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Development of Matching Priors for $P(X < Y)$ in Exponential distributions †

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ABSTRACT

In this paper, matching priors for $P(X < Y)$ are investigated when both distributions are exponential distributions. Two recent approaches for finding noninformative priors are introduced. The first one is the Berger and Bernardo's forward and backward reference priors that maximizes the expected Kullback–Liebler divergence between posterior and prior density. The second one is the matching prior identified by matching the one sided posterior credible interval with the frequentist's desired confidence level. The general forms of the second-order matching prior are presented so that the one sided posterior credible intervals agree with the frequentist's desired confidence levels up to $O(n^{-1})$. The frequentist coverage probabilities of confidence sets based on several noninformative priors are compared for small sample sizes via the Monte–Carlo simulation.

Keywords: exponential distribution; confidence interval; frequentist coverage probability; reference priors; matching priors; higher order asymptotics; Monte–Carlo simulation

1. INTRODUCTION

In Bayesian analysis, inference problems are not simple because of problems associated with selection of priors as well as computational difficulties. Recently, development of noninformative prior has received a lot of attention. Notably among these are matching priors leading to Bayesian confidence regions with frequentist's desired confidence level. The following definition is used for matching priors.

Definition 1.1. *(Sun, 1997) A prior π is said to be an i-the order matching prior for θ if*

$$P_\theta\left(\theta \le \hat{\theta}_n(\alpha)\right) = \alpha + O\left(n^{-i/2}\right),$$

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where $\hat{\theta}_n(\alpha)$ is the α-th quantile of the posterior density for θ based on n observations $\mathbf{X}$ and $P_\theta(\cdot)$ is the joint probability measure of $\mathbf{X}$ given θ.

Note that $P_\theta(\cdot)$ is the frequentist probability considering θ as fixed constant and $\hat{\theta}_n(\alpha)$ as random variable.

A commonly used noninformative prior is the Jeffreys prior (1961) utilizing a data translated likelihood. In one parameter case, an exact data translated likelihood can be derived for the Jeffreys prior. Also Welch and Peers (1963) proved that the Jeffreys prior is a second-order matching prior in one parameter case. Although its successful commitment, Berger and Bernardo (1989) argue that the Jeffreys prior has serious deficiencies in multiparameter case. To overcome these deficiencies, Berger and Bernardo (1992) provide a general algorithm that maximizes the expected Kullback-Liebler divergence between posterior and prior density. The algorithm is proceeded dividing the parameter space into several groups according to the inferential importance which represents as the parameters of interest and the nuisance parameters. In a discussion of Ghosh and Mukerjee (1992), Berger (1992) introduced the backward reference prior by simply interchanging the roles of inferential importance.

Let

$$\Sigma = \begin{bmatrix} a_{20} & a_{11} \\ a_{11} & a_{02} \end{bmatrix}$$

be the Fisher information matrix for (θ_1, θ_2) per observation where θ_1 is a parameter of interest and θ_2 is a nuisance parameter. Peers (1965) proved that a prior π is a second-order matching prior for θ_1 if

$$\frac{\partial}{\partial\theta_2}\left(\frac{a_{11}\pi}{a_{02}\sqrt{B}}\right) - \frac{\partial}{\partial\theta_1}\left(\frac{\pi}{\sqrt{B}}\right) = 0, \tag{1.1}$$

where $B = a_{20} - a_{11}^2/a_{02}$. Tibshirani (1989) shows that if θ_1 and θ_2 are orthogonal in the sense of Cox and Reid (1987), a_{11} becomes 0 so that the solution of (1.1) is of the form:

$$\pi(\theta_1, \theta_2) = g(\theta_2)\sqrt{a_{20}}, \tag{1.2}$$

where $g(\theta_2)$ is a continuously differentiable positive function.

