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# A Note on the Modified Mild-Slope Equation
# 修正 緩傾斜方程式에 대한 小考

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**Abstract** □ Recently the modified mild-slope equation has been developed by several researchers using different approaches, which, compared to the Berkhoff's mild-slope equation, includes additional terms proportional to the square of bottom slope and to the bottom curvature. By examining this equation, it is shown that both terms are equally important in intermediate-depth water, but in shallow water the influence of the bottom curvature term diminishes while that of the bottom slope square term remains significant. In order to examine the importance of these terms in more detail, the modified mild-slope equation and the Berkhoff's mild-slope equation are tested for the problems of wave reflection from a plane slope, a non-plane slope, and periodic ripples. It is shown that, when only the bottom slope is concerned, the mild-slope equation can give accurate results up to a slope of 1 in 1 rather than 1 in 3, which, until now, has been known as the limiting bottom slope for its proper application. It is also shown that the bottom curvature term plays an important role in modeling wave propagation over a bottom topography with relatively mild variation, but, where the bottom slope is not small, the bottom slope square term should also be included for more accurate results.

**Keywords :** mild-slope equation, numerical models, water waves, wave propagation, wave reflection

**要 旨** : 최근 몇몇 연구자들이 서로 다른 방법을 이용하여 修正 緩傾斜方程式을 개발하였는데, 이는, Berkhoff의 완경사방정식과 비교해 볼 때, 바닥 傾斜의 제곱 및 바닥 曲率에 비례하는 항들을 추가로 포함하고 있다. 이 식을 검토한 결과, 遷移海域에서는 두 항들이 다같이 중요하지만, 淺海에서는 바닥 경사 제곱항의 영향은 중요한 반면 바닥 곡률항의 영향은 작아짐을 보였다. 이 항들의 중요성을 좀더 면밀히 검토하기 위하여, 一定 斜面, 非一定 斜面 및 주기성을 갖는 물결진 바닥으로부터의 波의 반사 문제에 대하여 수정 완경사방정식과 Berkhoff의 완경사방정식을 적용하였다. 바닥 경사만을 생각할 때, 완경사방정식이 지금까지 그 적용 한계로 알려져 왔던 1:3보다 더 급한 1:1의 경사까지 정확한 결과를 나타냄을 보였다. 또한, 비교적 변화가 적은 해저면 위에서의 파의 전파를 모의할 때는 바닥 곡률항만이 중요한 역할을 하지만, 바닥 경사가 작지 않은 경우에는 보다 정확한 결과를 얻기 위하여 바닥 경사의 제곱항도 포함시켜야 함을 보였다.

**핵심용어** : 완경사방정식, 수치모형, 파랑, 파의 전파, 파의 반사

## 1. INTRODUCTION

The mild-slope equation, since first proposed by Berkhoff (1972), has been widely used for the computation of combined refraction and diffraction of waves in coastal waters. It has not only been used in its original form of elliptic equation but also provided the basic governing equation for the development of other wave equations such as parabolic equation (Radder, 1979) and hyperbolic equations (Copeland, 1985).

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The mild-slope equation assumes that the water depth varies slowly over a wavelength, that is, $|\nabla h|/kh << 1$, where $\nabla$ = horizontal gradient operator, $h$ = water depth, and $k$ = wavenumber. Booij (1983) showed that the mild-slope equation gives accurate results up to the bottom slope of 1 in 3 through the numerical tests for wave reflection by a plane slope. However, some researchers have reported that the mild-slope equation breaks down on a rapidly varying topography with local slopes less than $O(1)$ (Kirby (1986), O'Hare and Davies (1993) among others).

Recently efforts have been made to improve the mild-slope equation by including the terms proportional to the square of bottom slope and to the bottom curvature, which were neglected in the derivation of the mild-slope equation. Using the Galerkin-eigenfunction method, Massel (1993) developed an extended refraction-diffraction equation which includes these higher-order bottom effect terms and the evanescent modes as well, although he did not give any numerical examples for the equation including the evanescent modes. Chamberlain and Porter (1995) proposed a modified mild-slope equation which includes the higher-order bottom effect terms as in the Massel's equation but not the evanescent modes. On the other hand, using the Green's formula method and Lagrangian formulation, Suh *et al.* (1997) developed two equivalent time-dependent wave equations including these higher-order bottom effect terms, which reduce to the modified mild-slope equation of Chamberlain and Porter for a monochromatic wave. Neglecting the evanescent modes, for a monochromatic wave, the equations of Massel, Chamberlain and Porter, and Suh *et al.*, despite different approaches of derivation, are all equivalent, which will be referred to as the modified mild-slope equation in the present paper as named by Chamberlain and Porter (1995).

