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A Generalization of the Discrete Feedback Adjustment by Rational Subgrouping †

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ABSTRACT

Process adjustment has been widely used in production processes in order to set the output characteristic as close as to the target. Box and Kramer(1992) developed a feedback adjustment control procedure for process adjustment. We generalize their procedure by using a rational subgrouping of sequential observations. In this paper the feedback control rule of the rational subgrouping is proposed and the overall expected cost is evaluated. Also properties of the proposed control scheme are illustrated and compared to Box and Kramer's in the context of the expected cost.

Keywords: Process Adjustment; Automatic Process Control; Rational Subgrouping; Feedback Adjustment.

1. INTRODUCTION

Process monitoring and process adjustment are two complementary approaches to process control. In particular, Shewhart, cumulative sum(CUSUM), and exponentially weighted moving average(EWMA) charts are frequently employed for process monitoring as part of what is called statistical process control(SPC). By use of such charts we can continually check the state of the production system, and eliminate assignable causes pointed to by discrepant behavior.

By contrast, various forms of feedback and feedforward controls are used for process adjustment in what is often called automatic process control(APC). APC attemps to adjust a manipulated variable, whose effect on some output quality characteristic is already known, so as to maintain the process as close as possible

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to the desired target value(see Box, Jenkins and Reinsel(1994, Part IV), and Box and Luceño(1997b)). In this paper, we consider only feedback control. APC has been discussed for many years since contributions by Box and Jenkins(1962, 1963) and Box, Jenkins and MacGregor(1974). It has recently gained wider interests and has been studied by Kramer(1989), Box and Kramer(1992), Luceño(1993), Box and Luceño(1994, 1997a), and Luceño, Gonzalez and Puig-Pey(1996).

Suppose that observations and opportunities for adjustment occur at discrete times $\cdots, t-1, t, t+1, \cdots$ equispaced at what we define as unit intervals. The conventional Box and Kramer(1992)'s APC procedure, by considering the adjustment, the monitoring and the off-target cost, assumed that one observation is sampled at intervals of m units. In this paper, we generalize this APC procedure by using the rational subgrouping, which is often recommended as a factor to consider in economic designs of SPC. The sampling scheme with the rational subgrouping is based on the subgroup size and the sampling interval. Basically we assume that one item is produced at each unit interval. The subgroup size is the number of consecutive observations taken as a sample from the production process. The sampling interval is the number of unit intervals elapsed from the end of one subgroup to the start of the next. If the subgroup size is equal to 1, the rational subgrouping method is equivalent to Box and Kramer(1992)'s. We also propose the feedback control rule and derive the overall expected cost function when the rational subgrouping is used.

2. BOX AND KRAMER'S APC PROCEDURE

Let X_t be the manipulate input variable and y_t be the output quality characteristic at time t. We assume that one unit change in X_t will produce g units, which is called the system gain, of change in y_t. Moreover, we define the disturbance z_t as $y_t - T$, that is, the deviation from some target value T that would occur if no attemp at control were made.

A widely used disturbance model in APC is IMA(0,1,1) process defined as

$$z_{t+1} - z_t = a_{t+1} - \theta_a\, a_t, \tag{2.1}$$

where the random shocks a_t's are iid $N(0, \sigma_a^2)$, and $\theta_a(= 1 - \lambda_a)$ is a smoothing constant. The validity of IMA(0,1,1) for disturbance models is illustrated by the variogram, defined by $Var(z_{t+m} - z_t)/Var(z_{t+1} - z_t)$. It is reasonable to think that, if no corrective action is taken, observations spaced further apart will differ more and more, and thus the variogram will increase monotonically in m. As was pointed out by Box and Kramer(1992, pp. 254-255) and Box, Jenkins and Reinsel(1994, pp. 493-495), the IMA(0,1,1) process satisfies this property.