Mukerjee and Dey (1993) consider the higher order matching priors. They show that π is a third-order matching prior for θ_1 if π satisfies (1.1) and

$$\sum_{i=1}^{4} L_i(\pi; \theta_1, \theta_2) = 0,$$

where

$$L_1(\pi;\theta_1,\theta_2) = \left[D_1^2 B^{-1} - 2D_1^1 D_2^1 (a_{11}a_{02}^{-1}B^{-1}) + D_2^2(a_{11}^2 a_{02}^{-2}B^{-1})\right]/2 ,$$

$$L_2(\pi;\theta_1,\theta_2) = -\left(D_1^2\pi - 2a_{11}a_{02}^{-1}B^{-1} + a_{11}^2 a_{02}^{-2} D_2^2\pi\right)/(2\pi B) ,$$

$$L_3(\pi;\theta_1,\theta_2) = -\left[D_2^1\left((a_{02}B)^{-1}(K_{21} - 2a_{11}a_{02}^{-1}K_{12} + a_{11}^2 a_{02}^{-2}K_{03})\pi\right)\right]/(2\pi) ,$$

$$L_4(\pi;\theta_1,\theta_2) = -\left[D_1^1\psi(\pi;\theta_1,\theta_2) - D_2^1\left(a_{11}a_{02}^{-1}\psi(\pi;\theta_1,\theta_2)\right)\right]/\pi ,$$

$$\begin{aligned}\psi(\pi;\theta_1,\theta_2) &= \left(K_{30} - 3a_{11}a_{02}^{-1}K_{21} + 3a_{11}^2 a_{02}^{-2}K_{12} + a_{11}^3 a_{02}^{-3}K_{03}\right)\pi/\left(6B^2\right) \\ &- \left(D_1^1\pi - a_{11}a_{02}^{-1}D_2^1\pi\right)/(B) ,\end{aligned}$$

and

$$D_i^k = \frac{\partial^k}{\partial\theta_i^k} , \quad K_{ij} = E_{\theta_1,\theta_2}\left[D_1^i D_2^j \log p(X_1;\theta_1,\theta_2)\right] .$$

When π is a second-order matching prior for θ_1, one might be interested in a second-order jointly matching prior for (θ_1,θ_2), that is,

$$\begin{aligned}&P_{\theta_1,\theta_2}\left(\sqrt{n}(\theta_1 - \hat{\theta}_1) \le z_1 , \sqrt{n}(\theta_2 - \hat{\theta}_2) \le z_2\right) \\ &= P_\pi\left[\sqrt{n}(\theta_1 - \hat{\theta}_1) \le z_1 , \sqrt{n}(\theta_2 - \hat{\theta}_2) \le z_2\right] + O(n^{-1}) \qquad (1.3)\end{aligned}$$

for all (z_1,z_2) and $\hat{\theta}_i$ is the posterior mode or maximum likelihood estimator of θ_i. Note that the left side of (1.3) is a frequentist's joint coverage probability and the right side of (1.3) is a Bayesian joint posterior probability. It is shown that (1.3) holds when θ_1 and θ_2 are orthogonal. However Datta (1996) found that the common second-order matching prior for each θ_1, θ_2 is not sufficient condition to hold the second-order jointly matching prior when θ_1 and θ_2 are not orthogonal. He provided an additional partial differential equation to be a second-order jointly matching prior which is not presented here.

When the parameter of interest is $t(\theta)$, an arbitrary function of p-dimensional parameter vector θ, Datta and Ghosh (1996) consider the following second-order asymptotic property:

$$P_\theta\left[\frac{\sqrt{n}\left(t(\theta) - t(\hat{\theta})\right)}{\sqrt{b}} \le z\right] = P_\pi\left[\frac{\sqrt{n}\left(t(\theta) - t(\hat{\theta})\right)}{\sqrt{b}} \le z|data\right] + O_p(n^{-1})$$

for all z, where $\hat{\theta}$ is the posterior mode or maximum likelihood estimator of θ and b is the asymptotic posterior variance of $\sqrt{n}\left(t(\theta) - t(\hat{\theta})\right)$ up to $O_p(n^{-1})$. Here

$P_\theta(\cdot)$ represents the frequentist's joint probability measure of $X_1, X_2, \cdots, X_n$ for fixed θ and $P_\pi(\cdot)$ is the Bayesian posterior probability measure of θ under the second-order matching prior density, $\pi(\theta)$, with fixed $X_1, X_2, \cdots, X_n$. They proved that if $t(\theta)$ is a twice continuously differentiable function, $\pi(\theta)$ is a second-order matching prior if and only if $\pi(\theta)$ satisfies the following probability matching equation:

$$\sum_{i=1}^{p} \frac{\partial}{\partial \theta_i}\left[\frac{\rho_i^T I^{-1}(\theta)\nabla t(\theta)}{\sqrt{\nabla t(\theta)^T I^{-1}(\theta)\nabla t(\theta)}}\,\pi(\theta)\right] = 0\,,$$

where

$$\nabla t(\theta) = \left[\frac{\partial}{\partial \theta_1}t(\theta), \frac{\partial}{\partial \theta_2}t(\theta), \cdots, \frac{\partial}{\partial \theta_p}t(\theta)\right]^T,$$

ρ_i^T is the i-th unit column p-vector and $I^{-1}(\theta)$ is the inverse of the per unit observation information matrix of θ.