In the present study, first the higher-order bottom effect terms in the modified mild-slope equation are examined to compare their relative importance with respect to the relative depth, $kh$. Second the Booij's (1983) problem is revisited not only to examine the performance of the modified mild-slope equation but also to re-assess the accuracy of the mild-slope equation. Third, since the Booij's problem which involves a plane slope each end of which is connected to a constant-depth region is not appropriate for exactly evaluating the effect of the bottom curvature term, these equations are tested for wave reflection from a non-plane slope on which both the bottom slope square term and the bottom curvature term are equally important. Fourth these equations are applied to the problem of resonant Bragg reflection of surface waves by periodic ripples to examine the importance of the higher-order bottom effect terms on a more realistic bathymetry. For all these problems, additional numerical tests are made to examine the relative importance of the bottom slope square term and the bottom curvature term. Finally major conclusions follow.

## 2. MODIFIED MILD-SLOPE EQUATION

The modified mild-slope equation can be written as

$$\nabla^2\phi + \frac{\nabla(CC_g)}{CC_g}\cdot\nabla\phi + k^2\left[1 + R_1(\nabla h)^2 + \frac{1}{k_o}R_2\nabla^2 h\right]\phi = 0 \quad (1)$$

where $C$ and $C_g$ = local phase and group velocities, respectively, $k_o$ = deep water wavenumber, and $\phi(x, y)$ = horizontal spatial variation of the wave potential which is related to the total velocity potential, $\Phi(x, y, z, t)$, by

$$\Phi(x, y, z, t) = -\frac{ig}{\omega}\phi(x, y)\frac{\cosh k(h+z)}{\cosh kh}e^{-i\omega t} \quad (2)$$

where $i = \sqrt{-1}$, $g$ = gravitational acceleration, $\omega$ = wave angular frequency, and the vertical coordinate $z$ is measured vertically upwards from the still water line. The wavenumber, $k$, is the solution of the following dispersion relationship:

$$\omega^2 = gk\tanh kh \quad (3)$$

$R_1$ and $R_2$ are the functions representing the effects of the square of bottom slope and of the bottom curvature, respectively, and are given as rather complicated functions in the papers of Massel (1993) and Suh *et al.* (1997). Simpler expressions for these functions are given in Chamberlain and Porter (1995) as follows:

$$R_1 = \frac{u_1}{n} \tag{4}$$

$$R_2 = \frac{u_2}{n} \tag{5}$$

where

$$n = \frac{1}{2}\left(1 + \frac{2kh}{\sinh 2kh}\right) \tag{6}$$

$$u_1 = \frac{\operatorname{csch} kh \operatorname{sech} kh}{12(2kh + \sinh 2kh)^3}$$
$$[(2kh)^4 + 4(2kh)^3 \sinh 2kh - 9 \sinh 2kh \sinh 4kh$$
$$+ 6kh(2kh + 2\sinh 2kh)(\cosh^2 2kh - \cosh 2kh + 3)] \tag{7}$$

$$u_2 = \frac{\operatorname{sech}^2 kh}{4(2kh + \sinh 2kh)}(\sinh 2kh - 2kh \cosh 2kh) \tag{8}$$

With $R_1 = R_2 = 0$, the modified mild-slope equation (1) reduces to the Berkhoff's mild-slope equation.