Kramer(1989) and Box and Kramer(1992) suggested the APC procedure by considering the adjustment cost C_A, the monitoring cost C_M and the off-target cost C_T. When adjustment and monitoring costs are substantial, it is not optimal to sample and adjust the process at every unit interval. Instead it would be better to take observations at every $m(\geq 1)$ number of unit intervals and make adjustments when a certain criterion is satisfied. Reductions in costs for monitoring and adjustment will compensate for the increment in cost for off-target.

Let the process sampled at every m unit intervals be $n_t(= z_{tm})$ and the one-step ahead forecast of n_{t+1} at time tm be $\hat{n}_{t+1}$. Suppose that the last adjustment is made at time tm, then the control procedure is to make an adjustment at time $(t+N)m$ when $|\hat{n}_{t+N+1} - \hat{n}_{t+1}| \geq L$ for the action limit L. Here N denotes the number of observations taken until the next adjustment. If the disturbance at every unit interval is generated by the IMA(0,1,1) model in (2.1), then the sampled process at every m unit intervals n_t is also an IMA(0,1,1) process but with the smoothing constant $\theta_m(= 1 - \lambda_m)$ and the variance of random shocks σ_m^2, which satisfy

$$\lambda_m^2\sigma_m^2 = m\lambda_a^2\sigma_a^2 \quad \text{and} \quad \theta_m\sigma_m^2 = \theta_a\sigma_a^2. \tag{2.2}$$

Thus the one-step ahead forecast of n_{t+N+1} can be obtained as an EWMA of the current and past observations, i.e.,

$$\hat{n}_{t+N+1} = \lambda_m(n_{t+N} + \theta_m n_{t+N-1} + \theta_m^2 n_{t+N-2} + \cdots) \tag{2.3}$$

or

$$\hat{n}_{t+N+1} = \lambda_m n_{t+N} + \theta_m \hat{n}_{t+N}, \tag{2.4}$$

and the optimal control rule is to set $X_{(t+N)m} = -\hat{n}_{t+N+1}/g$ for the system gain g. Notice that the forecast in (2.3) or (2.4) is the minimum mean squared error forecast when the process follows an IMA(0,1,1).

The average adjustment interval(AAI) is the average number of unit intervals between adjustments, that is, $mE(N)$, and the mean squared deviation(MSD) is the expectation of squared deviation of the quality from the target per unit interval, which is defined as

$$\text{MSD} = \frac{1}{\text{AAI}} E\Big[\sum_{i=1}^{N}\sum_{j=1}^{m}\Big\{z_{(t+i-1)m+j} + gX_{tm}\Big\}^2\Big]. \tag{2.5}$$

Then the overall expected cost C per unit interval is expressed as

$$C_{BK} = \frac{C_A}{\mathrm{AAI}_{BK}} + \frac{C_M}{m} + \frac{C_T}{\sigma_a^2}\,\mathrm{MSD}_{BK}, \tag{2.6}$$

where subscripts BK denote corresponding functions by Box and Kramer(1992).

Kramer(1989) also showed the equation (2.6) could be written in the form

$$\begin{aligned} C_{BK} &= \frac{C_A}{m\,h[\,L/(\lambda_m\sigma_m)]} + \frac{C_M}{m} \\ &\quad + C_T\left\{\frac{\theta_a}{\theta_m} + m\lambda_a^2\,g[\,L/(\lambda_m\sigma_m)] - \frac{(m-1)\lambda_a^2}{2}\right\}, \end{aligned} \tag{2.7}$$

where $h(\cdot)$ and $g(\cdot)$ are functions related to AAI and MSD, respectively. Several methods for obtaining approximations for these functions have been given in the literature. For example, Kramer(1989) suggested functions based on extensive simulation, and Box and Luceño(1994) used integral equations in expressing these functions to solve numerically. Since the approximation by Kramer(1989) is of a simple form and accurate, we use his approximation in this paper. The approximations are given by

$$h(B) = (1 + 1.1B + B^2)\left\{1 - 0.115\,\exp\left[-9.2\left(B^{0.3} - 0.88\right)^2\right]\right\},$$

$$g(B) = \frac{1 + 0.06B^2}{1 - 0.647\,\Phi\left\{1.35\,[\,\ln(B) - 0.67]\right\}} - 1, \tag{2.8}$$

where $\Phi(\cdot)$ is the standard normal distribution function.