This paper is arranged as follows. In Section 2, exponential stress-strength model is introduced with additional theorems based on the results of Thompson and Basu (1993). In Section 3, several noninformative priors are investigated and a general form of a second-order matching prior is presented. In Section 4, small sample size simulations are performed to compare with several noninformative priors. The method is to investigate the frequentist properties of the procedure in finding 0.05, 0.5, 0.95-th posterior quantiles via the Gauss quadrature numerical method.

2. EXPONENTIAL STRESS-STRENGTH MODEL

Suppose that a unit of random strength Y is subjected to the random stress X. The unit fails if $X > Y$. In this case the reliability of the unit is given by

$$r_1 = P(X < Y).$$

This model was first considered by Birnbaum (1956). In his paper, it is shown that the non-parameteric UMVUE of r_1 is the Mann-Whitney U-statistic and provides the distribution free upper bound confidence bound for r_1. The comprehensive review of frequentist inference for this model is given by Basu (1981).

The stress-strength model has found many applications in structural and aircraft industries. For example, a solid propellant rocket engine is successfully fired if the chamber pressure, X is below the burst pressure Y. In this case a torsion stress is the most critical type of stress for a rotating steal shaft on a computer.

Suppose that X and Y are independent and exponentially distributed with means θ and ϕ respectively. In this case, the reliability of the unit is given by

$$r_1 = P(X < Y|\theta, \phi) = \frac{\phi}{\theta + \phi}. \tag{2.1}$$

A conjugate prior in this exponential stress–strength model was found by Enis and Geisser (1971). With independent inverted gamma priors as conjugate prior, they have found the exact expression of the posterior for r_1. It is known that the Jeffreys prior of (θ, ϕ) is $1/(\theta\phi)$. It is easy to show that the Jeffreys prior is the one of independent inverted gamma priors (conjugate priors) with parameter setting $\alpha = 0, \beta = 0$. By the transformation of variables, the Jeffreys prior of (r_1, r_2) becomes

$$\pi(r_1, r_2) \propto \frac{1}{r_1(1 - r_1)r_2}, \tag{2.2}$$

where r_1 is the parameter that represents the reliability in the exponential stress–strength model, the parameter of interest, and $r_2 = \theta + \phi$ is the nuisance parameter. Thompson and Basu (1993) employed the Berger and Bernardo reference prior algorithm and showed that the Berger and Bernardo reference prior for (r_1, r_2) is identical to the Jeffreys prior.

Next, we consider the inference problem when we have data set from random stress X and random strength Y. Let $X_1, X_2, \cdots, X_m$ be the random variables which is independent and identically distributed as exponential with mean θ and independently, $Y_1, Y_2, \cdots, Y_n$ be the random variables which is independent and identically distributed as exponential with mean ϕ. If the prior density of r_1 is the beta density with parameters α, β and the independent prior density of nuisance parameter,$r_2 = \theta + \phi$, is proportional to $r_2^{-\delta}$, then the marginal posterior density of r_1 becomes

$$\pi(r_1|\mathbf{x}, \mathbf{y}) \propto \frac{r_1^{\alpha+\delta+m-2}(1 - r_1)^{\beta+\delta+n-2}}{\left[1 - \left(1 - \frac{\sum_{i=1}^m x_i}{\sum_{i=1}^n y_i}\right) r_1\right]^{\delta+m+n-1}}, \quad 0 < r_1 < 1 \tag{2.3}$$

Thompson and Basu (1993) investigated that the Berger and Bernardo reference prior of r_1 in (2.1) with the choice of nuisance parameter, $r_2 = \theta + \phi$, and with the choice of the sequence of compact sets such that $\Omega_K = [1/K, K]$, is the same as the Jeffreys prior in (2.2). The theorem is as follows:

Theorem 2.1. *(Thompson and Basu, 1993) The Berger and Bernardo's forward reference posterior for* r_1 *with the nuisance parameter,* r_2 *is identical to the Jeffreys reference posterior.*

However, Berger and Bernardo (1992) note that the choice of the sequence of compact sets is important since reference prior may depend on Ω_K. The following theorem provides the uniqueness of the Berger and Bernardo's forward reference prior.

