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PERIODICITY ON CANTOR SETS

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ABSTRACT. In this paper we construct a homeomorphism on a Cantor set which is nearly periodic such that $h(a) = b$ for given $a, b \in D_p$. We also give an example which is not almost periodic and we discuss when a homeomorphism on a Cantor set is periodic.

1. Introduction

p-adic topological group can be constructed by using different methods. Let p be a prime number and let D_p be the set of all formal series in powers of p:

$$g = a_0 + a_1 p + \ldots + a_n p^n + \ldots, \text{ each } a_n = 0, \ldots, p-1.$$

If we add elements with infinite carry-over, then D_p forms an abelian group and the topology is determined by the following choice of neighborhoods of the identity:

$$U_m = \{g \in D_p | a_i = 0 \text{ if } i < m\},\ m = 1, 2, \ldots .$$

We call D_p the *p-adic topological group* or simply a *Cantor set (group)*.

Recall that the infinite carry over operation is the following: Let $g = a_0 + a_1 p + a_2 p^2 \ldots$ and $h = b_0 + b_1 p + b_2 p^2 \ldots$ be elements of D_p.
$g \odot h = c_0 + c_1 p + c_2 p^2 \ldots$, where $c_o \equiv a_0 + b_0 (mod\, p)$,

$$c_i \equiv a_i + b_i (mod\, p) \text{ if } a_{i-1} + b_{i-1} < p$$
$$c_i \equiv a_i + b_i + 1 (mod\, p) \text{ if } a_{i-1} + b_{i-1} \geq p.$$

A Cantor set also can be defined as follows: Let

$$A_i = \{0, 1, \ldots, p-1\} \text{ with discrete topology.}$$

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If we give the product topology on $\prod_{i=1}^{\infty} A_i$, then $\prod_{i=1}^{\infty} A_i$ is homeomorphic to the Cantor set which is constructed by a geometric method [1, page. 104] and it is totally disconnected and compact.

We define a map

$$\phi : \prod_{i=1}^{\infty} A_i \longrightarrow D_p, \text{ by } \phi(a_0, \ldots , a_n, \ldots) = (a_0 + \ldots + a_n p^n + \ldots).$$

Then the map ϕ is continuous and one to one since

$$\phi^{-1}(U_n) = \{< 0, 0, \ldots , 0, a_n, a_{n+1}, \ldots >\} \text{ for } U_n \subset D_p,$$

and the homeomorphism follows from the fact that D_p is a Hausdorff space and $\prod_{i=1}^{\infty} A_i$ is compact. By the above construction, we also denote an element of a Cantor set D_p with $(a_0, \ldots , a_n, \ldots)$ where $a_n = 0, 1, \ldots , p-1$.

Another important construction of the *p-adic* group is the following: Let D_p be the p-adic group which we already constructed. Then

$$U_m = \{g \in D_p | a_i = 0 \text{ if } i < m\},\ m = 1, 2, \ldots$$

form open subgroups and hence closed subgroups, since the cosets of U_m are open in D_p. We consider the sequence of quotient groups

$$D_p/U_1, D_p/U_2, \ldots , D_p/U_n, \ldots .$$

We remark that a group operation on this quotient group is induced by infinite carry over operation on D_p.

For $j > i$, let

$$h_{i,j} : D_p/U_j \longrightarrow D_p/U_i$$

be the continuous homomorphisms defined by $gU_j \longrightarrow gU_i$. Then we have

$$D_p \simeq \varprojlim\{D_p/U_j\} \text{ with bonding map } h_{i,j}.$$

We notice that D_p/U_i is a cyclic group of order p^i. Therefore we can also define the *p-adic* group as the inverse limit of cyclic groups of order p^i for $i = 1, 2, \ldots$.

NOTATION. We will denote an element of D_p/U_i with $(a_0, a_1, \ldots , a_{i-1})$ instead of$(a_0, a_1, \ldots , a_{i-1})U_i$.

In this paper we construct a homeomorphism on a Cantor set which is nearly periodic such that $h(a) = b$ for given $a, b \in D_p$. We also give a simple example which is not almost periodic and we discuss when a homeomorphism on a cantor set is periodic.

One motivation for this paper is the following question, raised by Seung-Hyeok Kye: Can we classify homeomorphisms on a Cantor set?

2. Main theorems

In this section, we state some definitions and lemmas which will be used for proofs of our main theorems. And, we prove our main theorems and give an example which shows that the existence of a homeomorphism which is not even almost periodic.