3. APC WITH RATIONAL SUBGROUPING

The rational subgrouping of sequential observations has been widely used in process control procedures. By the rational subgrouping we obtain samples of size $n(\geq 1)$ rather than a single observation. It is expected that the use of subgroup sample means will reduce the process variation and will produce more efficient control scheme than the use of single observation. A subgroup is defined as n consecutive observations taken during the process and the next subgroup is taken after h intervals. That is, the sample size is n and the sampling interval is h. At the time of the last observation in each subgroup a subgroup mean is calculated as

$$p_t = \sum_{i=tm-n+1}^{tm} z_i/n$$

Let $m = h + n$, then we observe subgroup means at every m intervals. Suppose that the last adjustment is made at some time tm, then we make an adjustment at time $(t+N)m$ when

$$|\,\hat{p}_{t+N+1} - \hat{p}_{t+1}| \geq L.$$

The adjustment is made by setting $X_{(t+N)m} = -\hat{p}_{t+N+1}/g$. The overall expected cost C per unit interval can be modified as

$$C_R = \frac{C_A}{\mathrm{AAI}_R} + \frac{n\,C_M}{m} + \frac{C_T}{\sigma_a^2}\,\mathrm{MSD}_R. \tag{3.1}$$

Note that if $n = 1$ then the proposed control scheme is equivalent to that of Box and Kramer(1992). The properties of the rational subgrouping by using p_t are shown by the following theorem and corollary.

Theorem 3.1. *The process p_t is also an IMA(0,1,1) time series model, that is,*

$$p_{t+1} - p_t = b_{t+1} - \theta_p\, b_t,$$

where the random shocks b_t's are iid $N(0,\sigma_p^2)$ and θ_p is a smoothing constant.

Proof: Let $\nabla p_t = p_t - p_{t-1}$. The difference ∇p_t can be written as

$$\begin{aligned}
\nabla p_t &= (1/n)\Big[\sum_{i=tm-n+1}^{tm} z_i - \sum_{i=(t-1)m-n+1}^{(t-1)m} z_i\Big] \\
&= (1/n)\Big[\sum_{k=0}^{n-1}\{1+k\lambda_a\}\,a_{tm-k} + n\lambda_a \sum_{k=n}^{m-1} a_{tm-k} \\
&\quad + \sum_{k=m}^{m+n-1}\{(m+n-1-k)\lambda_a - \theta_a\}\,a_{tm-k}\Big].
\end{aligned}$$

Then, for ∇p_t, the autocovariance $\gamma_k(m,n) = \mathrm{Cov}(\nabla p_t \cdot \nabla p_{t-k})$ are given by

$$\begin{aligned}
\gamma_0(m,n) &= \frac{\sigma_a^2}{n}\left\{(1+\theta_a^2) + (m-n)n\lambda_a^2 + \frac{2}{3}(n-1)(n+1)\,\lambda_a^2\right\}, \\
\gamma_1(m,n) &= \frac{\sigma_a^2}{n}\left\{\frac{1}{6}(n-1)(n+1)\lambda_a^2 - \theta_a\right\}, \\
\gamma_j(m,n) &= 0, \quad j \geq 2.
\end{aligned}$$

It follows that the process p_t is an IMA process of order (0,1,1). □

Corollary 3.1. *The parameters $\theta_p(= 1-\lambda_p)$, σ_p^2 of p_t and θ_a, σ_a^2 of z_t are related as follows.*

$$\begin{aligned} \lambda_p^2\sigma_p^2 &= m\lambda_a^2\sigma_a^2, \\ \theta_p\sigma_p^2 &= \theta_a\frac{\sigma_a^2}{n} - \frac{1}{6}(n-1)(n+1)\lambda_a^2\frac{\sigma_a^2}{n}. \end{aligned}$$

