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A Historical Remark on Nuclearity

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Abstract

The paper deals with compressed historical developments of certain invariant characterization of λ-nuclearity including approximable property of λ-nuclear representation and the normality of λ by means of C_o-permanence.

0. Historical background and introduction

Let E and F be Banach spaces, then $N(E,F)$ denotes the Banach space of all nuclear operators $S=\sum_1^\infty a_i \otimes y_i$ with $a_i \in E'$, the dual of E, and $y_i \in F$ such that $\sum_1^\infty \|a_i\| \|y_i\|$ is finite, and the nuclear norm $\nu(S)$ is the infimum of all sums $\sum_i \|a_i\| \|y_i\|$. This concept of ingenious ideas of the nuclearity in Banach spaces was independently initiated by the classical papers of A. Grothendieck [5] and A.F. Ruston [15] in 1951.

Historically, concerning on Banach ideals of some special operators which have played a crucial role in the development of functional analysis, it is generally accepted [14] that the finite and approximable operators were created by E. Goursat and E. Schmidt at the begining of 20C, and D. Hilbert studied the completely continuous operators at about the same time.

The notion of compact operators obviously goes back to F. Riesz's paper about 1918. The weakly compact operators were used by S. Kakutani and K. Yosida in 1938. The concept of unconditionally summing operators was introduced by A. Pelczynski in the early 1960's. The strictly singular and strictly consingular operators were investigated by

T. Kato in 1958 and A. Pelczynski in 1965, respectively. In 1972 E. Dubinsky and M.S. Ramanujan [2] introduced the notion of λ-nuclearity, and after, G. Walker [16] described the compactness of λ-nuclear operators by means of sequence spaces. And it is a result of Grothendieck [6] that a Banach space E has the approximation property if and only if for every nuclear $S \in N(E, F)$ the value $\sum_1^\infty a_i \otimes y_i$ does not depend on the particular choice of nuclear representation of $S = \sum a_i \otimes y_i$.

In particular, since Grothendieck's classical paper "Résumé", demonstrating the possibilities for the using of tensor products and local theory in the study of Banach spaces, was published in 1956(Bull. Math. Soc. São. Paulo), Lindenstrauss and Pelczyński in Studia Math.(1968), introduced the concept of the absolutely summing operators in $\mathcal{L}_p$ -spaces which presented the main result of the "Résumé". Since then, by drawing new attention to the tensor product approach there have been many and various studies of describing the relationship between Banach tensor products with tensor norms and Banach ideals of special operators [4] [7] [12] [13].

It is well known that the nuclearity of operators is used as a suitable tool in local and analytic invariants of the infinite dimensional Banach space theory. In particular, Persson and Pietsh [13] showed that the class of nuclear operators depends on the space of absolutely convergent series, which has been extended and applied to prove analogous results to more general context.

In this note, some reformed results of λ-nuclear operators are characterized by replacing ℓ_1 with an arbitrary Banach sequence space λ and then we reconstruct the normality of λ in terms of C_o-permanence.

For the Banach ideal and tensor of operators referred to [14] and sequence spaces with Köthe dual mainly to [8].

1. λ-nuclearity with DK-space

Throughout this note λ will denote a Banach sequence space, λ^* for the Köthe dual of λ, and $\mathcal{L}(E, F)$ the space of all bounded linear operators from E to F containing the finite rank operators $\mathcal{F}(E, F)$. We recall [16] that $S \in \mathcal{L}(E, F)$ is said to be λ -nuclear if $S = \sum_{i=1}^\infty p_i a_i \otimes y_i$, where $p = (p_i) \in \lambda$, $a_i \in E'$, and $y_i \in F$ with $(g(y_i)) \in \lambda^*$

for all $g \in B_{F'}$ (=closed unit ball in F').

Obviously the λ-nuclear representation for S is required to converge on F in the norm topology. We put $\nu(S) = \inf \sum \|p\| \|a_i\| \|g(y_i)\|_{\lambda^*}$, then ν determines the norm on $N_\lambda(E, F)$(=the space of all λ-nuclear operators).

All Banach sequence spaces will be assumed to include ϕ, the collection of all finitely non-zero sequences and to be normal, which implies that $a \in \lambda$ if $b \in \lambda$ and $|a_i| \leq |b_i|$ for all i. λ is called DK-space if each of the coordinate maps $P_i : p \to p_i$ $(p \in \lambda)$ is continuous and possess the property that p is the limit of its finite sections $(p_1, p_2, \cdots, p_n, 0, \cdots)$.

Analogously as with nuclearity, we have the following:

Lemma 1.1. Let λ be a DK-Space. Then every $S \in N_\lambda(E, F)$ is compact Pietsh integral.

