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STABILITY OF EQUIVALENT PROGRAMMING PROBLEMS OF THE MULTIPLE OBJECTIVE LINEAR STOCHASTIC PROGRAMMING PROBLEMS

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ABSTRACT. In this paper the stochastic multiple objective programming problems where the right-hand-side of the constraints is stochastic are considered. We define the equivalent scalar-valued problem and study the stability of the equivalent scalar-valued problem with respect to the weight parameters and probability mesures under reasonable assumptions.

1. Introduction

This paper is a sequel to [1], where the following multiple objective two-stage programming problem with complete fixed recourse was investigated.

$$\begin{aligned} &\text{VMIN} \;\; g(x) + E_\xi[\min d'y]\cdot \mathbf{1} \\ &\text{subject to} \;\; Ax = b \\ &\qquad Dx + Wy = \xi, \quad \xi \text{ on } (\Xi, \Im, \mu) \\ &\qquad x \geq 0, \quad y \geq 0, \end{aligned} \tag{1}$$

where A is an $m \times n$ matrix, D is an $\bar{m} \times n$ matrix, x is an $n-$vector, d and y are $\bar{n}-$vectors, $\mathbf{1} = (1, \cdots, 1)'$ is a $r-$vector, b is an $m-$vector and ξ is a random vector defined on the probability measure space $(\Xi, \Im, \mu)$, $g(x) = (g_1(x), \cdots, g_r(x))'$, $g_i(x) = c_{i1}x_1 + \cdots + c_{in}x_n$, $1 \leq i \leq r$, is a linear function, $W \in L(\mathbf{R}^{\bar{n}}, \mathbf{R}^{\bar{m}})$ such that for all $\xi \in \Xi$, $\{y \in$

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$\mathbf{R}^{\bar{n}} \mid Wy = \xi - Dx,\ y \geq 0\} \neq \emptyset$ and $\{u \in \mathbf{R}^{\bar{m}} \mid W'u \leq d\} \neq \emptyset$, and E_ξ denotes the mean operator with respect to ξ.

All quantities considered here belong to the reals. We use the notation ξ to denote a random vector of dimension $\bar{m}$, as well as the specific values assumed by this random variable.

In [1], Cho defined an equivalent scalar-valued problem for the problem (1) and observed the stability of the problem with respect to the weight parameter and probability distributions of the random vector, respectively. In this paper we study the stability of the equivalent scalar-valued problem with respect to the weight parameters and the probability distributions.

In the following we will introduce the parametric programming analysis due to D. Klatte([3]) which we are going to use to prove our main result. Let us consider the following programming problem with a parameter $t \in T$:

$$p(t) : \min\{f(x,t) : x \in M(t)\},$$

where T is a metric space with distance function $d(\cdot,\cdot)$, M is a closed-valued multifunction from T into $\mathbf{R}^n$, and $f : \mathbf{R}^n \times T \to \mathbf{R}$ is a continuous function. Given $Q \subset \mathbf{R}^n$, for any $t \in T$, we define

$$M_Q(t) = M(t) \cap clQ,$$

$$\rho_Q(t) = \inf\{f(x,t) \mid x \in M_Q(t)\},$$

$$\psi_Q(t) = \{x \in M_Q(t) \mid f(x,t) = \rho_Q(t)\},$$

where clQ denotes the closure of Q. We call ρ_Q the optimal value function with respect to clQ and ψ_Q the optimal set function with respect to clQ. Now we shall give some basic definitions and theorems.

DEFINITION 1.1. Given $t^o \in T$, a nonempty set $X \subset \mathbf{R}^n$ is called a *complete local minimizing set* (CLM set) for $f(\cdot,t^o)$ on $M(t^o)$ if there is an open set Q containing X such that $X = \psi_Q(t^o)$.

Note that a CLM set with respect to $p(t^o)$ is always a subset of the set of all local minimizers of $p(t^o)$, and CLM set are closed under our general assumptions on the problem $p(\cdot)$.