Theorem 2.2. *The Berger and Bernardo's forward reference prior of r_1 with the nuisance parameter, r_2 is unique.*

Proof: From the following Fisher information matrix of r_1, r_2

$$I(r_1, r_2) = \begin{bmatrix} \frac{n}{r_1^2} + \frac{m}{(1-r_1)^2} & \frac{1}{r_2}\left(\frac{n}{r_1} - \frac{m}{1-r_1}\right) \\ \frac{1}{r_2}\left(\frac{n}{r_1} - \frac{m}{1-r_1}\right) & \frac{m+n}{r_2^2} \end{bmatrix},$$

$\pi(r_2|r_1)$ and $h_1(r_1, r_2)$ become

$$\pi(r_2|r_1) \propto \sqrt{\frac{m+n}{r_2^2}} \propto \frac{1}{r_2}$$

$$h_1(r_1, r_2) = \frac{1}{r_1(1-r_1)}.$$

Construct an arbitrary sequence of compact sets $\Omega_K = \{a_K \leq r_2 \leq b_K\}$, where $a_K \to 0$ and $b_K \to \infty$ as $K \to \infty$. Then $\pi^K(r_2|r_1) = C^K(r_1)/r_2$, where

$$C^K(r_1) = \frac{1}{\int_{\Omega_K} 1/r_2 \, dr_2} = \frac{1}{\log b_K - \log a_K}$$

It then follows that

$$\pi^K(r_1) \propto \exp\left[\int_{\Omega_K} \pi^K(r_2|r_1) \log \frac{1}{r_1(1-r_1)} \, dr_2\right] = \frac{1}{r_1(1-r_1)}$$

Since $\pi^K(r_1)$ and $C^K(r_1)$ do not depend on an arbitrary sequence of compact sets Ω_K, the desired result follows. □

Corollary 2.1. *The reverse reference prior of r_1 with the nuisance parameter, r_2 is unique and identical to the Jeffreys reference posterior.*

Proof: After exchanging the elements of the Fisher information matrix, the desired result follows. □

3. DEVELOPMENT OF THE MATCHING PRIOR

In this section, the second-order matching priors are presented so that the asymptotic frequentist coverage probability of one sided posterior credible interval agree with frequentist's desired confidence level up to $O(n^{-1})$ when r_1 is the parameter of interest.

Lemma 3.1. *A prior $\pi(\theta, \phi)$ is the second-order matching prior for r_1 if and only if:*

$$\frac{\partial}{\partial \theta}\left[\theta \pi(\theta, \phi)\right] - \frac{\partial}{\partial \phi}\left[\phi \pi(\theta, \phi)\right] = 0. \tag{3.1}$$

Proof: Since the parameter of interest is $r_1 = \phi/(\theta + \phi)$, it follows

$$\nabla(r_1) = \left[\frac{-\phi}{(\theta+\phi)^2}, \frac{\theta}{(\theta+\phi)^2}\right]^T,$$

$$I^{-1}(\theta, \phi)\nabla(r_1) = \left[\frac{-\theta^2\phi}{(\theta+\phi)^2}, \frac{\theta\phi^2}{(\theta+\phi)^2}\right]^T,$$

$$\sqrt{\nabla(r_1)^T I^{-1}(\theta, \phi)\nabla(r_1)} = \frac{\sqrt{2}\theta\phi}{(\theta+\phi)^2}.$$

Hence, the probability matching equation in (3.1) simplified to

$$\frac{\partial}{\partial \theta}\left[\frac{-\theta}{\sqrt{2}}\pi(\theta, \phi)\right] + \frac{\partial}{\partial \phi}\left[\frac{\phi}{\sqrt{2}}\pi(\theta, \phi)\right] = 0.$$

□

Theorem 3.1. *In the exponential stress-strength model, the second-order matching prior is of the form:*

$$\pi(\theta, \phi) = \frac{1}{\theta\phi} g(\theta\phi), \tag{3.2}$$

where g is a continuously differentiable positive function.