In order to examine the importance of the higher-order bottom effect terms (i.e., the terms proportional to the square of bottom slope and to the bottom curvature), $R_1$ and $R_2$ are plotted in Fig. 1 in terms of the relative depth, $kh$. Both $R_1$ and $R_2$ approach to zero as $kh \to \infty$, indicating that the effects of both terms become negligible in deep water, as expected. In shallow water, the effect of bottom curvature is small, but the effect of the square of bottom slope is still significant. Massel (1993) presented a figure similar to Fig. 1 in his paper, showing that $R_1$ also approaches to zero as $kh \to 0$. However, Massel's equation has some errors (see erratum in p. 348, Coastal Engineering, vol. 20).

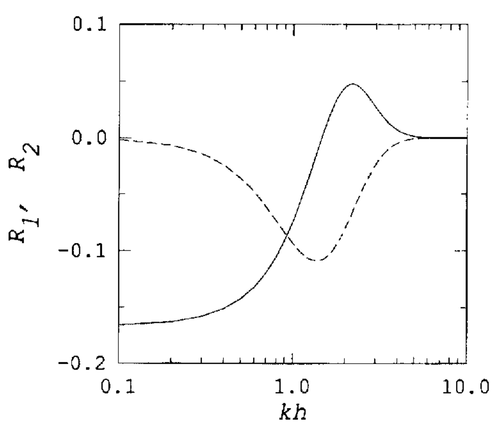


**Fig. 1.** $R_1$ and $R_2$ versus $kh$; —— = $R_1$, – – – = $R_2$.

The performance of the modified mild-slope equation will be tested for one-dimensional problems in next sections. In a one-dimensional problem for waves propagating over straight and parallel bottom contours, the modified mild-slope equation can be reduced to an ordinary differential equation. Let us consider a bottom geometry whose contours are straight in the $y$-direction and parallel one another. Furthermore, let us assume that the water depth varies only in the region of $0 \le x \le l$ (Region 2) and that, for $x \le 0$ (Region 1) and $x \ge l$ (Region 3), the water depth is constant and equal to $h_1$ and $h_3$, respectively. In other words,

$$h(x) = \begin{cases} h_1, & x \le 0 \\ h_2(x), & 0 \le x \le l \\ h_3, & x \ge l \end{cases} \tag{9}$$

Considering plane regular waves traveling with an angle $\theta_1$ with respect to the $x$-axis in Region 1, the solutions in each region may be assumed as

$$\phi_1(x,y,z) = [\exp(ik_1 x \cos\theta_1) + K_r \exp(-ik_1 x \cos\theta_1)]$$
$$\exp(i\chi y)\frac{\cosh k_1(z+h_1)}{\cosh k_1 h_1} \tag{10}$$

$$\phi_2(x,y,z) = \varphi(x)\exp(i\chi y)\frac{\cosh k_2(z+h_2)}{\cosh k_2 h_2} \tag{11}$$

$$\phi_3(x,y,z) = K_t \exp[ik_3(x-l)\cos\theta_3]\exp(i\chi y)$$
$$\frac{\cosh k_3(z+h_3)}{\cosh k_3 h_3} \tag{12}$$

where $K_r$ and $K_t$ = complex reflection and transmission coefficients, respectively, and

$$\chi = k_j \sin\theta_j = \text{constant} \quad (j = 1, 2, 3) \tag{13}$$

is the wavenumber in the $y$-direction. Noting that $\phi(x, y) = \varphi(x)\exp(i\chi y)$ in Eq. (11), substitution into Eq. (1) yields the following ordinary differential equation:

$$\frac{d^2\varphi}{dx^2} + D(x)\frac{d\varphi}{dx} + E(x)\varphi = 0 \tag{14}$$

where

$$D(x) = \frac{G(kh)}{h}\frac{dh}{dx} \tag{15}$$

$$E(x)=k^2\left[1+R_1\left(\frac{dh}{dx}\right)^2+\frac{1}{k_o}R_2\frac{d^2h}{dx^2}-\frac{\chi^2}{k^2}\right] \tag{16}$$

In Eq. (15),

$$G(kh)=\frac{kh}{\tau+kh(1-\tau^2)}\left[1-3\tau^2+\frac{2\tau}{\tau+kh(1-\tau^2)}\right] \tag{17}$$

where $\tau=\tanh(kh)$.

