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The Shape Operator of the Tubular Hypersurfaces*

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Abstract

Using Fermi coordinates and the principle curvature on the tubula hypersurfaces, we characterize space of constant sectional curvature by analysing the shape operator on the tubular hypersurfaces.

0. Historical Background and Introduction

In 1992 Fermi [1] introduced Fermi coordinates to describe the geometry of a Riemannian manifold in a neighborhood of a curve. He wrote this paper while a student at the Scuola Normale di Pisa and later became more famous as a physicist.

It was soon utilized by Levi-Civita, Eisenhart and other differential geometers in the 1920's.

Normal coordinates are the natural coordinates to use in the study of a geodesic ball, which is a simple but important special case of a tube. Fermi coordinates are a generalization of normal coordinates. It turns out that many facts about normal coordinates have generalization to Fermi coordinates.

Thus we are concerned with the geometry of tubes using Fermi coordinates.

In this paper we deal with the tubular hypersurfaces P_t in the Riemannian manifold M and study how the properties of P_t on M determine the ambient space.

In 1995, B.J. Papantoniou [5] proved the Theorem 3.2 by means of the Jacobi vector fields. It is the problem about the principle curvatures of the tubula hypersurfaces about

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every topologically embedded submanifold P of M provided that the shape operator $S(t)$ of P_t has a parallel eigenspaces of dimension $n-q-1$ along a geodesic ξ meeting P orthogonally.

However we use a Riccati differential equation (Lemma 2.3) instead of the Jacobi differential equation because it has more geometric content and gives direct information about the principal curvatures of the tubular hypersurfaces.

1. Tubular Hypersurfaces

Basic notions and facts used here can be founded in the paper [2].

In this paper we assume that all maps and manifolds are C^{∞}.

Let M be a Riemannian manifold of dimension n with the Levi-Civita connection ∇ and the curvature tensor R, which is defined by

$$R_{XY}=[\nabla_X, \nabla_Y]-\nabla_{[X,Y]} \qquad \text{for } X, Y\in \chi(M).$$

Let P be a q-dimensional connected topologically embedded submanifold of M. Denote by ν the normal bundle of P in M. Then $\exp_\nu: \nu \rightarrow M$, which is given by $\exp_\nu((p, v))=\exp_p(v)$ for $p\in P$ and $v\in P_p^{\perp}$, maps on a neighborhood Ω_P of the zero section of ν diffeomorphically.

Let $p\in P$ and let $E_{q+1}, \cdots, E_n$ be orthonormal sections of ν defined in a neighborhood $V\subset P$ of p. Let $(y_1, \cdots, y_n)$ be an arbitrary system of coordinates for P in M.

Definition 1.1. The Fermi coordinates $(x_1, \cdots, x_n)$ of $P\subset M$ centered at p in $\exp_\nu(\Omega_P)$ are defined by

$$x_a(\exp_\nu(\sum_{j=q+1}^{n} t_jE_j(p')) = y_a(p'), \qquad a=1, \cdots, q.$$

$$x_i(\exp_\nu(\sum_{j=q+1}^{n} t_jE_j(p')) = t_i, \qquad i=q+1, \cdots, n.$$

for $p'\in V$, provided that the numbers $t_{q+1}, \cdots, t_n$ are small enough so that $\sum_{j=q+1}^{n} t_jE_j(p')\in \Omega_P$.

With respect to the Fermi coordinates $(x_1, \cdots, x_n)$ for $P \subset M$ we put

$$N = \sum_{i=q+1}^{n} \frac{x_i}{\sigma} \frac{\partial}{\partial x_i}, \qquad \sigma^2 = \sum_{i=q+1}^{n} x_i^2.$$

Then for any unit speed geodesic ξ normal to P and $m \in M$,

$$\sigma(m) = d(m, P) \qquad N_{\xi(t)} = \xi'(t).$$

Therefore σ is defined on $\exp_\nu(\Omega_P)$ and N is defined on $\exp_\nu(\Omega_P) - P$.

Definition 1.2. A tube $T(P, r)$ of radius $r \geq 0$ about P is the set

$$\begin{aligned} T(P, r) &= \bigcup_{p \in P} \{ \exp_p(v) \mid v \in P_p^\perp,\ \| v \| \leq r \} \\ &= \{ m \in M \mid \text{there exist a geodesic } \xi \text{ of length } L(\xi) \leq r \\ &\qquad \text{from } m \text{ meeting } P \text{ orthogonally} \} \end{aligned}$$

We call a hypersurface of the form

$$P_t = \{ m \in T(P, r) \mid d(m, P) = t \}$$

the tubular hypersurface at a distance t from P.

