+1}, t\right).$$

(5) *Let* $0 < \alpha < \infty$, $0 < s \leq 1/\alpha$ *and* $0 < t \leq \infty$. *If* $|a_1| \geq |a_2| \geq \ldots \geq 0$, *then* $\{a_n\} \in l(s,t)$ *if and only if*

$$\{n^{\alpha}a_n\} \in l\left(\frac{s}{1-s\alpha}, t\right)$$

with the understanding that $s/(1-s\alpha) = \infty$ *if* $s\alpha = 1$.

Proof. (1) : If $\{a_n\}$ satisfies

$$\sum_{1}^{M} n^{\gamma}|a_n|^{\gamma} = O\ (M^{\gamma}),$$

then

$$\sum_{k\varepsilon I_n} |a_k|^{\gamma} \leq 2^{-(n-1)\gamma} \sum_{k\varepsilon I_n} k^{\gamma}|a_k|^{\gamma} = O(1).$$

Thus $\{a_n\}\ \varepsilon\ l(\gamma,\infty)$. Conversely, if

$$\sum_{I_n} |a_k|^{\gamma} \leq C < \infty$$

then

$$\sum_{I_n} k^{\gamma}|a_k|^{\gamma} \leq 2^{\gamma n} C$$

for $n = 0, 1, 2, \ldots$. Hence, if $M\ \varepsilon\ I_m$ then

$$\begin{aligned}\sum_{n=1}^{M} n^{\gamma}|a_n|^{\gamma} &\leq \sum_{n=1}^{m}\sum_{k\varepsilon I_n} k^{\gamma}|a_k|^{\gamma}\\ &\leq \sum_{n=1}^{m} 2^{\gamma n} C\\ &= O\ (M^{\gamma}).\end{aligned}$$

(2) : Obvious by definition.
(3) : By (1),

$$\sum_{n=1}^{M} n^{\gamma}(n^{-1+\alpha/\gamma}|a_n|^{\beta/\gamma})^{\gamma} = \sum_{n=1}^{M} n^{\alpha}|a_n|^{\beta} = O(M^{\gamma})$$

if and only if

$$\{n^{-1+\alpha/\gamma}|a_n|^{\beta/\gamma}\} \in l(\gamma,\infty).$$

This is equivalent to

$$\{|a_n|^{\beta/\gamma}\} \in I^{(\alpha-\gamma)/\gamma}\ l(\gamma,\infty)$$

by definition, which is in turn equivalent to

$$\{a_n\} \in I^{(\alpha-\gamma)/\beta}\, l(\beta,\infty)$$

by (2).
(4) : It is easy to see that

$$(I^\alpha)^\wedge = \{(n+1)^{-\alpha}\} \in l\left(\frac{1}{\alpha},\infty\right),$$

so that

$$I^\alpha l(s,t) \subset l\left(\frac{1}{\alpha},\infty\right) * l(s,t).$$

Hölder's inequality gives

$$l\left(\frac{1}{\alpha},\infty\right) * l(s,t) \subset l(u,t),$$

where $u = s/(\alpha s+1)$. Indeed, if $a = \{a_n\} \in l\left(\frac{1}{\alpha},\infty\right)$ and $b = \{b_n\} \in l(s,t)$, $0 < s,t < \infty$, then

$$\begin{aligned}\|a*b\|^t_{(u,t)} =& \sum_n \left(\sum_{k\varepsilon I_n} |a_k b_k|^u\right)^{t/u} \\ \leq& \sum_n \left(\sum_{I_n} |a_k|^{1/\alpha}\right)^{\alpha t} \left(\sum_{I_n} |b_k|^s\right)^{t/s} \\ \leq& \|a\|^t_{(1/\alpha,\infty)}\ \|b\|^t_{(s,t)};\end{aligned}$$