DEFINITION 1.2. $M : T \to \mathbf{R}^n$, is said to be *closed* at t^o if and only if $t^k \to t^o$, $x^k \to x^o$, as $k \to \infty$, and $x^k \in M(t^k) \Rightarrow x^o \in M(t^o)$.

DEFINITION 1.3. A multifunction M from T to $\mathbf{R}^n$ is said to be *pseudo-Lipschitzian* at (x^o, t^o), where $t^o \in T$ and $x^o \in M(t^o)$, if there are neighborhoods $U = U(t^o)$ and $V = V(x^o)$ and a positive real number L such that both

$$M(t) \cap V \subset M(t^o) + L \cdot d(t, t^o) \cdot B_n \text{ and}$$
$$M(t^o) \cap V \subset M(t) + L \cdot d(t, t^o) \cdot B_n$$

hold for all $t \in U$, where B_n is the closed unit ball in $\mathbf{R}^n$, and

$$X + \beta \cdot B_n = \{x + \beta \cdot u \mid x \in X, \; u \in B_n\},$$

for $X \subset \mathbf{R}^n$ and $\beta \in \mathbf{R}$.

THEOREM 1.4. ([3]) *Consider the parametric program $p(t)$: fix some $t^o \in T$ with the following conditions:*

$(C1)$ Assume that there exists a bounded open subset V of $\mathbf{R}^n$ and a nonempty subset X of V such that $X = \psi_V(t^o)$.

$(C2)$ Let the multifunction M be closed-valued and closed at t^o.

$(C3)$ Let M be a pseudo-Lipschitzian at each pair $(x^o, t^o) \in \psi_V(t^o) \times \{t^o\}$.

$(C4)$ Suppose there are real numbers $p \in (0, 1]$, $L_f > 0$ and $\delta_f > 0$ such that

$$\mid f(x, t^o) - f(y, t) \mid \leq L_f(\| x - y \| + d(t, t^o)^p)$$

for each x, $y \in clV$ and each t satisfying $d(t, t^o) < \delta_f$.

Then the following conclusions hold:

(a) The multifunction ψ_V is upper semicontinuous at t^o, i.e., for each $\epsilon > 0$ there exists $\eta > 0$ such that

$$\psi_V(t) \subset \psi_V(t^o) + \epsilon \cdot B_n \quad \text{when } d(t, t^o) < \eta.$$

(b) There exist positive real numbers δ_ρ and L_ρ such that $\psi_V(t) \neq \emptyset$ is a CLM set for $f(\cdot, t)$ on $M(t)$ and such that

$$\mid \rho_V(t) - \rho_V(t^o) \mid \leq L_\rho \cdot d(t, t^o)^p \quad \text{whenever } d(t, t^o) < \delta_\rho.$$

It is clear that we could also write problem (1) as:

$$(2)\qquad \begin{array}{ll} \text{VMIN} & g(x) + E_\xi[\min d'y : Wy = \xi - Dx, y \geq 0] \cdot \mathbf{1} \\ \text{subject to} & Ax = b \\ & x \geq 0. \end{array}$$

Now we define a feasible solution to problem (1).

DEFINITION 1.5. ([5]) A *feasible solution* to (1) is a vector x such that it satisfies the first stage constraints and such that for any $\xi \in \Xi$, it is always possible to find a feasible solution to the second stage problem $\min\{d'y \mid Wy = \xi - Dx,\ y \geq 0\}$.

Let K be the set of feasible solutions of (1), then

$$K = \{x \mid Ax = b, x \geq 0\} \cap \{x \mid \forall\xi,\ \exists\, y \geq 0 \text{ such that } Wy = \xi - Dx\}.$$

Then K is a convex polyhedron. Define

$$Q(x,\xi) = \min\{d'y \mid Wy = \xi - Dx,\ y \geq 0\}$$

and

$$Q(x) = E_\xi[Q(x,\xi)].$$

Then $Q(x)$ is convex and continuous. So we have an equivalent programming problem to (1).

$$(3)\qquad \begin{array}{ll} \text{VMIN} & F(x) = (g_1(x) + Q(x), \cdots, g_r(x) + Q(x))' \\ \text{subject to} & x \in K. \end{array}$$

DEFINITION 1.6. ([2]) The vector x^* is an *efficient solution* of

$$\begin{array}{ll} \text{VMIN} & F(x) = (f_1(x), \cdots, f_r(x))' \\ \text{subject to} & x \in K \end{array}$$

if and only if there exists no $x \in K$ such that $f_i(x) \leq f_i(x^*)$ for $i = 1, \cdots, r$ and such that for at least one i_o one has $f_{io}(x) < f_{io}(x^*)$.