In order to solve Eq. (14) in Region 2, we need the boundary conditions at $x=0$ and $x=l$. The potential $\phi_j$ $(x, y, z)$ must satisfy the matching conditions which provide continuity of pressure and horizontal velocity, normal to the vertical planes separating the fluid regions, i.e.

$$\phi_1=\phi_2,\quad \frac{\partial\phi_1}{\partial x}=\frac{\partial\phi_2}{\partial x}\quad (x=0,\ -h_1\le z\le 0) \tag{18}$$

$$\phi_2=\phi_3,\quad \frac{\partial\phi_2}{\partial x}=\frac{\partial\phi_3}{\partial x}\quad (x=l,\ -h_3\le z\le 0) \tag{19}$$

Substitution of Eqs. (10) to (12) into the preceding equations gives

$$\varphi(0)=1+K_r \tag{20}$$

$$i(1-K_r)k_1\cos\theta_1=\frac{d\varphi(0)}{dx} \tag{21}$$

$$\varphi(l)=K_t \tag{22}$$

$$iK_tk_3\cos\theta_3=\frac{d\varphi(l)}{dx} \tag{23}$$

Eliminating $K_r$ and $K_t$ in the preceding equations, the boundary conditions at $x=0$ and $x=l$ are obtained as follows:

$$\frac{d\varphi(0)}{dx}=i[2-\varphi(0)]k_1\cos\theta_1 \tag{24}$$

$$\frac{d\varphi(l)}{dx}=i\varphi(l)k_3\cos\theta_3 \tag{25}$$

The differential equation (14) with the preceding two boundary conditions can be solved using the finite-difference method. Using the forward-differencing for $d\varphi(0)/dx$, backward-differencing for $d\varphi(l)/dx$, and central-differencing for the derivatives in Eq. (14), the boundary value problem [Eqs. (14), (24) and (25)] may be approximated by a system of linear equations, **AY**= **B**, where **A** is a tridiagonal band type matrix, **Y** is a column vector, and **B** is also a column vector. The subroutines given in the book of Press *et al.* (1992) can be used to solve this matrix equation. After solving the equation, the reflection and transmission coefficients can be obtained by taking the real part of $K_r$ and $K_t$, respectively, in Eqs. (20) and (22).

## 3. BOOIJ'S PROBLEM

By numerical computation for the reflection coefficient of a monochromatic wave normally incident on a plane slope each end of which is connected to a constant-depth region, Booij (1983) has shown that the mild-slope equation gives accurate results up to a slope of 1 in 3. He compared the solution of the mild-slope equation with a finite element model solution which can be considered as an exact solution for linear water waves. In the Booij's test, the wave period was 2 s, and the water depths on the upwave and downwave sides of the slope were 0.6 and 0.2 m, respectively, so that the difference of the water depth between the two constant-depth regions was 0.4 m.

In the Booij's (1983) paper, the finite element model solution is provided only for the slopes steeper than about 1 in 3, and enough information is not given on the accuracy of the mild-slope equation for milder slopes. In the present study, therefore, in order to re-assess the accuracy of the mild-slope equation especially for milder slopes, a new finite element model was constructed for the Booij's problem. An example of the finite element mesh is shown in Fig. 2 for the slope of 1 in 3. The finite element model is also based on linear potential wave theory. The near-field solution including the inclined slope was discretized by 8-noded isoparametric elements with quadratic shape functions. The far-field region was modeled by using infinite elements whose shape function was derived from the usage of the progressive and first evanescent wave components in the analytical boundary series solutions (Park *et al.*, 1991). The shape function of the infinite elements satisfies the radiation boundary condition at infinity. To properly model the behavior of the scat-

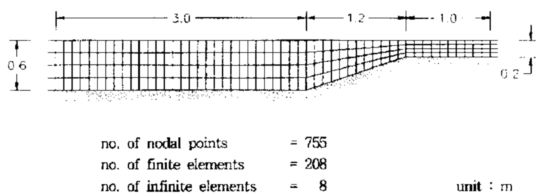


**Fig. 2.** Finite element mesh for the bottom slope of 1:3 of the Booij's (1983) problem.

tered waves, the infinite elements on the upwave and downwave sides were located at a distance of five times the constant water depth from each end of the slope.