Proof: To find a matching prior π satisfying the probability matching equation in (3.1), let $\pi^* = \pi/(\theta\phi)$. Then the probability matching equation simplifies to

$$\frac{\partial}{\partial \theta}\left[\pi^*/\phi\right] - \frac{\partial}{\partial \phi}\left[\pi^*/\theta\right] = 0.$$

Since the equation satisfies with $\pi^* = 1$, the corresponding matching prior becomes $\pi(\theta,\phi) = \theta\phi$. Let $\pi(\theta,\phi) = \frac{1}{\theta\phi}g(\theta\phi)$. It can be shown that the prior, $\frac{1}{\theta\phi}g(\theta\phi)$ is the solution of the probability matching equation since

$$\frac{\partial}{\partial\theta}\left[\theta\pi(\theta,\phi)\right] = \frac{\partial}{\partial\phi}\left[\phi\pi(\theta,\phi)\right] = g\prime(\theta\phi).$$

□

Corollary 3.1. *If θ and ϕ have independent inverted gamma priors with parameters (α_1, β_1) and (α_2, β_2), respectively, then a prior of θ and ϕ is a second-order matching prior for r_1 if and only if $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2 = 0$.*

Proof: From the fact that

$$\frac{\partial}{\partial\theta}\left[\theta\pi(\theta,\phi)\right] = \frac{\beta_1^{\alpha_1}}{\Gamma(\alpha_1)}(\beta_1 - \alpha_1\theta)\theta^{-(\alpha_1+2)}\phi^{-(\alpha_2+1)}\exp\left[\frac{-\beta_1}{\theta} - \frac{\beta_2}{\phi}\right],$$

$$\frac{\partial}{\partial\phi}\left[\phi\pi(\theta,\phi)\right] = \frac{\beta_2^{\alpha_2}}{\Gamma(\alpha_2)}(\beta_2 - \alpha_2\phi)\theta^{-(\alpha_1+1)}\phi^{-(\alpha_2+2)}\exp\left[\frac{-\beta_1}{\theta} - \frac{\beta_2}{\phi}\right],$$

the probability matching equation is simplified to

$$\frac{\beta_1^{\alpha_1}}{\Gamma(\alpha_1)}(\beta_1 - \alpha_1\theta)\phi - \frac{\beta_2^{\alpha_2}}{\Gamma(\alpha_2)}(\beta_2 - \alpha_2\phi)\theta = 0.$$

Thus, the desired result follows. □

Corollary 3.2. *If the prior density of r_1 is the beta density with parameters α, β and the independent prior density of nuisance parameter, $r_2 = \theta + \phi$, is proportional to $r_2^{-\delta}$, then a prior for (r_1, r_2) is the second-order matching prior for r_1, if and only if:*

$$\alpha = \beta = \frac{1-\delta}{2}.$$

Proof: By the transformation of variables, since the Jacobian term becomes $(\theta+\phi)^{-1}$, the functional form of prior has

$$\pi(\theta,\phi) \propto \theta^{\beta-1}\phi^{\alpha-1}(\theta+\phi)^{1-(\alpha+\beta+\delta)}.$$

By Theorem 3.1, the desired result follows. □

It is interesting to note that with the choice of $\alpha = \beta = 0$ and $\delta = 1$, the second-order matching prior becomes the Berger and Bernardo's forward and backward reference prior. In the case of $\delta = 0$ - assigning the density to be proportional to constant over R^+ - to be a second-order matching prior, the prior density of r_1 should be the arcsine distribution. In this case another second-order matching prior can be found when $\alpha = \beta = 1/2$ and $\delta = 0$, that is,

$$\pi_M(r_1, r_2) \propto \left(\frac{1}{r_1(1-r_1)}\right)^{\frac{1}{2}}, \; 0 < r_1 < 1 \text{ and } r_2 > 0. \tag{3.3}$$

For the inference of exponential stress-strength model with small sample size, it is important to check the asymptotic efficiency since the first order asymptotic based on the large sample theory provide sampling distribution on which the width of the confidence interval is smaller than that on the true sampling distribution. Corollary 3.2 shows that if prior setting is $\alpha = \beta = (1 - \delta)/2$, theses discrepancies became smaller with the posterior density in (2.3) suggested by Thompson and Basu (1993). In next section, two second-order matching priors are chosen to evaluate the performance of confidence level via the Monte-Carlo simulation by counting that true reliability is in one-sided 0.05, 0.5 and 0.95 confidence intervals.