DEFINITION 1.7. ([2]) The vector x^* is a *properly efficient solution* of

$$\begin{aligned} \text{VMIN} \quad & F(x) = (f_1(x), \cdots, f_r(x))' \\ \text{subject to} \quad & x \in K \end{aligned}$$

if and only if it is efficient and if there exists a scalar $M > 0$ such that for each i and each $x \in K$ satisfying $f_i(x) < f_i(x^*)$, there exists at least one j such that $f_j(x^*) < f_j(x)$ and $(f_i(x^*) - f_i(x))/(f_j(x) - f_j(x^*)) \leq M$.

We define an equivalent scalar-valued problem for the problem (1) as follows;

$$\begin{aligned} \text{minimize} \quad & \sum_{i=1}^{r} \lambda_i g_i(x) + Q(x) \\ \text{subject to} \quad & x \in K, \end{aligned} \tag{4}$$

where $\sum_{i=1}^{r} \lambda_i = 1, \ \lambda_i > 0, \ i = 1, \cdots, r$.

CORORALLY 1.8. [1] *x^* is properly efficient in problem* (3) *if and only if x^* is optimal in problem* (4) *for some λ with strictly positive components.*

2. Stability analysis

When we consider the stochastic programming problems, their stability with respect to the perturbations of the distributions of the underlying random variables plays an essential role. And since the weight parameter depends on decision makers, the stability with respect to weight parameter is also important. So we investigate the stability of the optimal solution set and optimal value functions to problem (4) with respect to the weight parameter λ and the probability measure μ by applying the analysis of D. Klatte.
Define for $q \in (1, \infty)$ and $k \in (0, \infty)$,

$$\wp(\mathbf{R}^{\bar{m}}; q, k) = \{\mu \in \wp(\mathbf{R}^{\bar{m}}) \mid \int_{\mathbf{R}^{\bar{m}}} \| \xi \|^{2q} \mu(d\xi) \leq k\},$$

where $\wp(\mathbf{R}^{\bar{m}})$ is the set of all Borel probability measures on $\mathbf{R}^{\bar{m}}$. We restrict on $\wp(\mathbf{R}^{\bar{m}}; q, k)$ in our stability analysis and consider the weight parameter space as a discrete probability measure space because of its characteristics. Now we define suitable metrics on the weight parameter space and probability measure space. Define the bounded Lipschitz metric β on $\wp(\mathbf{R}^{\bar{m}})$ as follows:

$$\beta(\mu,\nu) = \sup\{\mid \int_{\mathbf{R}^{\bar{m}}} g(\xi)\mu(d\xi) - \int_{\mathbf{R}^{\bar{m}}} g(\xi)\nu(d\xi) \mid : g : \mathbf{R}^{\bar{m}} \to \mathbf{R}, \quad \| g \|_{BL} \leq 1\},$$

for any μ and $\nu \in \wp(\mathbf{R}^{\bar{m}})$, where

$$\| g \|_{BL} = \sup_{\xi \in \mathbf{R}^{\bar{m}}} \mid g(\xi) \mid + \sup_{\xi \neq \tilde{\xi}} \frac{\mid g(\xi) - g(\tilde{\xi}) \mid}{d(\xi, \tilde{\xi})} < \infty,$$

and

$$d(\xi, \tilde{\xi}) = \sqrt{(\xi_1 - \tilde{\xi}_1)^2 + \cdots + (\xi_{\bar{m}} - \tilde{\xi}_{\bar{m}})^2},$$

for

$$\xi = (\xi_1, \cdots, \xi_{\bar{m}}),\ \tilde{\xi} = (\tilde{\xi}_1, \cdots, \tilde{\xi}_{\bar{m}}).$$