LEMMA 2.1. *Let $a = (a_0, a_1, a_2, \dots) \in D_p$ with $a \neq 0$ and let h_i be a map of D_p/U_i onto itself such that $h_i(b_0, b_1, \dots, b_{i-1}) = (b_0, b_1, \dots, b_{i-1}) \odot (a_0, a_1, \dots, a_{i-1})$ where $\odot$ is the operation induced by infinite carry over operation on D_p and such that $h_i \circ h_{i,j} = h_{i,j} \circ h_j$ where $h_{i,j}$ is the natural bonding map from D_p/U_j onto D_p/U_i in the above and $j > i$. Then h_i is periodic homeomorphism of D_p/U_i onto itself with period p^{k_i} for $k_i \leq i$. And ${h_i}^{-1}(b_0, b_1, \dots, b_{i-1}) = (b_0, b_1, \dots, b_{i-1}) \odot (a_0, a_1, \dots, a_{i-1})^{-1}$.*

PROOF. Recall that the operation $\odot$ is induced by infinite carry over operation on D_p. Therefore D_p/U_i is a finite cyclic group of order p^i. And h_i is periodic of order p^{k_i} for $k_i \leq i$. We also recall that the topology on D_p/U_i is discrete and therefore h_i is homeomorphism of D_p/U_i onto itself with period p^{k_i}. □

LEMMA 2.2. *Let h be a map of D_p/U_i onto itself induced by h_i's; i.e. $h(a_0, a_1, \dots a_i \dots) = \lim_{\leftarrow} h_i(a_0, a_1, \dots a_{i-1})$. Then h is homeomorphic.*

PROOF. The proof is obvious by properties of inverse limit and the fact that h_i is homeomorphism of D_p/U_i onto itself and $h_i \circ h_{i,j} = h_{i,j} \circ h_j$. [See, for example, 1 page. 431]. □

Let h be a homeomorphism of a metric space (X, d) onto itself. h is said to be *nearly periodic* iff there exists a complete system $\{\mathcal{U}_i\}_{i=1}^{\infty}$ of finite covers which are invariant under h [8]. The sequence $\{\mathcal{U}_i\}_{i=1}^{\infty}$ is called a *complete system* iff $\{mesh(\mathcal{U}_i)\}$ has limit 0.

A homeomorphism h of a metric space (X, d) onto itself is said to be *almost periodic* iff, for every $\epsilon > 0$, there exists a relatively dense sequence $\{n_i\}$ of integers (i.e. the gaps are bounded) such that $d(x, h^{n_i}(x)) < \epsilon$ for all $x \in X$ and $i = \pm 1, \pm 2, \ldots$. In particular, if, for every $\epsilon > 0$, there exists a positive integer n_ϵ such that $d(x, h^k(x)) < \epsilon$ for all $x \in X$ and for all $k \in n_\epsilon Z$, we say that the homeomorphism h is *regularly almost periodic* [2].

We now state and prove one of our main theorems.

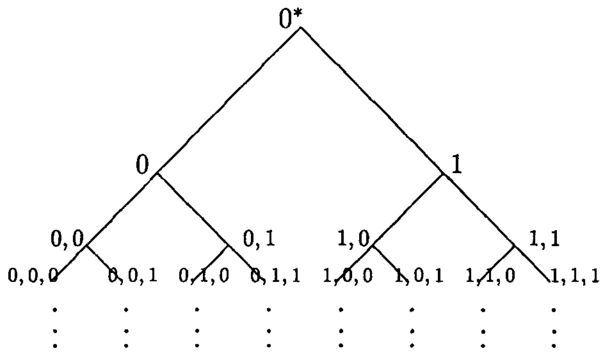
THEOREM 2.1. *Let $a = (a_0, a_1, \ldots ,) \in D_p$. Then, there exists a nearly periodic homeomorphism h of D_p onto itself, which is not periodic, such that $h(0) = a$*

PROOF. Let h_i be a homeomorphism of D_p/U_i onto itself in Lemma 2.1 and let h be the homeomorphism of D_p onto itself in Lemma 2.2. Then $h(U_i) = h_i(0_0, 0_1, \ldots 0_{i-1})U_i$. For example, in case $p = 2$, if we consider Cantor dyadic tree as shown in Figure 2.1, then we can easily identify an element of Cantor set with an infinite path whose end point is 0^* and $aU_i = (a_0, a_1, \ldots , a_{i-1})$ is the unique coset in D_2/U_i such that the path, corresponding to a, pass through.

Recall that D_p/U_i is finite group of order p^i and h_i is periodic of order p^{k_i} for $k_i \leq i$. We now consider a finite open cover $\{(a_0, a_1, \ldots , a_{i-1})\}$ of D_p/U_i with discrete topology, i.e. the set of all elements of D_p/U_i. Then $\{h|_{U_i}^{-1}(a_0, a_1, \ldots , a_{i-1})\}$ forms a finite open cover $\{(a_0, a_1, \ldots , a_{i-1})U_i\}$ of D_p where $h|_{U_i}$ is a natural quotient map from D_p onto D_p/U_i.