Proof: The following properties of IMA(0,1,1) time series models give the above results(See Box, Jenkins and Reinsel(1994, pp. 526-529)).

$$\begin{aligned} \frac{\gamma_0(m,n)+2\gamma_1(m,n)}{\gamma_1(m,n)} &= -\frac{(1-\theta_p)^2}{\theta_p}, \\ \gamma_1(m,n) &= -\theta_p\sigma_p^2. \end{aligned}$$

□

Notice that from Corollary 1 and (2.2) we can see $\lambda_p\sigma_p = \lambda_m\sigma_m$. The explicit expression of the parameter θ_p is obtained as follows by Corollary 1 and the invertibility condition of IMA models.

$$\theta_p = \begin{cases} A_p - \sqrt{A_p^2 - 1} & \text{if } n = 1 \text{ or } \left(\frac{n^2+2-\sqrt{6n^2+3}}{n^2-1} < \theta_a < 1 \ \text{ for } n \geq 2\right) \\ A_p + \sqrt{A_p^2 - 1} & \text{if } 0 < \theta_a < \frac{n^2+2-\sqrt{6n^2+3}}{n^2-1} \ \text{ for } n \geq 2 \end{cases},$$

where $A_p = 1 + \dfrac{mn\lambda_a^2}{2\theta_a - \frac{1}{3}(n-1)(n+1)\,\lambda_a^2}$.

4. THE EXPECTED COST OF THE RATIONAL SUBGROUPING PROCEDURE

In calculating the expected cost (3.1), we need to derive the AAI and MSD of the proposed control scheme in Section 3. Then the overall expected cost function is used to find optimal parameters, such as the subgroup size n, the sampling interval h, and the action limit L, which produce the minimum overall cost.

Theorem 4.1. *The AAI of the rational subgrouping is obtained as*

$$\text{AAI}_R = m\,h[\,L/(\lambda_p\sigma_p)].$$

Proof: Suppose that the last adjustment is made at time tm. Then the next adjustment should be made at time $(t+N)m$ when $|\,\hat{p}_{t+N+1} - \hat{p}_{t+1}| \geq L$. Here

$$\hat{p}_{t+N+1} - \hat{p}_{t+1} = \lambda_p(b_{t+N} + b_{t+N-1} + \cdots + b_{t+1}).$$

The action criterion(adjustment rule) is now expressed as

$$|\,u_{t+N} + u_{t+N-1} + \cdots + u_{t+1}| \geq L/(\lambda_p\sigma_p),$$

where the random shocks u_t are iid $N(0,1)$. Because $h(B)$ is defined by the expected first passage time for a standardized random walk with barrier B, $E(N)$ becomes $h(L/\lambda_p\sigma_p)$. Therefore

$$\text{AAI}_R = mE(N) = m\,h(\,L/\lambda_p\sigma_p).$$

□

Notice that the AAI_R does not depend on the subgroup size n. From Theorem 4.1 and the fact that $\lambda_p\sigma_p = \lambda_m\sigma_m$, we see that AAI_R=AAI_{BK}.

Theorem 4.2. *The MSD of the rational subgrouping is obtained as*

$$\begin{aligned}\text{MSD}_R &= \sigma_p^2\Big\{1 + \lambda_p^2\, g[\,L/(\lambda_p\sigma_p)]\Big\}\\ &+ \frac{\sigma_a^2}{mn}\Big[\,\frac{1}{6}\Big\{(m-n)(n^2-1-3mn) + n(n-1)(n+1)\Big\}\lambda_a^2\\ &+ m(n-1)\theta_a\Big].\end{aligned}$$

Proof: See APPENDIX. □

From the fact that MSD_{BK} in equation (2.7),

$$\text{MSD}_{BK} = \sigma_m^2 + m\lambda_a^2\sigma_a^2\, g[\,L/(\lambda_m\sigma_m)] - \frac{(m-1)\lambda_a^2\sigma_a^2}{2}$$

and $\lambda_p\sigma_p = \lambda_m\sigma_m$, the difference of MSD_{BK} and MSD_R can be written as

$$\text{MSD}_{BK} - \text{MSD}_R = \sigma_m^2 - \sigma_p^2 - \frac{n-1}{6n}\left\{(4n+1)\lambda_a^2 + 6\theta_a\right\}\,\sigma_a^2.$$