Note that the metric β metrizes the topology of weak convergence on $\wp(\mathbf{R}^{\bar{m}})$. Let s be a random variable defined on the sample space $\Omega = \{s_1, s_2, \cdots, s_r\}$ and $s_i \in \mathbf{R}$. Define a metric d on Ω by

$$d(s_i, s_j) = \mid s_i - s_j \mid .$$

Then (Ω, d) is a metric space. We may treat $\Lambda = \{\lambda \in \mathbf{R}^r \mid \sum_{i=1}^r \lambda_i = 1, \lambda_i > 0\}$ as a discrete probability measure space on Ω.
Define a metric d on Λ by

$$d(\lambda, \lambda_o) = \sqrt{(\lambda_1 - \lambda_{o1})^2 + (\lambda_2 - \lambda_{o2})^2 + \cdots + (\lambda_r - \lambda_{or})^2},$$

for any $\lambda,\ \lambda_o \in \Lambda$. Then (Λ, d) is a metric space and for any $q > 1$,

$$\sum_{i=1}^{r} \lambda_i \cdot \mid s_i \mid^q \leq M^q,$$

where $M = \max\{| s_i || s_i \in \Omega\}$. Let $\lambda \in \Lambda$, $\mu \in \wp(\mathbf{R}^{\bar{m}}; q, k)$ and $q > 1$. Then for $z := (s, \xi) \in \Omega \times \mathbf{R}^{\bar{m}}$ we obtain

$$\begin{aligned}\int_{\Omega\times\mathbf{R}^{\bar{m}}} \|z\|^{2q} d(\lambda \times \mu) &= \int_{\mathbf{R}^{\bar{m}}} \sum_{i=1}^{r} \lambda_i \sqrt{| s_i |^2 + \|\xi\|^2}^{\,2q} \mu(d\xi) \\ &\leq \sum_{i=1}^{r} \lambda_i \int_{\mathbf{R}^{\bar{m}}} 2^q (M^{2q} + \|\xi\|^{2q}) \mu(d\xi) \\ &= 2^q (M^{2q} + k) < \infty.\end{aligned}$$

For $q > 1$ define a metric $\tilde{d}$ on $\Lambda \times \wp(\mathbf{R}^{\bar{m}}; q, k)$ by

$$\tilde{d}((\lambda, \mu), (\lambda_o, \mu_o)) = d(\lambda, \lambda_o) + \beta(\mu, \mu_o)^{1-\frac{1}{q}}.$$

Then $(\Lambda \times \wp(R^{\bar{m}}; q, k), \tilde{d})$ is a metric space. We apply Theorem 1.4 to $(T, d) = (\Lambda \times \wp(R^{\bar{m}}; q, k), \tilde{d})$. Take $M(t) = K$, fixed. Denote the objective function by

$$F(x, \lambda, \mu) = \sum_{i=1}^{r} \lambda_i g_i(x) + \int_{\mathbf{R}^{\bar{m}}} Q(x, \xi) \mu(d\xi).$$

Define an optimal value function $\varphi \colon \Lambda \times \wp(\mathbf{R}^{\bar{m}}; q, k) \longrightarrow \mathbf{R}$ and an optimal set function $\psi \colon \Lambda \times \wp(\mathbf{R}^{\bar{m}}; q, k) \longrightarrow K$ by

$$\begin{aligned}\varphi(\lambda, \mu) &= \inf_{x\in K} F(x, \lambda, \mu), \\ \psi(\lambda, \mu) &= \{x \in K \mid F(x, \lambda, \mu) = \varphi(\lambda, \mu)\}.\end{aligned}$$

LEMMA 2.1. *Let $B \subset \mathbf{R}^n$ be a nonempty and compact set. Fix $(\lambda_o, \mu_o) \in \Lambda \times \wp(\mathbf{R}^{\bar{m}}; q, k)$ and assume that $\psi(\lambda_o, \mu_o)$ is nonempty and bounded. Then there exist $p \in (0, 1], L_F > 0$ such that*

$$| F(x_o, \lambda_o, \mu_o) - F(x, \lambda, \mu) | \leq L_F (\|x - x_o\| + \tilde{d}((\lambda, \mu), (\lambda_o, \mu_o))^p),$$

for each $x, x_o \in B$ and each $(\lambda, \mu) \in \Lambda \times \wp(\mathbf{R}^{\bar{m}}; q, k)$.