Fig. 3 shows the comparison among the modified mild-slope equation, the mild-slope equation, and the finite element model results. The abscissa, $b$, in the figure indicates the width of the plane slope in the direction of wave propagation. First it should be mentioned that, for the relatively steep slope range, the present and the Booij's finite element model results are almost identical in spite of the usage of different finite elements and shape functions. In Fig. 3, it is shown that the modified mild-slope equation gives reflection coefficients very close to those of the finite element model, but the mild-slope equation underpredicts the reflection coefficients for steeper slopes. Even for very mild slopes, the modified mild-slope equation and the mild-slope equation show some difference, and the finite element model results coincide well with the modified mild-slope equation rather than the mild-slope equation.

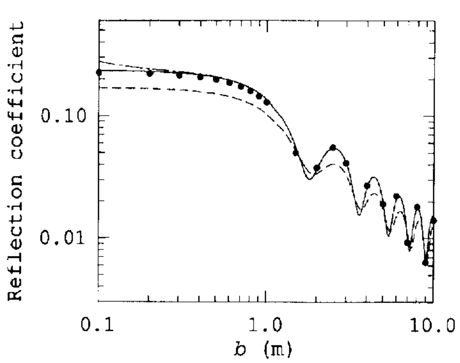


**Fig. 3.** Reflection coefficient versus horizontal length of a plane slope; ● = finite element model, —— = modified mild-slope equation, – – – = mild-slope equation, – · – · – = mild-slope equation plus the bottom curvature term.

It has been well known that the mild-slope equation gives accurate results up to a slope of 1 in 3, but the results shown in Fig. 3 indicate that this is not true. However, it should be noted that the bottom configuration of the Booij's test includes the effects of not only the bottom slope but also the slope discontinuities at both ends of the slope. The latter effect may be represented by the bottom curvature term of the modified mild-slope equation, which is non-zero only at the ends of the slope in the Booij's problem (In fact the bottom curvature at the ends of the slope is infinite, but in this study it is approximated by central-differencing $d^2h/dx^2$, i.e., $d^2h/dx^2=[h(x+\Delta x)-2h(x)+h(x-\Delta x)]/\Delta x^2$ where $\Delta x$ is the grid spacing in the wave propagation direction). The original mild-slope equation does not take into account the effect of bottom curvature and for the bottom slope it includes only the first-order effect represented by $\nabla h$. On the other hand, the modified mild-slope equation includes the second-order bottom slope effect represented by $(\nabla h)^2$ as well as the first-order effect, and in addition it includes the effect of bottom curvature. Therefore, if we want to assess the accuracy of these equations in view of only bottom slope, we have to compare the equation including only the first-order bottom slope effect with that including both the first- and second-order effects. If we directly compare the original and modified mild-slope equations, it is impossible to compare purely the effect of bottom slope because the former does not take into account the effect of bottom curvature while the latter does. Therefore, for this purpose of comparison it is necessary to include the effect of bottom curvature in both equations. Conclusively, if we want to assess the accuracy of the mild-slope equation merely for the bottom slope, we have to compare the mild-slope equation including the bottom curvature term, i.e., Eq. (1) with $R_1=0$, with the modified mild-slope equation.

The result of the mild-slope equation including the bottom curvature term is shown in Fig. 3 by a dash-dotted line, which gives somewhat larger reflection coefficient than the modified mild-slope equation for steeper slopes but is almost identical with the modified mild-slope equation for milder slopes. Noting that the effect of the bottom curvature is included in all the results in Fig. 3 except the original mild-slope equation, it is observed that, when only the bottom slope is concerned, the mild-slope equation can give accurate results up to a slope of 1 in 1 rather than 1 in 3. But it should not be overlooked that without the bottom curvature term, the mild-slope equation could not give accurate results even for the slopes milder than 1 in 3, though the reflection coefficient is very small there.