By using the expressions of AAI_R and MSD_R in Theorems 4.1 and 3, respectively, the overall expected cost function can be expressed as

$$\begin{aligned}C_R &= \frac{C_A}{m\,h[\,L/(\lambda_p\sigma_p)]} + \frac{n\,C_M}{m} + \frac{C_T}{\sigma_a^2}\,\Big[\,\sigma_p^2\Big\{1 + \lambda_p^2\, g[\,L/(\lambda_p\sigma_p)]\Big\}\\ &+ \frac{\sigma_a^2}{mn}\Big[\,\frac{1}{6}\Big\{(m-n)(n^2-1-3mn) + n(n-1)(n+1)\Big\}\lambda_a^2\\ &+ m(n-1)\theta_a\Big]\,\Big].\end{aligned}$$

In Table 4.1, the values of the subgroup size n, which provides the minimum MSD_R, and the difference of MSD_{BK} and the minimum MSD_R are obtained for some given values of λ_a and m. The function $g(\cdot)$ of (2.8) is used in calculating MSD_R. Note that the value of n to provide the minimum MSD_R does not depend on the action limit L for fixed λ_a, σ_a^2 and m because $L/(\lambda_p\sigma_p) = L/(\sqrt{m}\lambda_a\sigma_a)$.

Table 4.1 shows that the subgroup size n which produces the minimum MSD_R is larger than 1 for small and moderate values of λ_a (≤ 0.6). We see that the subgroup size increases monotonically as λ_a decreases for given m. We also see that the difference between the two MSD's increases monotonically as m increases for given optimal $n > 1$. This indicates that Box and Kramer(1992)'s procedures may produce larger MSD than the rational subgrouping with subgroup size $n > 1$, and thus their procedure is not optimal in the context of the expected cost.

5. CONCLUSIONS AND REMARKS

APC schemes are designed to minimize the overall expected cost function, and this minimum-cost schemes depend on the adjustment, the monitoring and the off-target cost. However because the off-target cost is generally larger than the other costs, the control method to reduce MSD is very meaningful.

In this paper we generalize Box and Kramer(1992)'s APC procedures by using the rational subgrouping. In calculating the subgroup size to provide the minimum MSD, we can see that the proposed control scheme gives good performances in reducing the MSD especially when λ_a is small or moderate. Determination of optimal control parameters(the subgroup size, the sampling interval, and the action limit) which produce the minimum expected cost in the rational subgrouping is left as a further study.

Table 4.1 Values of the subgroup size n which produces the minimum MSD_R

λ_a	m	n	$MSD_{BK} - MSD_R$	λ_a	m	n	$MSD_{BK} - MSD_R$
	5	5	0.098		5	2	0.053
	10	10	0.160		10	2	0.077
0.1	20	13	0.236	0.6	20	2	0.092
	30	14	0.289		30	2	0.097
	40	15	0.330		40	2	0.100
	50	15	0.363		50	2	0.102
	5	5	0.144		5	1	0.000
	10	6	0.221		10	2	0.011
0.2	20	7	0.311	0.7	20	2	0.019
	30	7	0.364		30	2	0.022
	40	7	0.400		40	2	0.023
	50	7	0.426		50	2	0.023
	5	4	0.147		5	1	0.000
	10	4	0.221		10	1	0.000
0.3	20	5	0.293	0.8	20	1	0.000
	30	5	0.330		30	1	0.000
	40	5	0.353		40	1	0.000
	50	5	0.369		50	1	0.000
	5	3	0.127		5	1	0.000
	10	3	0.185		10	1	0.000
0.4	20	3	0.233	0.9	20	1	0.000
	30	3	0.254		30	1	0.000
	40	3	0.266		40	1	0.000
	50	3	0.274		50	1	0.000
	5	2	0.092		5	1	0.000
	10	2	0.127		10	1	0.000
0.5	20	3	0.153	1.0	20	1	0.000
	30	3	0.165		30	1	0.000
	40	3	0.172		40	1	0.000
	50	3	0.176		50	1	0.000