Proof. We have

$$\begin{aligned}
&\mid F(x_o,\lambda_o,\mu_o) - F(x,\lambda,\mu) \mid \\
\leq &\mid F(x_o,\lambda_o,\mu_o) - F(x,\lambda_o,\mu_o) \mid + \mid F(x,\lambda_o,\mu_o) - F(x,\lambda,\mu) \mid \\
\leq &\mid \sum_{i=1}^{r} \lambda_{oi} g_i(x_o) - \sum_{i=1}^{r} \lambda_{oi} g_i(x) \mid + \mid \int_{\mathbf{R}^{\bar{m}}} (Q(x_o,\xi) - Q(x,\xi))\, \mu_o(d\xi) \mid \\
&+ \mid \sum_{i=1}^{r} \lambda_{oi} g_i(x) - \sum_{i=1}^{r} \lambda_i g_i(x) \mid \\
&+ \mid \int_{\mathbf{R}^{\bar{m}}} Q(x,\xi)\, \mu_o(d\xi) - \int_{\mathbf{R}^{\bar{m}}} Q(x,\xi)\, \mu(d\xi) \mid
\end{aligned}$$

Now let $\bar{c} = \max\{\|c_1\|, \cdots, \|c_n\|\}$ then we have

$$\begin{aligned}
\mid \sum_{i=1}^{r} \lambda_{oi} g_i(x_o) - \sum_{i=1}^{r} \lambda_{oi} g_i(x) \mid &\leq \sum_{i=1}^{r} \mid \lambda_{oi} \mid\mid g_i(x_o) - g_i(x) \mid \\
&\leq \sum_{i=1}^{r} \lambda_{oi} \|c_i\| \parallel x_o - x \parallel \\
&\leq \bar{c} \parallel x_o - x \parallel, \\
&\quad \text{by Cauchy-Schwarz inequality.}
\end{aligned}$$

Since $Q(x,\cdot)$ is convex in x, $Q(x,\cdot)$ is Lipschitzian on B. Therefore there exists some constant $\bar{b} > 0$ such that

$$\mid \int_{\mathbf{R}^{\bar{m}}} (Q(x_o,\xi) - Q(x,\xi))\, \mu_o(d\xi) \mid \leq \bar{b} \parallel x_o - x \parallel .$$

Since B is bounded, there exists $\bar{p} > 0$ with $\bar{p} = \max\{ \parallel x \parallel \mid x \in B\}$ and we have

$$\begin{aligned}
\mid \sum_{i=1}^{r} \lambda_{oi} g_i(x) - \sum_{i=1}^{r} \lambda_i g_i(x) \mid &\leq \sum_{i=1}^{r} \mid \lambda_{oi} - \lambda_i \mid\mid g_i(x) \mid \\
&\leq r\bar{c} \parallel \lambda_o - \lambda \parallel\parallel x \parallel, \\
&\leq r\bar{c} \parallel \lambda_o - \lambda \parallel \bar{p}
\end{aligned}$$

For the recourse part it is not hard to verify that for $x \in B$, there exists $M > 0$ such that

$$
\begin{aligned}
& | Q(x,\xi) - Q(x,\tilde{\xi}) | \\
\leq & M \cdot \max\{\| d \| + \| \xi - Dx \|, \| d \| + \| \tilde{\xi} - Dx \|\} \cdot \| \xi - \tilde{\xi} \| \\
\leq & M \cdot K \cdot \max\{\| d \| + \| \xi \| + \| D \|, \| d \| + \| \tilde{\xi} \| + \| D \|\} \cdot \| \xi - \tilde{\xi} \|,
\end{aligned}
$$

where $K = \max\{1, \bar{p}\}$.