Recently Porter and Staziker (1995) also showed that the mild-slope equation gives accurate results up to a slope of 1 in 1 by testing the mild-slope equation and the modified mild-slope equation for the Booij's problem. They showed that these equations do not ensure continuity of mass flow at locations where the bed slope is discontinuous and the use of interfacial jump conditions at such locations improves the accuracy of these equations. The solution technique of Porter and Staziker is, however, different from that of the present study. In the present study, the whole domain including the slope and the horizontal bed regions was modeled as one, that is, the model boundaries were taken to be located not at the locations of slope discontinuity but on the horizontal bed regions. However Porter and Staziker divided the domain into three regions (i.e., two horizontal bed regions and the sloping bed region) and imposed matching conditions at the vertical planes separating the regions. Therefore, in the solution of Porter and Staziker, the effect of bottom curvature is not included directly in the calculation but it is included through the use of the interfacial jump condition. A comparison between Fig. 3 and the corresponding figure in Porter and Staziker paper (Fig. 2(a)) shows that the two results are almost identical. This means that the use of the interfacial jump condition in the Porter and Staziker solution is equivalent to the inclusion of the bottom curvature term in our solution.

## 4. WAVE REFLECTION FROM A NON-PLANE SLOPE

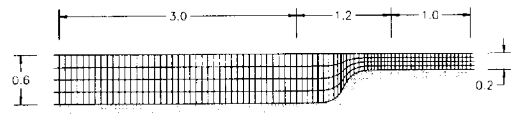


**Fig. 4.** Finite element mesh for the width of 1.2 m of a non-plane slope.

The Booij's (1983) problem tested in the previous section involves a constant slope on the sloping section and the effect of bottom curvature only at the two points at the ends of the slope. In order to assess the simultaneous influence of the steepness and curvature of slope, we consider now a non-plane slope where both the steepness and curvature of the slope vary continuously in space. The water depth on the slope is given by (see Fig. 4)

$$h_2(x) = 0.5(h_1 + h_3) - 0.5(h_1 - h_3)\tanh p(x) \tag{26}$$

where

$$p(x) = 3\pi\left(\frac{x}{b} - \frac{1}{2}\right) \tag{27}$$

$b$ is the width of the slope where the water depth varies. The waves are assumed to propagate normal to the slope. As in the Booij's problem, the constant depths on the upwave and downwave sides of the slope are chosen to be $h_1 = 0.6$ m and $h_3 = 0.2$ m, respectively, and the wave period is 2 s. Here, again, the mild-slope equation and the modified mild-slope equation are compared with the finite element model. An example of the finite element mesh is shown in Fig. 4 for the width of the slope of 1.2 m.

Fig. 5 shows the comparison among the modified mild-slope equation, the mild-slope equation, and the finite element model results. Again, it is shown that the modified mild-slope equation gives reflection coefficients very close to those of the finite element model, but the mild-slope equation systematically underpredicts the reflection coefficients. Compared to the plane slope (see Fig. 3), the oscillatory behavior of

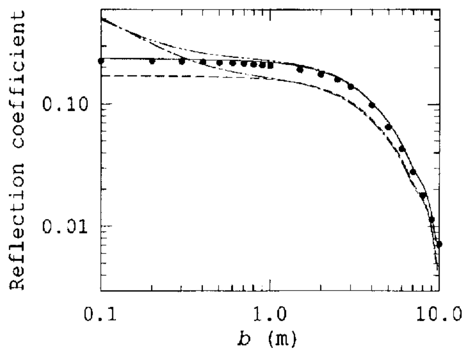


**Fig. 5.** Reflection coefficient versus horizontal length of a non-plane slope; ● = finite element model, —— = modified mild-slope equation, – – – = mild-slope equation, – · – · – = mild-slope equation plus the bottom slope square term, – ·· – ·· – = mild-slope equation plus the bottom curvature term.

the reflection coefficient for the milder slopes disappears, showing monotonous decrease of the reflection coefficient with the decreasing steepness of the slope.

In order to examine the relative importance of the bottom slope square term and the bottom curvature term, additional calculations were made by including only the slope square term or the bottom curvature term to the mild-slope equation. Each result is shown in Fig. 5 by a dash-dot line and dash-dot-dot line, respectively. As expected, the inclusion of the slope square term gives some difference from the mild-slope equation for very steep slopes, while the difference is minute for milder slopes. Without showing the result, it is just stated that the same trend has been observed for the Booij's plane slope. On the other hand, when only the bottom curvature term is included to the mild-slope equation, the solution closely follows that of the finite element model for milder slopes, but the deviation becomes larger for steeper slopes where the effect of the bottom slope square term is important as well. As a result, it can be stated that the mild-slope equation plus only the bottom curvature term may give sufficiently accurate results for a bottom topography with relatively mild variation, but where the bottom slope is not small the bottom slope square term should also be included for more accurate results.

