Generalized Random Walk in a Random Environment on Integer
- 정수상에서 랜덤환경에 있어서 랜덤워크의 일반화

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요 지

본 논문은 Kalikow가 제시한 랜덤환경에서의 랜덤워크를 정수상으로 일반화 하였으며 Solomon의 극한정리를 랜덤 환경별로 분해하여 제시하고, 증명하였다.

1. Introduction

Let $X_1, X_2, \cdots X_n$ be independent and identically distributed (iid) with $E[\,|X_i|\,]<\infty$. Let $Z_0=0$, $Z_n=\sum_1^n X_i$, $n\geq 1$. The process $\{Z_n,\ n\geq 0\}$ is called a random walk process.

Random walks are quite useful for modeling various phenomena. For instance, we have previously encountered the simple random walk— $P\{X_i=1\}=p=1-P\{X_i=-1\}$— in which Z_n can be interpreted as the winnings after the nth bet of a gambler who either wins or loses 1 unit on each bet. We may also use random walks to model more general gambling situations; for instance, many people believe that the successive prices of a given company listed on the stock market can be modeled as a random walk. As we will see, random walks are also useful in the analysis of queuing and ruin systems.

Let $\{\alpha_n\}$ be a sequence of iid random variables with $0\leq \alpha_n \leq 1$ for all n. The RWIRE on the integers is the sequence $\{X_n\}$ where $X_0=0$ and $X_{n+1}=X_n+1, (X_n-1)$ with probability $\alpha_{x_n}, (1-\alpha_{x_n})$. By the law of large numbers

$$\alpha_j = P(X_n=j+1 \mid X_{n-1}=j, X_{n-2}, = i_{n-2}, \cdots, X_0=i_0\ ; environment\{\alpha_n\})$$

is completely determined as the number of hits at j approaches infinity.

If X_n denotes the position of the random walk at time n, then $\{X_n\}$ is not, in general, a Markov Chain(Kalikow, 1981). But the limit behavior of $\{X_n\}$ can be obtained by fixing the environment and considering the limit behavior of the resulting Markov Chain. Now, we briefly summarize the contents of each chapter.

In chapter 2, We discuss the basic concept and mathematical definition of RWIRE. In

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chapter 3, We illustrate some examples of RWIRE. Limit theorem for the RWIRE is established in chapter 4(main part of this paper).

2. Basic concept of Mathematical definition of RWIRE

Consider the stochastic process which is the path of a particle which moves along an axis with steps of one unit at time intervals of one unit also. Suppose that the probability is p of any steps being taken to the right, and is $q=1-p$ of being to left. Suppose also that each step is taken independently of every other step. Then this process is called the unrestricted random walk. If the particle is in position 0 at time 0, determine the probability that it will be in position k after n steps.

Let $\{Z_n\}$ be the stochastic process, where Z_n is the position of the particle at time n that is, after n steps from its starting point 0. This stochastic process has a discrete time parameter space $\{0,1,2,\cdots\}$ and a discrete state space $\{-\infty,\cdots,-1,0,1,\cdots,\infty$. Now each step X is an independent rv having distribution

$$pr(X=1)=p, \qquad pr(X=-1)=q$$

Initially $Z_0=0$ after n steps

$$Z_n=X_1+X_2+\cdots+X \tag{1}$$

Where each X_t is independently distributed as X. We can write equation (1) as

$$Z_n=Z_{n-1}+X_t \tag{2}$$

Where each X_n is idependent of Z_{n-1}.

We wish to determine the value of

$$p_{ok}^{(n)}=pr(Z_n=k \mid Z_0=0) \tag{3}$$

Let random variable

$$Y_i=\begin{cases}1 & \text{if } X_i=1\\ 0 & \text{if } X_i=-1\end{cases} \qquad (i=1,2,\cdots,n)$$

That is let random variable $Y_i=\frac{1}{2}(X_i+1)$; then each Y_t is an independent Bernoulli trial with probability of success p. Then random variable $R_n=Y_1+\ldots+Y_n=\frac{1}{2}(Z_n+n$ is Bin (n,p).