4. MONTE-CARLO SIMULATION STUDY

In the Monte-Carlo simulation study, two priors are investigated to check the agreement of frequentist coverage probability with desired frequentist's confidence level when sample size is small. The first prior is the Berger and Bernardo's forward and backward prior, π_B which is a second-order matching prior with the prior setting $\alpha = \beta = 0$ and $\delta = 1$. The second one is another second-order matching prior, π_M with the prior setting $\alpha = \beta = 1/2$ and $\delta = 0$. This prior, π_M is used for evaluation purpose.

Let $\omega_\alpha(\mathbf{x}, \mathbf{y})$ be the α-th marginal posterior quantile of r_1 for a given data set defined by $F(\omega_\alpha|\mathbf{x}, \mathbf{y}) = \alpha$, where $F(\cdot|\mathbf{x}, \mathbf{y})$ is the marginal posterior distribution function as given in prior π_B, π_M. The following theorem allows us to choose arbitrary values of θ and ϕ since the frequentist coverage probability of $\omega_\alpha(\mathbf{x}, \mathbf{y})$ that r_1 is no more than $\omega_\alpha(\mathbf{x}, \mathbf{y})$ does not depend on the choice of θ and ϕ.

Lemma 4.1. *For fixed θ and ϕ with prior settings $\alpha = \beta = 0$ and $\delta = 1$, $\alpha = \beta = 1/2$ and $\delta = 0$, the marginal posterior density of r_1 in (2.3) does not depend on the choice of θ and ϕ.*

Proof: Define two independent random variables X and Y that have the Gamma densities with parameters $(m,1)$ and $(n,1)$, respectively. Then $\sum_{i=1}^{m} X_i$ becomes θX and $\sum_{i=1}^{n} Y_i$ becomes ϕY. From the relation $r_1 = \phi/(\theta+\phi)$, the marginal posterior distribution function of r_1, $F(r_1|\mathbf{x},\mathbf{y})$, in (3.3) can be rewritten as:

$$\begin{aligned} F(r_1|\mathbf{x},\mathbf{y}) &= \int_0^{\frac{\phi}{\theta+\phi}} \frac{\left[\frac{\theta X}{\phi Y}\right]^m}{B(m,n)} \frac{s^{m-1}(1-s)^{n-1}}{\left[1-\left(1-\frac{\theta X}{\phi Y}\right)s\right]^{m+n}}\, ds \\ &= \int_0^{r_1} \frac{\left[\frac{1-r_1}{r_1}\frac{X}{Y}\right]^m}{B(m,n)} \frac{s^{m-1}(1-s)^{n-1}}{\left[1-\left(1-\frac{1-r_1}{r_1}\frac{X}{Y}\right)s\right]^{m+n}}\, ds. \end{aligned}$$

Let $z = s/(1-s)$. By the transformation of variable,

$$F(r_1|\mathbf{x},\mathbf{y}) = \int_0^{\frac{r_1}{1-r_1}} \frac{\left[\frac{1-r_1}{r_1}\frac{X}{Y}\right]^m}{B(m,n)} \frac{z^{m-1}}{\left(1+\frac{1-r_1}{r_1}\frac{X}{Y}z\right)^{m+n}}\, dz.$$

Let $t = \frac{1-r_1}{r_1}z$. By the transformation of variable,

$$F(r_1|\mathbf{x},\mathbf{y}) = \int_0^{1} \frac{\left[\frac{X}{Y}\right]^m}{B(m,n)} \frac{t^{m-1}}{\left(1+\frac{X}{Y}t\right)^{m+n}}\, dt.$$

Since $F(r_1|\mathbf{x},\mathbf{y})$ does not depend on θ and ϕ, the desired result follows.