## 5. RESONANT BRAGG REFLECTION BY RIPPLES

The numerical tests in the previous sections showed that the effect of bottom curvature is important for wave propagation on bed with mild slopes of practical interest. In order to assess the simultaneous influence of the steepness and curvature of bottom on a more realistic bathymetry, numerical test is made for wave reflection by a patch of periodic ripples. When surface waves are normally incident on a region of long-crested periodic bottom undulation, a significant amount of incident wave energy is reflected at the point where the wavenumber of the periodic bottom undulation ($K$) is twice the wavenumber of surface wave ($k$), that is, $2k/K = 1$. This wave reflection, which has been known as Bragg reflection, has been studied by both laboratory experiments and theoretical or numerical models.

Davies and Heathershaw (1984) reported experimental data for the reflection of waves due to sinusoidal ripple patches with different numbers of ripples. In their experiment, the ripple wavelength and amplitude were 1 m and 5 cm, respectively, and the number of ripples was 2, 4, and 10. The water depth at the constant-depth region was 15.6 cm for the cases of 2 and 4 ripples and 31.3 cm for the case of 10 ripples. This experimental data has been used for comparison with various numerical models by a number of researchers including Kirby (1986), Massel (1993), Chamberlain and Porter (1995), and Suh *et al.* (1997). All of them showed that the mild-slope equation gives a good agreement with the experimental data for the cases of 2 and 4 ripples, but, for the case of 10 ripples, it fails to predict the magnitude of resonant reflection. Therefore, in the present study, a numerical test is made only for the case of 10 ripples.

Fig. 6 shows the reflection coefficients calculated by the modified mild-slope equation and the mild-slope equation along with the experimental data. Again, in order to examine the relative importance of the bottom slope square term and the bottom curvature term, the

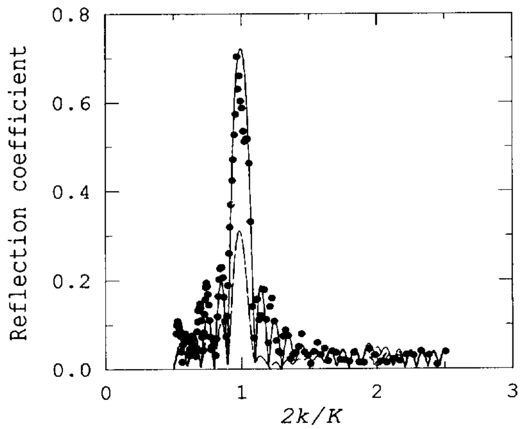


**Fig. 6.** Comparison between numerical results and the experimental data of Davies and Heathershaw (1984) for wave reflection by ripples; ● = experimental data, —— = modified mild-slope equation, – – – = mild-slope equation, – · – · – = mild-slope equation plus the bottom slope square term, – ·· – ·· – = mild-slope equation plus the bottom curvature term.

results of the mild-slope equation including only the slope square term or the bottom curvature term are also presented in Fig. 6. The results of the modified mild-slope equation and the mild-slope equation including only the bottom curvature term show some difference only in the vicinity of $2k/K=2.0$, and for other values of $2k/K$ they are almost identical so that the difference is undistinguishable in the figure. Both the modified mild-slope equation and the mild-slope equation including only the bottom curvature term describe the resonant peak very well. The mild-slope equation and that including only the slope square term (shown to be almost identical in the figure), however, while correctly positioning the resonant reflection, completely fail to predict its magnitude. Some of the researchers have interpreted this failure to be attributed to the violation on the mild-slope assumption that the depth must vary slowly over a wavelength. However the results shown in Fig. 6 suggest that this interpretation is not appropriate. The failure of the mild-slope equation may be not because the depth varies rapidly but because it does not include the effect of bottom curvature. It is also worthwhile to note that, for this Bragg problem, the depth perturbation about the mean bed level is of the form of $A\sin(Kx)$, where $A$ is the ripple amplitude, so that $(dh/dx)^2=O(\varepsilon^2)$ where $\varepsilon=A/\lambda<<1$ ($\lambda$ is the ripple wavelength). Therefore, as for the Bragg problem, it is not surprising that the bottom curvature term is more important than the slope square term.