APPENDIX

Proof of Theorem 4.2. From AAI $= mE(N)$ and $X_{tm} = -\hat{p}_{t+1}/g$, the MSD of (2.5) becomes

$$\begin{aligned}
\mathrm{MSD}_R &= \frac{1}{mE(N)} E\Big[\sum_{i=1}^{N}\sum_{j=1}^{m}\Big\{z_{(t+i-1)m+j} - \hat{p}_{t+1}\Big\}^2\Big] \\
&= \frac{1}{mE(N)} E\Big[\sum_{i=1}^{N}\sum_{j=1}^{m}\Big\{p_{t+i} - \hat{p}_{t+1} + z_{(t+i-1)m+j} - p_{t+i}\Big\}^2\Big] \\
&= \frac{1}{mE(N)} E\Big[\sum_{i=1}^{N}\sum_{j=1}^{m}\Big\{p_{t+i} - \hat{p}_{t+1}\Big\}^2\Big] \\
&\quad + \frac{1}{mE(N)} E\Big[\sum_{i=1}^{N}\sum_{j=1}^{m}\Big\{z_{(t+i-1)m+j} - p_{t+i}\Big\}^2\Big] \\
&\quad + \frac{2}{mE(N)} E\Big[\sum_{i=1}^{N}\sum_{j=1}^{m}\Big\{p_{t+i} - \hat{p}_{t+1}\Big\}\Big\{z_{(t+i-1)m+j} - p_{t+i}\Big\}\Big] \qquad (A.1)
\end{aligned}$$

For convenience, let the three terms in the right hand side of (A.1) be denoted by (T1), (T2), and (T3), respectively. That is, MSD_R=(T1)+(T2)+(T3). Then it is easily seen that

$$(\mathrm{T1}) = \sigma_p^2 \Big\{1 + \lambda_p^2\, g[\, L/(\lambda_p\sigma_p)]\Big\}, \qquad (A.2)$$

by simple algebra and the definition of $g(\cdot)$(See Kramer(1989, pp. 35 or pp. 86-87)).

In (T2), $z_{(t+i-1)m+j} - p_{t+i}$ can be written as

$$\begin{aligned}
& z_{(t+i-1)m+j} - \Big\{z_{(t+i)m} + z_{(t+i)m-1} + \cdots + z_{(t+i)m-(n-1)}\Big\}/n \\
&= \begin{cases} -(1/n)\sum_{k=0}^{n-1}\Big\{z_{(t+i)m-k} - z_{(t+i)m-(m-j)}\Big\} & \text{if } m-j \ge n \\ -(1/n)\sum_{k=0}^{m-j-1}\Big\{z_{(t+i)m-k} - z_{(t+i)m-(m-j)}\Big\} & \\ \quad +(1/n)\sum_{k=m-j+1}^{n-1}\Big\{z_{(t+i)m-(m-j)} - z_{(t+i)m-k}\Big\} & \text{if } m-j < n \end{cases} \\
&= \begin{cases} -(1/n)\Big[\sum_{l=0}^{n-1}(1+l\lambda_a)a_{(t+i)m-l} & \\ \quad +\sum_{l=n}^{m-j-1} n\lambda_a a_{(t+i)m-l} - n\theta_a a_{(t+i)m-(m-j)}\Big] \equiv A & \text{if } m-j \ge n \\ -(1/n)\Big[\sum_{l=0}^{m-j-1}(1+l\lambda_a)a_{(t+i)m-l} & \\ \quad -\Big\{(m-j)\theta_a + (n-m-1+j)\Big\}a_{(t+i)m-(m-j)} & \\ \quad -\sum_{l=m-j+1}^{n-1}\Big\{(n-1-l)\lambda_a - \theta_a\Big\}a_{(t+i)m-l}\Big] \equiv B & \text{if } m-j < n, \end{cases}
\end{aligned}$$