In the case of the prior setting $\alpha=\beta=1/2$ and $\delta=0$, with similar procedure, $F(r_1|\mathbf{x},\mathbf{y})$ can be simplified as:

$$\begin{aligned} F(r_1|\mathbf{x},\mathbf{y}) &= \int_0^{r_1} \frac{\left[\frac{1-r_1}{r_1}\frac{X}{Y}\right]^{m-\frac{1}{2}}}{B\left(m-\frac{1}{2},n-\frac{1}{2}\right)} \frac{s^{m-\frac{1}{2}-1}(1-s)^{n-\frac{1}{2}-1}}{\left[1-\left(1-\frac{1-r_1}{r_1}\frac{X}{Y}\right)s\right]^{m+n-1}}\, ds \\ &= \int_0^{1} \frac{\left[\frac{X}{Y}\right]^{m-\frac{1}{2}}}{B\left(m-\frac{1}{2},n-\frac{1}{2}\right)} \frac{t^{m-\frac{1}{2}-1}}{\left(1+\frac{X}{Y}t\right)^{m+n-1}}\, dt. \end{aligned}$$

the desired result follows similarly. □

Theorem 4.1. *With prior settings $\alpha=\beta=0$ and $\delta=1$, $\alpha=\beta=1/2$ and $\delta=0$, the frequentist coverage probability, $P_{(\theta,\phi)}\left((\mathbf{x},\mathbf{y})|r_1\le\omega_\alpha(\mathbf{x},\mathbf{y})\right)$ is independent of the choice of θ and ϕ, where $\omega_\alpha(\mathbf{x},\mathbf{y})$ is the α-th random marginal posterior quantile of r_1.*

Proof: From the following probability relation:

$$P_{(\theta,\phi)}\left((\mathbf{x},\mathbf{y})|r_1 \le \omega_\alpha(\mathbf{x},\mathbf{y})\right) = P_{(\theta,\phi)}\left((\mathbf{x},\mathbf{y})|F(r_1|\mathbf{x},\mathbf{y}) \le \alpha\right),$$

the joint probability measure of $(\mathbf{x},\mathbf{y})$, $P_{(\theta,\phi)}(\cdot)$ is independent of the choice of θ and ϕ by Lemma 4.1. Hence the desired result follows. □

Theorem 4.2 indicates that the simulation results are invariant under the parameter values of θ and ϕ. Therefore, no different setting for values of θ and ϕ is necessary. Hence in this simulation study, we set both values of θ and ϕ are 1 and true reliability becomes 0.5. The simulation was done by the following procedure: 1) generating 50,000 data sets, 2) finding the posterior quantiles for each data sets, and 3) determining the proportion that ω_α is smaller than the true value $r_1 = 0.5$ after fixing $\theta = \phi = 1$. This was done on a SPARC workstation. The standard error of the entries is 0.002. For the numerical integration, the Gauss quadrature method are used. The simulation program was written as Fortran77 linked with the IMSL Math and Stat Library. In this study, 0.05, 0.5 and 0.95-th marginal posterior quantiles of r_1 are considered. The results are shown at Table 4.1.

Table 4.1: Frequentist Coverage Probabilities for Exponential

m	n	π_B			π_M		
		0.05	0.50	0.95	0.05	0.50	0.95
2	2	0.0529	0.5012	0.9504	0.0282	0.5012	0.9737
2	3	0.0493	0.4991	0.9492	0.0210	0.4406	0.9563
3	2	0.0501	0.4997	0.9510	0.0428	0.5559	0.9791
5	5	0.0509	0.4986	0.9495	0.0417	0.4986	0.9587
5	10	0.0506	0.5019	0.9502	0.0354	0.4599	0.9463
10	5	0.0492	0.4986	0.9495	0.0541	0.5408	0.9648
10	10	0.0509	0.5004	0.9504	0.0469	0.5004	0.9546

Table 4.1 shows that the posterior quantiles for π_B yield frequentist error rate that are closer to the ideal 0.05, 0.5 and 0.95 than those for π_M. For small sample sizes (e.g. $m, n < 5$), the posterior quantiles for π_M seem to be small, implying that the event that the true value of r_1 is greater than the 0.05-th quantile, occurs more often than one would desire. Therefore, we can conclude that, for

small sample size with π_M, the width of confidence interval is larger than that of the frequentist's desired confidence interval. This seems to be reasonable in the support of π_B when small sample size are used. However, as sample size increase, both π_B and π_M provide the ideal posterior quantiles that are close to the ideal frequentist probabilities.

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