## 6. CONCLUSIONS

By examining the modified mild-slope equation which, compared to the Berkhoff's mild-slope equation, includes additional terms proportional to the square of bottom slope and to the bottom curvature, it has been shown that both terms are equally important in intermediate-depth water, but in shallow water the influence of the bottom curvature term diminishes while that of the bottom slope square term remains significant. In deep water, the effects of both terms are negligible, as expected.

In order to examine the importance of these terms in more detail, the modified mild-slope equation and the Berkhoff's mild-slope equation were tested for the problems of wave reflection from a plane slope, a non-plane slope, and periodic ripples. In addition, in order to compare the relative importance of these terms, the mild-slope equation plus only the bottom slope square term or the bottom curvature term was tested for the same problems. It was shown that, when only the bottom slope is concerned, the mild-slope equation can give accurate results up to a slope of 1 in 1 rather than 1 in 3, which, until now, has been known as the limiting bottom slope for the proper application of the mild-slope equation. It was also shown that the bottom curvature term plays an important role in modeling wave propagation over a bottom topography with relatively mild variation, but, where the bottom slope is not small, the bottom slope square term should also be included for more accurate results.

In most practical applications, the influence of the bottom slope square term is negligible so that the mild-slope equation plus only the bottom curvature term may give sufficiently accurate results. However, the modified mild-slope equation already exists, which includes the bottom slope square term as well. Therefore, there is no reason not to use it, because its use may require only a little increase of computational time or programming effort.

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## REFERENCES

Berkhoff, J.C.W., 1972. Computation of combined refraction-diffraction, *Proc. 13th Int. Coastal Engrg. Conf.*, ASCE, Vol. 1, pp. 471-490.

Booij, N., 1983. A note on the accuracy of the mild-slope equation, *Coastal Engrg.*, **7**, pp. 191-203.

Chamberlain, P.G. and Porter, D., 1995. The modified mild-slope equation, *J. Fluid Mech.*, **291**, pp. 393-407.

Copeland, G.J.M., 1985. A practical alternative to the mild-slope wave equation, *Coastal Engrg.*, **9**, pp. 125-149.

Davies, A.G. and Heathershaw, A.D., 1984. Surface-wave propagation over sinusoidally varying topography, *J. Fluid Mech.*, **144**, pp. 419-443.

Kirby, J.T., 1986. A general wave equation for waves over rippled beds, *J. Fluid Mech.*, **162**, pp. 171-186.

Massel, S.R., 1993. Extended refraction-diffraction equation for surface waves, *Coastal Engrg.*, **19**, pp. 97-126.

O'Hare, T.J. and Davies, A.G., 1993. A comparison of two models for surface-wave propagation over rapidly varying topography, *Applied Ocean Res.*, **15**, pp. 1-11.

Park, W.S., Lee, D.S., Oh, Y.M. and Jeong, W.M., 1991. Infinite elements for analysis of diffraction and radiation problems in the vertical plane, *J. Korean Soc. of Coastal and Ocean Engrs.*, **3**(4), pp. 235-243 (in Korean).

Porter, D. and Staziker, D.J., 1995. Extensions of the mild-slope equation, *J. Fluid Mech.*, **300**, pp. 367-382.

Press, W.H., Teukolsky, S.A., Vetterling, W.T. and Flannery, B.P., 1992. *Numerical recipes in FORTRAN: the art of scientific computing*, 2nd Ed., Cambridge University Press, Cambridge, UK.

Radder, A.C., 1979. On the parabolic equation method for water-wave propagation, *J. Fluid Mech.*, **95**, pp. 159-176.

Suh, K.D., Lee, C. and Park, W.S., 1997. Time-dependent equations for wave propagation on rapidly varying topography, *Coastal Engrg.*, **32**, pp. 91-117.