where '≡' denotes 'is defined as'. By Wald's equation the second term becomes

$$
\begin{aligned}
(\mathrm{T2}) &= (1/m)\Big\{\sum_{j=1}^{m-n} E(A^2) + \sum_{j=m-n+1}^{m} E(B^2)\Big\} \\
&= \frac{\sigma_a^2}{mn}\Big[\sum_{j=1}^{m-n}\sum_{l=0}^{n-1}(1+l\lambda_a)^2 + \sum_{j=1}^{m-n-1}\sum_{l=n}^{m-j-1} n^2\lambda_a^2 + \sum_{j=1}^{m-n} n^2\theta_a^2 \\
&\quad + \sum_{j=m-n+1}^{m-1}\sum_{l=0}^{m-j-1}(1+l\lambda_a)^2 + \sum_{j=m-n+1}^{m}\Big\{(m-j)\theta_a + (n-m-1+j)\Big\}^2 \\
&\quad + \sum_{j=m-n+2}^{m}\sum_{l=m-j+1}^{n-1}\Big\{(n-1-l)\lambda_a - \theta_a\Big\}^2\Big] \equiv C.
\end{aligned}
$$

In (T3), by the fact that

$$p_{t+i} - \hat{p}_{t+1} = b_{t+i} + \lambda_p\,(b_{t+i-1} + \cdots + b_{t+1}),$$

we have

$$
\begin{aligned}
&E\Big[\sum_{i=1}^{N}\sum_{j=1}^{m}\Big\{p_{t+i} - \hat{p}_{t+1}\Big\}\Big\{z_{(t+i-1)m+j} - p_{t+i}\Big\}\Big] \\
&= E\Big[\sum_{i=1}^{N}\sum_{j=1}^{m} b_{t+i}\Big\{z_{(t+i-1)m+j} - p_{t+i}\Big\}\Big] \\
&= E\Big[\sum_{i=1}^{N}\sum_{j=1}^{m}\Big\{b_{t+i} - \theta_p b_{t+i-1}\Big\}\Big\{z_{(t+i-1)m+j} - p_{t+i}\Big\}\Big].
\end{aligned}
$$

Now,

$$
\begin{aligned}
b_{t+i} - \theta_p b_{t+i-1} &= p_{t+i} - p_{t+i-1} \\
&= (1/n)\Big[\sum_{l=0}^{n-1}(1+l\lambda_a)a_{(t+i)m-l} + \sum_{l=n}^{m-1} n\lambda_a a_{(t+i)m-l} \\
&\quad + \sum_{l=m}^{m+n-1}\Big\{(m+n-1-l)\lambda_a - \theta_a\Big\}\, a_{(t+i)m-l}\Big] \equiv D,
\end{aligned}
$$

then (T3) becomes

$$
\begin{aligned}
\text{(T3)} &= \frac{2}{m}\Big[\sum_{j=1}^{m-n} E(D\cdot A) + \sum_{j=m-n+1}^{m} E(D\cdot B)\Big] \\
&= -\frac{2\sigma_a^2}{mn^2}\Big[\sum_{j=1}^{m-n}\sum_{l=0}^{n-1}(1+l\lambda_a)^2 + \sum_{j=1}^{m-n-1}\sum_{l=n}^{m-j-1} n^2\lambda_a^2 - \sum_{j=1}^{m-n} n^2\theta_a\lambda_a \\
&\quad + \sum_{j=m-n+1}^{m-1}\sum_{l=0}^{m-j-1}(1+l\lambda_a)^2 \\
&\quad - \sum_{j=m-n+1}^{m}\Big\{(m-j)\theta_a + (n-m-1+j)\Big\}\Big\{1+(m-j)\lambda_a\Big\} \\
&\quad - \sum_{j=m-n+2}^{m}\sum_{l=m-j+1}^{n-1}\Big\{(n-1-l)\lambda_a - \theta_a\Big\}(1+l\lambda_a)\Big] \equiv E.
\end{aligned}
$$