Taking $L(t) = M \cdot K \cdot (\| d \| + \| D \|) + M \cdot K \cdot t$, there exists $C > 0$ such that for all for $q > 1$, we have

$$
\begin{aligned}
& | \int_{\mathbf{R}^{\tilde{m}}} Q(x,\xi)\, \mu_o(d\xi) - \int_{\mathbf{R}^{\tilde{m}}} Q(x,\xi)\, \mu(d\xi) | \\
\leq & C(1 + M_q(\mu) + M_q(\mu_o))\beta(\mu,\mu_o)^{1-\frac{1}{q}},
\end{aligned}
$$

where $M_q(\mu) = (\int_{\mathbf{R}^{\tilde{m}}} L_1(\| \xi \|)^q\, \mu(d\xi))^{\frac{1}{q}}$, $L_1(t) = L(t) \cdot t$.

Therefore

$$
\begin{aligned}
| F(x_o,\lambda_o,\mu_o) - F(x,\lambda,\mu) | \leq & (\bar{c} + \bar{b}) \| x_o - x \| + r\bar{c}\bar{p} \| \lambda_o - \lambda \| \\
& + C(1 + M_q(\mu) + M_q(\mu_o))\beta(\mu,\mu_o)^{1-\frac{1}{q}} \\
\leq & L_F(\| x_o - x \| + \tilde{d}((\lambda,\mu),(\lambda_o,\mu_o))),
\end{aligned}
$$

where

$$L_F = \max\{\bar{c} + \bar{b},\ r\bar{c}\bar{p},\ C(1 + M_q(\mu) + M_q(\mu_o))\}. \qquad \square$$

Since for any fixed parameter (λ, μ) problem (4) is a convex programming problem, local optimal solutions and local optimal values are global optimal solutions and global optimal values, respectively. We are now ready to deduce our main result:

THEOREM 2.2. *Consider problem* (4). *Fix* $(\lambda_o, \mu_o) \in \Lambda \times \wp(\mathbf{R}^{\tilde{m}}; q, k)$ *and assume that* $\psi(\lambda_o, \mu_o)$ *is nonempty and bounded. Then it follows that:*

(*a*) *For each* $\varepsilon > 0$ *there exists some* $\eta > 0$ *such that*

$$\psi(\lambda, \mu) \subset \psi(\lambda_o, \mu_o) + \varepsilon \cdot B_n,$$

whenever $(\lambda, \mu) \in \Lambda \times \wp(R^{\bar{m}}; q, k)$ *and* $\tilde{d}((\lambda, \mu), (\lambda_o, \mu_o)) < \eta$.

(*b*) *There exists positive reals* δ_φ *and* L_φ *such that*

$$\psi(\lambda, \mu) \neq \emptyset$$

$$| \varphi(\lambda, \mu) - \varphi(\lambda_o, \mu_o) | \leq L_\varphi \cdot \tilde{d}((\lambda, \mu), (\lambda_o, \mu_o)),$$

whenever $(\lambda, \mu) \in \Lambda \times \wp(\mathbf{R}^{\bar{m}}; q, k)$ *and* $\tilde{d}((\lambda, \mu), (\lambda_o, \mu_o)) < \delta_\varphi$.

Proof. We check the conditions of Theorem 1.4.

$(C1)$: Since $\psi(\lambda_o, \mu_o)$ is a global minimizing solution set, $\psi(\lambda_o, \mu_o)$ is a CLM set. And since $\psi(\lambda_o, \mu_o)$ is bounded by assumption, $\psi(\lambda_o, \mu_o)$ is a bounded complete local minimizing set.

$(C2)$: $M : \Lambda \times \wp(\mathbf{R}^{\bar{m}}; q, k) \to K$, fixed valued. Then M is a constant multifunction. Therefore M is closed valued and closed on $\Lambda \times \wp(\mathbf{R}^{\bar{m}}; q, k)$.

$(C3)$ is trivially held.

$(C4)$: This is proved in Lemma 2.1. The proof is complete. □

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