Similar pattern also holds when s or t is ∞.
(5) : We may assume $a_n \geq 0,\ n = 1,2,...$ If we assume

$$\{n^\alpha a_n\} \in l\left(\frac{s}{1-s\alpha},t\right),$$

then

$$\{a_n\} = \{n^{-\alpha}\} * \{n^\alpha a_n\} \in I^\alpha\, l\left(\frac{s}{1-s\alpha},t\right) \subset l(s,t)$$

by (4) in the last containment . Conversely, if we assume $\{a_n\} \in l(s,t)$, $0 < t < \infty, 0 < s < \frac{1}{\alpha}$, and set $s/(1-s\alpha) = u$ then

$$\begin{aligned}\sum_{n=1}^{\infty}\left(\sum_{k\in I_n} k^{\alpha u}a_k^u\right)^{t/u} &\leq \sum_{n=1}^{\infty} 2^{\alpha tn}\left(\sum_{k\in I_n} a_k^u\right)^{t/u}\\ &\leq 2^{\alpha t}\sum_{n=1}^{\infty}(2^n a_{2^n}^s)^{t/s}\\ &\leq 2^{\alpha t+(t/s)}\sum_n\left(\sum_{k\in I_n} a_k^s\right)^{t/s}\\ &< \infty,\end{aligned}$$

where we used the hypothesis $|a_1| \geq |a_2|...$ in the third inequality. When u or t is ∞, the same proof also holds by replacing the corresponding summations by suprema. □

3. Proof of main result

We recall the following useful result.

THEOREM A. ([7]) *If* $0 < p < \infty$, $0 < \alpha < \infty$, $a_n \geq 0$, $t_n = \sum_{k\in I_n} a_k$, *and* $f(x) = \sum_1^\infty a_n x^n$, *then there is a* $C = C(p,\alpha)$ *such that*

$$(1) \qquad \frac{1}{C}\sum_1^\infty 2^{-n\alpha}t_n^p \leq \int_0^1 (1-x)^{\alpha-1}f(x)^p dx \leq C\sum_1^\infty 2^{-n\alpha}t_n^p.$$

$$(2) \qquad (1)\ \textit{remains valid if } t_n \textit{ is replaced by } S_{2^n}, \textit{where} S_n = \sum_1^n a_k.$$

$$(3) \quad \frac{1}{C}\sum_1^\infty n^{-(\alpha+1)}S_n^p \leq \int_0^1 (1-x)^{\alpha-1}f(x)^p\, dx \leq C\sum_1^\infty n^{-(\alpha+1)}S_n^p.$$

For two real valued functions F and G, we say that F and G are equivalent if there is a constant C which is independent on b of the domain of definition such that $\frac{1}{C}F(b) \leq G(b) \leq CF(b)$.

Using our new language, Theorem A can be expressed as

THEOREM B. *Let $0 < p < \infty$ and $0 < \alpha < \infty$. Then any two of the following quantities are equivalent as functions of sequences $f^\wedge = \{a_n\}_{n=1}^\infty$.*

$$F := \|I^{\alpha/p} f^\wedge\|_{(1,p)}^p$$

$$G := \sum_1^\infty 2^{-n\alpha} \left(\sum_{k=1}^{2^n} |a_k| \right)^p$$

$$H := \sum_1^\infty n^{-(\alpha+1)} \left(\sum_{k=1}^{n} |a_k| \right)^p$$

$$I := \int_0^1 (1-r)^{\alpha-1} \|f_r^\wedge\|_{(1,1)}^p \, dr.$$

Now Theorem 1 and Theorem B are parts of the following, where empathesis should be given at the point that the quantities are independent on the index t.

THEOREM 3. *Let $0 < p < \infty, -1 < \alpha < \infty$ and $0 < s,t \leq \infty$. Then any two of the following quantities are equivalent as functions of sequences $f^\wedge = \{a_n\}_{n=1}^\infty$.*

$$J := \|I^{(\alpha+1)/p} f^\wedge\|_{(s,p)}^p$$

$$K := \sum_1^\infty 2^{-n(\alpha+1)} \left(\sum_{k=1}^{2^n} |a_k|^s \right)^{p/s}$$

$$L := \sum_1^\infty n^{-(\alpha+2)} \left(\sum_{k=1}^{n} |a_k|^s \right)^{p/s}$$

$$M := \int_0^1 (1-r)^\alpha \|f_r^\wedge\|_{(s,s)}^p \, dr$$

$$P := \int_0^1 (1-r)^\alpha \|f_r^\wedge\|_{(s,t)}^p \, dr$$

with the obvious understanding of K and L when $s = \infty$.