Proof. This is well known, for completeness, we sketch a proof, following a suggestion of [10]. Let $S \in N_\lambda(E, F)$ with λ-nuclear representation $Sx = \sum_1^\infty p_i a_i(x) y_i$, for $x \in E$, and let S_n be in $\mathcal{F}(E, F)$ defined by $S_n x = \sum_1^n p_i a_i(x) y_i$. Now we consider

$$\begin{aligned} \|S - S_n\| &= \sup_{x \in B_E, g \in B_{F'}} \left| \sum_{n+1}^\infty p_i a_i(x) g(y_i) \right| \\ &\leq \sup \|a_i\| \ \|P_n p - p\|_\lambda \sup_{g \in B_{F'}} \|g(y_i)\|_{\lambda*}. \end{aligned}$$

Since $\|P_n p - p\|_\lambda \to 0$, and the collection of $(g(y_i))_{g \in B_{F'}}$ is pointwise bounded [2] and hence norm bounded in λ^*, this proves the first assertion. Next, to see that $S \in N_\lambda$ is Pietsch integral, let $Sx = \sum_1^\infty p_i a_i(x) y_i$ with $\sum_1^\infty \|p\| \|a_i\| \|g(y_i)\|_{\lambda^*} \leq 1$ and $g \in B_{F'}$. As usual, define a F-valued vector measure G on the Borel σ-algebra of $B_{E'}$;

$$G(M) = \sum_1^\infty \|p\| \|a_i\| \delta_{a_i}(M) y_i \ ,$$

where $a_i \in B_{E'}$, $y_i \in F$ and δ_{a_i} denotes the Dirac functional at a_i.

We have obviously $|G|(B_{E'}) \leq \sum \|p\| \|a_i\| \|g(y_i)\|_{\lambda^*}$. Moreover, for each $x \in E$, there exists $p \in \lambda$, $a \in B_{E'}$, by Riesz Representation Theorem such that $Sx = \int_{B_{E'}} \|p\| a(x) dG$, which completes the proof. $\square$

The following factorization theorem can be found in [11], reduces a criterion that λ-nuclear operator factors through $\ell_p (p=1, \infty)$.

Theorem 1.2. An operator $S \in \mathcal{L}(E, F)$ is λ-nuclear if and only if there exists a factorization as $E \xrightarrow{V_1} \ell_\infty \xrightarrow{\Delta} \ell_1 \xrightarrow{V_2} F$ such that $\Delta \in \mathcal{L}(\ell_\infty, \ell_1)$ is a diagonal operator of the form $\Delta \xi = (\sigma_i \xi_i)$ with $\sigma = (\sigma_i) \in \ell_1$. In this case

$$\nu(S) = \inf \| V_2 \| \| \Delta \| \| V_1 \|,$$

where the infimum is taken over all possible factorizations.

2. Main result

We recall [16] that λ is said to be μ-invariant if $\lambda = \mu\lambda$, where $\mu\lambda$ denotes the set of products $(a_i b_i)$ formed by taking $a \in \mu$ and $b \in \lambda$. The following refined result, by a slight modification, shows that λ-nuclearity is characterized in terms of C_o:

Theorem 2.1. Let $\sigma = (\sigma_i) \in C_o$ and define $S\xi := (\gamma_i \sigma_i \xi_i)$ for $\gamma = (\gamma_i) \in \lambda$. Then S is λ-nuclearity from ℓ_1 to ℓ_∞ with $\nu(S) = \sup_i \| |\gamma| \|_\lambda |\sigma_i|$.

Proof. We set, for $\xi \in \ell_1$, $S_n \xi := P_n(\gamma_i \sigma_i \xi_i) = P_n \sigma(\gamma_i)(\xi_i)$. Then

$$\nu(S_n) \leq \sup_{1 \leq i \leq n} |\sigma_i| \| |\gamma| \| \cdot \nu(I : \ell_1^n \to \ell_\infty^n) \leq \sup_i |\sigma_i| \| |\gamma| \|_\lambda.$$

Analogously we have

$$\nu(S_m - S_n) \leq \sup_{n < i \leq m} |\sigma_i| \| |\gamma| - P_n |\gamma| \|_\lambda \text{ for } n < m.$$

Hence, (S_n) becomes a ν-Cauchy from the fact that $\lim \| |\gamma| - P_n |\gamma| \|_\lambda = 0$ in λ-nuclear norm topology. It follows that S belongs to $N_\lambda(\ell_1, \ell_\infty)$.

Next, $\nu(S) = \lim \nu(S_n) \leq \sup_i |\sigma_i| \| |\gamma| \|_\lambda$.

On the other hand, we get $\sup_i |\sigma_i| \| |\gamma| \|_\lambda = \|S\| \leq \nu(S)$. This completes the proof. □

The following Lemma, due to walker [16], can be reproduced using the DK-Space vein:

Lemma 2.2. Let λ be a DK-Space. Then λ implies $C_o \lambda$.

We now conclude with the following of under what restrictions λ to have normality:

Theorem 2.3. If λ is a norm-closed DK-subspace of C_o with the C_o-invariant, λ is normal.

Proof. If $a=(a_i)\in\lambda$, $b=(b_i)\in C_o$, and $|b_i|\le|a_i|$ for all i, then there exists a $g\in B_{C_o^{\cdot}}$ with $(g(b_i))\in C_o^{*}$.

Since λ is C_o-invariant, there is a $\xi=(\xi_i)\in\lambda$ such that $\| |\xi| - P_n|\xi| \|_{\lambda}=0$ and $|g(b)-g(\xi)|\rightarrow 0$. Hence $\|g\|_{C_0^{*}}\|b-\xi\|_{\lambda}\rightarrow 0$. Therefore λ is normal.

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