Therefore

$$p_{ok}^{(n)} = pr(Z_n = k \mid Z_0 = 0) = pr\left\{R_n = \frac{1}{2}(Z_n + n) = \frac{1}{2}(k+n)\right\}$$

$$= \begin{cases} \binom{n}{(k+n)/2} p^{(k+n)/2} q^{(n-k)/2} & (k+n)/2 \in S = \{0, 1, 2, \cdot\cdot\cdot, n\} \\ 0 & (otherwise) \end{cases}$$

Since $E(R_n) = np$

$V(R_n) = npq$

$E(Z_n) = E(2R_n - n) = 2E(R_n) - n = n(p-q)$ (since p+q=1)

$V(Z_n) = V(2R_n - n) = 4V(R_n) = 4npq$ (4)

We can call upon some quite deep theorems of probability theory to obtain the behaviour of Z_n when n is large.

For example, by the strong law of numbers, with probability one

$$\frac{1}{n} Z_n \rightarrow \frac{1}{n} EZ_n = p - q \quad as \quad n \rightarrow \infty$$

That is, for large n, the particle will, if $p > q$, almost certainly drift in a positive direction along the axis of motion, the mean step length being $p-q$. Also, by the central limit theorem,

$$W_n = \frac{Z_n - n(p-q)}{\sqrt{(4npq)}} \rightarrow \text{an N(0,1) random variable as } n \rightarrow \infty$$

So, from tables of N(0,1) distribution, for large n,

$$pr(-1.96 \le W_n \le 1.96) \approx 0.95$$

$$\text{i.e. } pr\{n(p-q) - 1.96\sqrt{(4npq)} \le Z_n \le n(p-q) + 1.96\sqrt{(4npq)}\} \approx 0.9$$

Now we discuss the definition of a random environment.

The definition of a random environment on an abelian group is given in Kalikow(1981). The following is a summary of that definition in the case where the abelian group is the integers. Let X be the set of all integer sequences by Z^+ and endowed with the σ-algebra F generated by the cylinder set in X. A RWIRE on the integers is a discrete time stochastic process $\{X_n ; n \in Z^+\}$ with the integer state space. The measure which defines the RWIRE is defined by a two step construction of a measure P(*) on (X, F). For L and R will be fixed positive integers, let G be the set of all probability measures on {-L, ⋯ ,R} and let μ be a probability measure G. It is assumed that L and R are minimal in the sense that $\mu\{g \in G : g(-L) > 0\} > 0$ and $\mu\{g \in G : g(R) > 0\} > 0$.

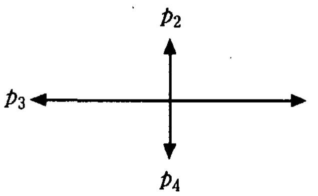
Definition 1. The random environment $\{\alpha_n\}$ defined by μ is an integer indexed sequence of iid G-valued random variables with common distribution μ.

Definition 2. An environment is a realization of the random environment $\{\alpha_n\}$.

Definition 3. Let X_n be a Markov Chain on Z. A finite subset A of Z is called a reflecting-to-the-right barrier for X_n if $X_N \geq \min(A)$ for some $N \geq 0$ implies $X_n \geq \min(A)$ for all. $n \geq N$. Reflecting-to-the-left barriers are defined similarly(Solomon, 1975).