Proof. We may assume $a_n \geq 0,\ n = 1, 2, ...$ First assume $s, t < \infty$. By Theorem B, we are suffice to prove that there exists $C = C(s, t, p, \alpha)$ such that

$$J \leq CP \leq CK.$$

If we set $r_n = 1 - 2^{-n},\ n = 1, 2, ...$ then it follows that

$$\begin{aligned}
P &= \int_0^1 (1-r)^\alpha \|f_r^\wedge\|_{(s,t)}^p \, dr \\
&\geq \sum_{n=1}^\infty \int_{r_n}^{r_{n+1}} (1-r)^\alpha \|f_r^\wedge\|_{(s,t)}^p \, dr \\
&\geq \sum_{n=1}^\infty (r_{n+1} - r_n)(1 - r_{n+1})^\alpha \left\{ \sum_{m=1}^\infty \left(\sum_{k \in I_m} |a_k r_n^k|^s \right)^{t/s} \right\}^{p/t} \\
&\geq 2^{-(\alpha+1)} \sum_{n=1}^\infty 2^{-(\alpha+1)n} \left(\sum_{k \in I_n} |a_k|^s \right)^{p/s} \left(r_n^{2^n} \right)^p \\
&\geq 2^{-(\alpha+1)-p} \|\{k^{-(\alpha+1)/p} a_k\}\|_{(s,p)}^p \\
&\geq C(p, \alpha) J,
\end{aligned}$$

where we used that $r_n^{2^n} \geq \frac{1}{2}$ in the last inequality.

Next, if we set $b_n = (\sum_1^n a_k^s)^{t/s}$ and $\gamma = min(\frac{t}{2}, 1)$ then

$$\|f_r^\wedge\|_{(s,t)}^p = \|\{a_n r^n\}\|_{(s,t)}^p \leq \left(\sum_1^\infty b_{2^n} r^{\gamma 2^n} \right)^{p/t}.$$

Since b_n is nondecreasing,

$$b_{2^n} (r^\gamma)^{2^n} \leq 2^{1-n} \sum_{k \in I_n} b_k r^{\gamma k} \leq 2 \sum_{k \in I_n} k^{-1} b_k r^{\gamma k},$$

so that $\|f_r^\wedge\|_{(s,t)}^p$ is bounded by

$$\left(2 \sum_1^\infty n^{-1} b_n r^{\gamma n} \right)^{p/t}.$$

Thus, by use of the fact that $\gamma \leq 1$ and that $(1-r^{1/\gamma}) \leq \frac{1}{\gamma}(1-r)$, $0 \leq r \leq 1$, we have

$$\begin{aligned} P &\leq \int_0^1 (1-r)^\alpha \left(2\sum_1^\infty n^{-1}b_n r^{\gamma n}\right)^{p/t} dr \\ &\leq 2^{p/t} \int_0^1 (1-r^{1/\gamma})^\alpha \left(2\sum_1^\infty n^{-1}b_n r^n\right)^{p/t} r^{1/\gamma-1}dr \\ &\leq C(\alpha,\gamma) \int_0^1 (1-r)^\alpha \left(2\sum_1^\infty n^{-1}b_n r^n\right)^{p/t} dr. \end{aligned}$$

Hence, by Theorem A-(1),

$$P \leq C(p,t,\alpha) \sum_{n=1}^\infty 2^{-n(\alpha+1)} \left(\sum_{k\in I_n} k^{-1}b_k\right)^{p/t}.$$

Now, an application of Hölder's inequality gives

$$\begin{aligned} \sum_{k\in I_n} k^{-1}b_k &= \sum_{k\in I_n} k^{-1} \left(\sum_{j=1}^k a_j^s\right)^{p/s} \\ &\leq \left(\sum_{k\in I_n} k^{-1}\right) \left(\sum_{j=1}^{2^n} a_j^s\right)^{p/s} \leq \left(\sum_{j=1}^{2^n} a_j^s\right)^{p/s}. \end{aligned}$$

Therefore $P \leq C(p,t,\alpha)K$. Our method of proof remains true when s or t is ∞. The proof is complete. □

4. Applications

To give illustrations, we introduce some definitions. For $-1 < \alpha$, $0 < p < \infty$, $0 < q < \infty$, $H^q = H^q(D)$ and $A^{p,q,\alpha} = A^{p,q,\alpha}(D)$ denote the space of those functions f holomorphic in D satisfying

$$\|f\|_q = \sup_{0\leq r<1} M_q(r,f) < \infty$$

and

$$\|f\|_{p,q,\alpha}^p = \int_0^1 (1-r)^\alpha M_q(r,f)^p dr < \infty$$

respectively, where

$$M_q(r,f) = \left\{ \int_0^{2\pi} |f(re^{i\theta})|^q \frac{d\theta}{2\pi} \right\}^{1/q}.$$