Summing (T2) and (T3) equals

$$
\begin{aligned}
&C+E \\
=\ & \frac{\sigma_a^2}{mn^2}\Big[-\sum_{j=1}^{m-n}\sum_{l=0}^{n-1}(1+l\lambda_a)^2 - \sum_{j=1}^{m-n-1}\sum_{l=n}^{m-j-1} n^2\lambda_a^2 + \sum_{j=1}^{m-n}\Big\{n^2\theta_a^2 + 2n^2\theta_a\lambda_a\Big\} \\
& - \sum_{j=m-n+1}^{m-1}\sum_{l=0}^{m-j-1}(1+l\lambda_a)^2 \\
& + \sum_{j=m-n+1}^{m}\Big\{(m-j)\theta_a + (n-m-1+j)\Big\} \\
& \quad \cdot\Big\{(m-j)\theta_a + (n-m-1+j) + 2 + 2(m-j)\lambda_a\Big\} \\
& + \sum_{j=m-n+2}^{m}\sum_{l=m-j+1}^{n-1}\Big\{(n-1-l)\lambda_a - \theta_a\Big\}\Big\{(n-1+l)\lambda_a - \theta_a + 2\Big\}\Big].
\end{aligned}
$$

After some tedious calculations and simplifications, we obtain

$$
\begin{aligned}
C+E &= \frac{\sigma_a^2}{mn}\Big[\frac{1}{6}\Big\{(m-n)(n^2-1-3mn) + n(n-1)(n+1)\Big\}\lambda_a^2 \\
&\quad + m(n-1)\theta_a\Big]. \qquad \text{(A.3)}
\end{aligned}
$$

Combining (A.2) and (A.3), we have the result.

References

Box, G. E. P. and Jenkins, G. M. (1962). "Some Statistical Aspects of Adaptive Optimization and Control," *Journal of the Royal Statistical Society*, Series B, **24**, 297-343.

Box, G. E. P. and Jenkins, G. M. (1963). "Further Contributions to Adaptive Optimization and Control : Simultaneous Estimation of Dynamics : Non-Zero Costs," *Bulletin of the International Statistical Institute*, 34th Session, Ottawa, 943-974.

Box, G. E. P., Jenkins, G. M. and MacGregor, J. F. (1974). "Some Recent Advanced in Forecating and Control," Part II, *Journal of the Royal Statistical Society*, Series C, **23**, 158-179.

Box, G. E. P., Jenkins, G. M. and Reinsel, G. C. (1994). *Time Series Analysis : Forecast and Control*, 3rd Edition, Prentice Hall, Englewood Cliffs, NJ.

Box, G. E. P. and Kramer, T. (1992). "Statistical Process Monitoring and Feedback Adjustment - A Discussion," *Technometrics*, **34**, 251-285.

Box, G. E. P. and Luceño, A. (1994). "Selection of Sampling Interval and Action Limit for Discrete Feedback Adjustment," *Technometrics*, **36**, 369-378.

Box, G. E. P. and Luceño, A. (1997a). "Discrete Proportional-Integral Adjustment and Statistical Process Control," *Journal of Quality Technology*, **29**, 248-260.

Box, G. E. P. and Luceño, A. (1997b). *Statistical Control by Monitoring and Feedback Control*, John Wiley & Sons, New York.

Kramer, T. (1989). Process Control From an Economic Point of View, unpublished Ph.D. dissertation, University of Wisconsin-Madison, Dept. of Statistics.

Luceño, A. (1993). "Performance of EWMA Versus Last Observation for Feedback Control," *Communications in Statistics - Theory and Methods*, **22**, 241-255.

Luceño, A., Gonzalez, F. J. and Puig-Pey, J. (1996). "Computing Optimal Adjustment Schemes for the General Tool-Wear Problem," *Journal of Statistical Computation and Simulation*, **54**, 87-113.