3. Examples of RWIRE

Consider the nearest neighbor random walk on Z where at each time the process moves to the left with probability 1/4 and to the right with probability 3/4. We abbreviate this probability law by writing 1/4$\leftrightarrow$3/4. If we say that 0 has environment 1/4$\leftrightarrow$3/4, we mean that for any n, if the process is at 0 at time n, then at time n+1, it will be at -1 with probability 1/4 and at 1 with probability 3/4. Now, we can be easily extended to the nearest neighbor process on Z^2 where an environment is



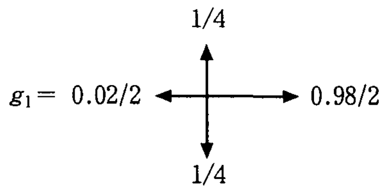
with $p_1+p_2+p_3+p_4=$

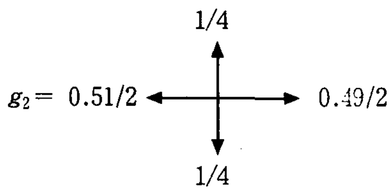
Example 1. Consider the one dimensional case where the environment at each element of Z is taken to be 0.02 $\leftrightarrow$ 0.98 with probability 0.99 and 0.51$\leftrightarrow$ 0.49 with probability 0.01. We denote this model in following fashion;

$$g_1 = 0.02 \leftrightarrow 0.98, \ g_2 = 0.51 \leftrightarrow 0.49$$
$$\mu(\{g_1\}) = 0.99, \ \mu(\{g_2\}) = 0.01$$

Then it is easily proved that the process is transient; that moves to the right with probability 1.

Example 2. Consider the two dimensional case where

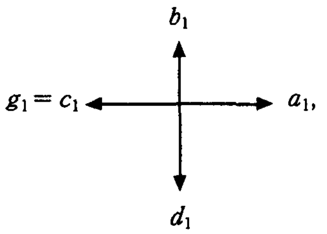




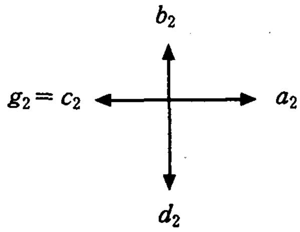
$\mu(\{g_1\}) = 0.99$ $\mu(\{g_2\}) = 0.01$

It seems still obvious, that this process is transient and moves to the right.

Generalizing example 2. Consider the two dimensional case where



$\mu(\{g_1\}) = p$



$\mu(\{g_2\}) = 1-p$

where $0 < p < 1,\ a_1 + b_1 + c_1 + d_1 = a_2 + b_2 + c_2 + d_2 = 1,$

$a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2 > 0,\ a_1 > c_1,\ a_2 < c_2,$

If

$$\frac{p(a_1 - c_1)}{(1-p)(c_2 - a_2)} > \max\left(\frac{a_1}{a_2}, \frac{b_1}{b_2}, \frac{c_1}{c_2}, \frac{d_1}{d_2}\right)$$

then the process moves to the right with probability 1.

4. Limit theorem for the RWIRE

When the environment $\{\alpha_n\}$ is fixed, Chung (1), page 65-71, uses systems of difference equations to derive results which we summarize in Lemma 1, and 2. The limit behavior of the RWIRE(Main theorem) will be obtained by applying fluctuation theory to Lemma 2.

Definition 4. Fix the environment $\{\alpha_n\}$ and let $\{X_n\}$ be the Markov Chain on Z with transition matrix.

$$M(n,\ n+1) = \alpha_n,\ M(n,\ n-1) = \beta_n = 1 - \alpha_n$$

For each integer $n \geq 1$

$$f_i{}^n{}_j = P(X_n = j, X_v \neq j, v = 1, 2. \cdot\cdot\cdot, n-1 \mid X_0 = i)$$

$$f_i{}^*{}_j = P(X_n = j, \textit{some}\ n \geq 1 \mid X_0 = i) = \sum_{n=1}^{\infty} f_i{}^n{}_j$$

$$m_{ij} = \sum_{n=1}^{\infty} n \cdot f_i{}^n{}_j$$

In other words, f_i^{n}, is the probability that, starting from state i, the first return to state j occurs at the n-th transition, and is the mean recurrence time from i to j.

Definition 5. State i is recurrent it $f_i^{*}{}_{i}=1$ otherwise called transient.

The following Lemma 1(Solomon,1975)is needed for the proof of main theorem.