Then the finite open cover $\{(a_0, a_1, \ldots , a_{i-1})U_i\}$ is invariant under h since h_i is periodic on D_p/U_i with period p^{k_i}. Now $\{(a_0, a_1, \ldots , a_{j-1})U_j\}$ refines $\{(a_0, a_1, \ldots , a_{i-1})U_i\}$ for $i < j$. Clearly $mesh(U_i)$ goes to 0. Therefore h is nearly periodic.

Note that h_i is periodic homeomorphism with period p^{k_i} and we can easily verify that if $i < j$ then the period of h_j is strictly bigger than the period of h_i. This means that h can *not* be periodic. This completes the proof. $\square$



Cantor Dyadic Tree

FIGURE 2.1

THEOREM 2.2. *Let $a, b \in D_p$. Then there exists a nearly periodic homeomorphism f of D_p onto itself which is not periodic such that $f(a) = b$.*

PROOF. Let h, g be the nearly periodic homeomorphism with $h(0) = a$ and $g(0) = b$ in Theorem 2.1. We consider $g \circ h^{-1}$. We note that $g_i \circ h_i^{-1}$ is periodic homeomorphism of D_p/U_i onto itself with period p^{k_i} for $k_i \leq i$. Then we can find $c = (c_0, c_1, c_2, \dots)$ in D_p such that $g_i \circ h_i^{-1}(0, 0, \dots 0_{i-1}) = (c_0, c_1, \dots c_{i-1})$ for each i. We now apply Theorem 2.1 to c. Then $g \circ h^{-1}$, which is the inverse limit of $\{g_i \circ h_i^{-1}\}_{i=1}^{\infty}$, is nearly periodic and $g \circ h^{-1}(a) = b$

Therefore $g \circ h^{-1}$ is desired nearly periodic homeomorphism of D_p onto itself and is *not* periodic. □

REMARK. The homeomorphism h on Theorem 2.2 is also regularly almost periodic [*4*]. Nearly periodic homeomorphism and regularly periodic homeomorphism on metric spaces are not equivalent in general. But they are equivalent on compact metric spaces [*L1*].

P. A. Smith [8] showed how to construct a compact 0-dimensional transformation group on a compact metric space M, generated by a given nearly periodic homeomorphism of M onto itself. See also [4] and [5].

Recall that the homeomorphism h in Theorem 2.2 is nearly periodic induced by h_i and complete system $\{\{a_{i,j}U_i\}_{j=1}^{p^i}\}_{i=1}^{\infty}$. Therefore we have the following theorem:

THEOREM 2.3. *A homeomorphism h in* Theorem 2.2 *is the generator of the p-adic transformation group.*

We now introduce how to construct a periodic homeomorphism. We consider a permutation g_i on D_p/U_i of order p^{k_i} for some $k_i \leq i$ and $g_i \circ h_{i,j} = h_{i,j} \circ g_j$ for $j > i$. If orders of p^{k_i}'s are different for only finitely many stages, then homeomorphism g, induced by those g_i's, is periodic.

In fact, let h be a nearly periodic homeomorphism on a compact metric space and let h_i is an induced periodic map on a finite open cover $\{U_{i,j}\}$ with period q_i. In general, if only finitely many q_i's are different in the sequence of integers $(q_i)_{i \in N}$, then h has to be periodic [8].

Now we consider an existence of homeomorphism of D_p onto itself which is not even almost periodic. Recall that a cantor set D_p can be embedded in unit interval $[0, 1]$. Let h be a homeomorphism of $[0, 1]$ onto itself with 2 fixed points $0, 1$. If h is almost periodic then $h = \mathit{identity}$. We deduce that the above statement is true from [4, 3]. Consequently we have the following theorem.

THEOREM 2.4. *Let h be a homeomorphism of* $[0, 1]$ *onto itself with only 2 fixed points* $0, 1$ *such that $h|_{D_p}$ is a homeomorphism of D_p onto itself. Then, $h|_{D_p}$ is not almost periodic.*

We can easily find such a homeomorphism. For example, we consider D_2, so called dyadic set which is embedded in unit interval $[0, 1]$. We describe homeomorphism h as follows: $h((0, 0)U_2) = (0)U_1$, $h((0, 1)U_2) = (1, 0)U_2$ and $h((1)U_1) = (1, 1)U_2$ such that h is order preserving map [See Figure 2.1]. Then h induces a homeomorphism of unit interval $[0, 1]$ onto itself with only 2 fixed point $0, 1$.

We remark that if h is an almost periodic homeomorphism of a compact metric space M onto itself then, for a given $\epsilon > 0$, there exists a nearly periodic homeomorphism of M onto itself such that $d(h(x), g(x)) < \epsilon$ for all $x \in M$ [2, 4]. But the author does not know whether there exists an almost periodic homeomorphism of D_p onto itself which is not nearly periodic.

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