The Hardy space H^p can be considered as the limiting case of the weighted Bergman space $A^{p,\alpha} = A^{p,p,\alpha}$ as $\alpha \to -1$. They are Banach spaces when $1 \leq p,\ q < \infty$ and Fréchet spaces when $min(p,q) < 1$. See [1] and [2] for H^p spaces and $A^{p,q,\alpha}$ spaces.

Now, the following follows almost directly from Theorem 3. It improves [6, Theorem 1].

THEOREM 4. *There are constants $C = C(p,q,\alpha)$ satisfying the followings respectively.*
(1) *If $1 \leq q \leq 2$ and $f \in A^{p,q,\alpha}(D)$, then*

$$\|I^{(\alpha+1)/p} f^\wedge\|_{(q',p)} \leq C \|f\|_{p,q,\alpha}.$$

(2) *If $2 \leq q < \infty$ and $b \in I^{-(\alpha+1)/p}\, l(q',p)$, then $f(z) = \sum b_k z^k \in A^{p,q,\alpha}(D)$ and*

$$\|f\|_{p,q,\alpha} \leq C \|I^{(\alpha+1)/p}\, b\|_{(q',p)}.$$

Proof. For (1), first apply Hausdorff-Young Theorem to $f_r \in H^p$, $0 < r < 1$, to obtain

$$\|f_r^\wedge\|_{(q',q')} \leq M_q(r,f),$$

then integrate p power of both sides with respect to $(1-r)^\alpha dr$ and apply the equivalence of J and M in Theorem 3 to obtain (1). Similar way gives (2). □

Here we have another application in order to illustrate the independence on t of the quantities in Theorem 3, where we set $(H^p)^\wedge = \{f^\wedge : f \in H^p\}$.

THEOREM 5. *Let* $0 < s \leq \infty$ *and* $0 < p < \infty$. *If* $(H^p)^\wedge \subset l(s,\infty)$ *then* $(H^p)^\wedge \subset l(s, max(p,2))$. *If* $(H^p)^\wedge \supset l(s,\infty)$ *then* $(H^p)^\wedge \supset l(s, max(p,2))$.

Proof. We only prove the first half left the similar second. Let $q = max(p,2)$. Since $(H^p)^\wedge \subset l(s,\infty)$, by the closed graph theorem, there is a C independent of f such that

$$\|f^\wedge\|_{(s,\infty)} \leq C\,\|f\|_p, \quad f \in H^p.$$

In particular,

$$\|(f_r')^\wedge\|_{(s,\infty)} \leq C\,M_p(r,f'), \quad 0 < r < 1.$$

Now, integrating q power of the both sides with respect to $(1-r)^{q-1}dr$ gives

$$\int_0^1 (1-r)^{q-1}\,\|(f_r')^\wedge\|^q_{(s,\infty)}\,dr \leq C\int_0^1 (1-r)^{q-1}M_p(r,f')^q\,dr.$$

By the equivalence of J and P in Theorem 3, the left side is equivalent to $\|f^\wedge\|^q_{(s,q)}$, while the right side integral is, by the Minkowski's inequality in continuous form, at most the q/p power of

$$\int_0^{2\pi}\left\{\int_0^1 (1-r)^{q-1}|f'(re^{i\theta})|^q\,dr\right\}^{p/q}\frac{d\theta}{2\pi},$$

which is, by Littlewood-Paley type inequality (see [3]), at most $C(p,q)$ $\|f\|_p^q$. Therefore we have

$$\|f^\wedge\|_{(s,q)} \leq C\|f\|_p, \quad f \in H^p$$

for some C independent on f. □

If Λ is a bounded linear operator from H^p into $l(s,t)$ satisfying the property

$$\Lambda(f_r) = \Lambda(f) * r^\wedge, \quad 0 < r < 1,$$

then $\Lambda(H^p) \subset l(s, max(p,2))$. This (and its dual result) follows by a way similar to that in the proof of Theorem 5.

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