Lemma 1. Fix $\{\alpha_n\}$ with $0<\alpha_n<1$ for all n; set $\sigma_n=\beta_n/\alpha_n$ and

$$\begin{aligned}\rho_n &= \sigma_1\cdots\sigma_n, \quad n>0\\ &= \sigma_{-1}\cdots\sigma_n, \quad n<0\end{aligned}$$

(i) Let i<j ; then

$$f_i^{*}{}_{j} = \left(\sum_{n=-\infty}^{i}\frac{1}{\sigma_n\ldots\sigma_j}\right)\left(\sum_{n=-\infty}^{j}\frac{1}{\sigma_n\ldots\sigma_j}\right)^{-1}<1 \quad \text{if } \sum_{n=1}^{\infty}\frac{1}{\rho_{-n}}<\infty$$

$$=1 \quad \text{if } \sum_{n=1}^{\infty}\frac{1}{\rho_{-n}}=\infty$$

(ii) Let i>j ; then

$$f_i^{*}{}_{j} = \left(\sum_{n=i_\infty}\sigma_j\ldots\sigma_n\right)\left(\sum_{n=j}^{\infty}\sigma_j\ldots\sigma_n\right)^{-1}<1 \quad \text{if } \sum_{n=1}^{\infty}\rho_n<\infty$$

$$=1 \quad \text{if } \sum_{n=1}^{\infty}\rho_n=\infty$$

(iii) *If* $f_{01}^{*}=1$, *then*

$$m_{01}=(1+\sigma_0)+\sum_{j=-\infty}^{0}(1+\sigma_{j-1})\sigma_j\ \ldots\ \sigma_0$$

Lemma 2. Fix $\{\alpha_n\}$ with $0<\alpha_n<1$ for all n, then

(i) $\sum_{n=1}^{\infty}(\rho_{-n})^{-1}=\infty$, $\sum_{n=1}^{\infty}\rho_n<\infty$ *implies* $\lim_{n\to\infty}X_n=\infty$ *a.e.*

(ii) $\sum_{n=1}^{\infty}(\rho_{-n})^{-1}<\infty$, $\sum_{n=1}^{\infty}\rho_n=\infty$ *implies* $\lim_{n\to\infty}X_n=-\infty$ *a.e.*

(iii) $\sum_{n=1}^{\infty}(\rho_{-n})^{-1}=\infty=$, $\sum_{n=1}^{\infty}\rho_n$ *implies* $\{X_n\}$ is recurrent.

In fact $-\infty=\lim_{n\to\infty}\inf X_n<\lim_{n\to\infty}\sup X_n=\infty$ a.e

Proof. (i) if

$$\sum_{n=1}^{\infty}\frac{1}{\rho_{-n}}=\infty \qquad \sum_{n=1}^{\infty}\rho_n<\infty\ ,$$

then Lemma 1. implies $f_{ij}^{*}=1$ for i<j, but $f_{ij}^{*}<1$ for i>j.

Therefore $\lim_{n\to\infty}X_n=\infty$ *a.e.* Cases (ii) , (iii) are also clear from Lemma 1.

Lemma 3. Let $\{Y_i\}_1^{\infty}$ be a sequence of iid, nondegenerate finite valued random variables; let $S_n = Y_1 + \cdot \cdot \cdot + Y_n$.

(*i*) $\sum_{n=1}^{\infty} \frac{1}{n} P(Sn>0)<\infty$ iff $\lim_{n\to\infty} Sn = -\infty$ *a.e*

in which case $\sum_{n=1}^{\infty} e^{Sn} < \infty$ *a.e*

(*ii*) $\sum_{n=1}^{\infty} \frac{1}{n} P(Sn>0) = \infty = \sum_{n=1}^{\infty} \frac{1}{n} P(Sn<0)$ iff

$-\infty = \lim_{n\to\infty} \inf S_n < \lim_{n\to\infty} \sup S_n = \infty$ a.e.

in which case $\sum_{n=1}^{\infty} e^{-Sn} = \infty = \sum_{n=1}^{\infty} e^{Sn}$ *a.e.*

Proof. The 'iff' parts follow from fluctuation theory (2, Chapter 8), and 'if $\lim_{n\to\infty} Sn = -\infty$ *a.e. then* $\sum_{n=1}^{\infty} e^{Sn} < \infty$ *a.e.*' part proved by stone(1969).