We use S and R_N for the tensor field defined on the set $\exp_\nu(\Omega_P) - P$ by

$$R_N U = R_{NU} N \ , \quad SU = -\nabla_U N$$

for $U \in \chi(\exp_\nu(\Omega_P) - P)$.

Lemma 1.3.[3] On $\exp_\nu(\Omega_P) - P$,

$$\nabla_N S = S^2 + R_N .$$

For each t let $S(t), R(t)$ and $S'(t)$ be the restrictions to the hypersurface P_t of S, R_N and $\nabla_N S$.

Then $S(t)$ is the shape operator of P_t and $S'(t) = S^2(t) + R(t)$.

2. Main Result

Let ξ be a unit speed geodesic normal to P at p with $\xi(0) = p$ and let $\{f_1, \cdots, f_q\}$ be an orthonormal basis of P_p that diagonalizes the shape operator of P

at p. We extend these tangent vectors $\{E_1, \cdots, E_q\}$ to unit vector fields $F_1(t), \cdots, F_q(t)$ along ξ such that each t and a, $F_a(t)$ is an eigenvector of $S(t)$. We put

$$S(t)F_a(t) = k_a(t)F_a(t) , \qquad a = 1, \cdots, q.$$

Then the $k_a(t)$ are eigenvalues of P_t. Let $k_{q+2}(t), \cdots, k_n(t)$ be the remaining eigenval ues. Then there are unit vector fields $F_{q+2}(t), \cdots, F_n(t)$ along ξ such that

$$S(t)F_i(t) = k_i(t)F_i(t), \qquad i = q+2, \cdots, n$$

If $F_{q+1}(t) = \xi'(t) = N_{\xi(t)}$, then $\{F_1, F_2, \cdots, F_n\}$ is an orthonormal frame field along ξ for M. For fixed t, $k_1(t), \cdots, k_q(t), k_{q+2}(t), \cdots, k_n(t)$ which are restricted to P_t are the eigenvalues of P_t at $\xi(t)$.

Lemma 2.1.[3] Suppose F_a is differentiable at t. Then

$$k_a'(t) = k_a^2(t) + R_{\xi'(t)F_a(t)\xi'(t)F_a(t)} , \qquad a = 1, \cdots, q, q+2, \cdots, n.$$

Theorem 2.2. Let P be a q-dimensional submanifold of a connected n-dimensional Riemannian manifold M.

If the shape operator $S(t)$ of the tubular hypersurface P_t has $(n-q-1)$ dimensional parallel eigenspace along the unit speed geodesic ξ of M meeting P orthogonally, then M has constant curvature.

Proof. Let ξ be the unit speed geodesic normal to P at p with $\xi(0) = p$. Let $k_1 = k_1(t)$ and $k_2 = k_2(t)$ be the distinct eigenfunctions of multiplicity q and $n-q-1$ of the shape operator $S(t)$ for P_t.

Let $\{F_1(t), F_2(t), \cdots, F_n(t)\}$ be the parallel orthonormal frame field along ξ, obtained by parallel translation of orthonormal basis $\{E_1, \cdots, E_n\}$ of M_p such that

$$S(t)F_a(t) = k_1(t)F_a(t), \qquad a = 1, \cdots, q$$

$$S(t)F_i(t) = k_2(t)F_i(t), \qquad i = q+2, \cdots, n$$

$$F_{q+1}(t) = \xi'(t).$$

By Lemma 1.3

$$(S'(t) - S^2(t))F_a(t) = R_N F_a(t) \qquad a = 1, \cdots, q, q+1, \cdots, n.$$

Hence

$$(k_1'(t) - k_1^2(t))F_a(t) = R(\xi', F_a(t))\xi', \qquad a = 1, \cdots, q$$

$$(k_2'(t) - k_2^2(t))F_i(t) = R(\xi', F_i(t))\xi', \qquad i = q+2, \cdots, n.$$

Thus $F_a(t)$ and $F_i(t)$ are eigenvectors of the mapping $R(\xi', \ )N$ along $\xi - p$ with eigenvalues $k_1'(t) - k_1'(t)$ of multiplicity q and $k_2'(t) - k_2^2(t)$ of multiplicity $n-q-1$.

At $t=0$, we have

$$R(E_{q+1}, E_a)E_{q+1} = k\ (E_{q+1,} E_a)E_a, \qquad a = 1, \cdots, q$$

$$R(E_{q+1}, E_i)E_{q+1} = k\ (E_{q+1}, E_i)E_i, \qquad i = q+2, \cdots, n.$$

Since k doesn't depend on the choice of E_α, k is constant on M by Schur's theorem.

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