A complete characterization of the limit behavior $\{X_n\}$ of can now be given in terms of the random environment $\{\alpha_n\}$ by combining Lemma 2 and 3.

Main theorem. Let $\{\alpha_n\}$ be a sequence of iid nondegenerate random variables with $0 \le \alpha_n < 1$ or $0 < \alpha_n \le 1$ for all n.

(*i*) *If* $\sum_{n=1}^{\infty} \frac{1}{n} P(\rho_n > 1) < \infty$, then $\lim_{n\to\infty} X_n = \infty$ *a.e.*

(*ii*) *If* $\sum_{n=1}^{\infty} \frac{1}{n} P(\rho_n < 1) < \infty$, then $\lim_{n\to\infty} X_n = -\infty$ *a.e.*

(*iii*) *If* $\sum_{n=1}^{\infty} \frac{1}{n} P(\rho_n < 1) = \infty = \sum_{n=1}^{\infty} \frac{1}{n} P(\rho_n > 1)$, then $\{X_n\}$ is resurrent ;

in fact $-\infty = \lim_{n\to\infty} \inf X_n < \lim_{n\to\infty} \sup X_n = \infty$ a.e.

Proof. If E $(1n\ \sigma)$ is defined, then (i), (ii), (iii) correspond respectively to

(i′) $E(1n\ \sigma) < 0$
(ii′) $E(1n\ \sigma) > 0$
(iii′) $E(1n\ \sigma) = 0$

Notice that the two series in (i) and (ii) cannot both converge simultaneously since $\sum_{n=1}^{\infty} \frac{1}{n} (\rho_n = 1) < \infty$ (2, chapter 8).

First suppose $0< \alpha_n <1$ for all n. We prove (i). So, suppose

$$\sum_{n=1}^{\infty}\frac{1}{n}P(\ln\sigma_1+\cdot\cdot\cdot+\ln\sigma_n>0)=\sum_{n=1}^{\infty}\frac{1}{n}P(\rho_n>1)<\infty$$

Then Lemma 3 implies

$$\sum_{n=1}^{\infty}\rho_n=\sum_{n=1}^{\infty}e^{S_n}<\infty \text{ a.e. where } S_n=\ln\sigma_1+\cdot\cdot\cdot+\ln\sigma_n$$

Now $\rho_n=\rho_{-n}$ in distribution. Thus $\sum_{n=1}^{\infty}\frac{1}{\rho_{-n}}=\infty$ a.e.

Hence Lemma 2 implies that for a.e. fixed environment $\lim_{n\to\infty}X_n=\infty$ a.e.

Now randomizing the environment gives

$$\sim \lim_{n\to\infty}X_n=\infty)=1$$

If $\alpha_n=1, (\alpha_n=0)$, with positive probability, but $\alpha_n>0, (\alpha_n<1)$, for all n, then it is clear that case (i), (case ii), holds.

Suppose that E ($\ln\sigma$) is defined. Set $S_n=\ln\sigma_1+\cdot\cdot\cdot+\ln\sigma_n$

Then E ($\ln\sigma$)<0 iff $\lim_{n\to\infty}S_n=-\infty$ a.e. iff

$$\sum_{n=1}^{\infty}\frac{1}{n}P(\sigma_n>1)=\sum_{n=1}^{\infty}\frac{1}{n}P(S_n>0)<\infty$$

by Lemma 3. Thus (i ') *corresponds to* (i), *Similarly* (ii ') *and* (iii ') *correspond* respectively to (ii) and (iii).

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