# $H^\infty$ State Feedback Control for Generalized Continuous/Discrete Time Delay System

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## Abstract

In this paper, we consider the problem of designing $H^\infty$ state feedback controller for the generalized time delay systems with delayed states and control inputs in continuous and discrete time cases, respectively. The generalized time delay system problems are solved on the basis of LMI(linear matrix inequality) technique considering time delays. The sufficient condition for the existence of controller and $H^\infty$ state feedback controller design methods are presented. Also, using some changes of variables and Schur complements, the obtained sufficient condition can be rewritten as a LMI form in terms of transformed variables. The proposed controller design method can be extended into the problem of robust $H^\infty$ state feedback controller design method easily.


## I. Introduction

Since the time delay is frequently a source of instability and encountered in various engineering systems such as chemical processes, long transmission lines in pneumatic systems, etc., the study of time delay systems has received considerable attention over the past years. Because some works of analytic $H^\infty$ controller design method[4,5] and software toolbox[6] have been developed, many state feedback controller design methods of time delay systems were presented[2,3,8,9,10,11,12,14]. Lee *et al.*[10] presented a memoryless $H^\infty$ controller which is a delay independent stabilizer for the state delayed system. And the work[3] proposed by Choi *et al.* was extended to the problem of memoryless $H^\infty$ controller design for linear systems with delayed state and control using the Riccati equation approach. But not only their works[2,3] but also other results[8,9,10,11,12,14] were conservative in pre-determination of some starting values determined whether there exists a positive symmetric definite solution, and were not considered delayed state and control input in the controlled signal output.

The first aim of our paper is to find solutions at a time without the pre-selection of some variables using LMI(linear matrix inequality) technique. Recently, many works[7,13,15] related robust problem or robust $H^\infty$ problem against parameter uncertainties were presented. Also, robust control problem with time delay[2,9,11,12], $H^\infty$ control problem with time delay[3,10] and robust $H^\infty$ control problem with time delay[8,14] were proposed. However many related works treated controller design method in continuous time case only. Therefore it is important to deal with a controller design method in discrete time case because most of engineering systems are controlled by digital computer. Garcia *et al.*[7] presented a robust stabilization of discrete time linear systems with norm-bounded time-varying uncertainty, and Yuan *et al.*[15] proposed a robust $H^\infty$ control for linear discrete time systems with norm-bounded time-varying uncertainty. But they did not consider time delay. Therefore the second objective of our paper is to present an $H^\infty$ state feedback controller design method of discrete time delay systems. Also, the proposed controller design method can be extended into the problem of robust $H^\infty$ state feedback controller design method for parameter uncertain time delay systems through some manipulations. To find an $H^\infty$ state feedback controller, we consider the bounded real lemma for the closed loop system as LMI problems.

In this paper, we propose $H^\infty$ state feedback controller design methods of the generalized time delay system in continuous time and discrete time cases, respectively. The existence conditions and design methods of state feedback $H^\infty$ controllers were given. Through some changes of variables and Schur complement, the obtained sufficient condition is changed into a LMI form in terms of each finding variable. The $H^\infty$ state feedback controller can be easily obtained using

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LMI toolbox[6] because the transformed sufficient condition is a LMI form in terms of variables. The state feedback $H^{\infty}$ controller guarantees not only the quadratic stability of the closed loop system but also the $H^{\infty}$ norm bound within a $\gamma$.

## II. Continuous Time Controller Design

Consider a continuous time linear system with time-varying delays

$$\begin{aligned} \dot{x}(t) &= Ax(t)+A_d x(t-d_1(t))+B_1 w(t)+B_2 u(t)+B_d u(t-d_2(t)) \\ z(t) &= Cx(t)+C_d x(t-d_1(t))+D_{11}w(t)+D_{12}u(t)+D_d u(t-d_2(t)) \\ x(t) &= 0,\ t\le 0 \end{aligned} \tag{1}$$

where $x(t)\in R^n$ is the state, $u(t)\in R^m$ is the control input, $w(t)\in R^l$ is the exogenous input, and $z(t)\in R^p$ is the controlled signal output. And we assume that all states are measurable. Here, time-varying delays are satisfied with

$$0\le d_i < \infty,\ \ \dot{d}_i(t)\le \beta_i < 1,\ \ i=1,2. \tag{2}$$

As an $H^{\infty}$ controller of delayed system (1), we propose a continuous time state feedback law

$$u(t)=Kx(t). \tag{3}$$

When we apply the control (3) to the delayed system (1), the closed loop system from $w(t)$ to $z(t)$ is given by

$$\begin{aligned} \dot{x}(t) &= A_K x(t)+A_d x(t-d_1(t))+B_1 w(t)+B_d Kx(t-d_2(t)) \\ z(t) &= C_K x(t)+C_d x(t-d_1(t))+D_{11}w(t)+D_d Kx(t-d_2(t)) \end{aligned} \tag{4}$$

where, $A_K=A+B_2K$ and $C_K=C+D_{12}K$.

**Lemma 1:** For a given constant $\gamma>0$, the system (1) is quadratically stable with an $H^{\infty}$ norm bound $\gamma$ with the controller (3) if there exist positive symmetric definite matrices $P$, $R_1$, and $R_2$ such that

$$\begin{bmatrix} A_K^TP+PA_K+R_1+K^TR_2K & PA_d & PB_d & PB_1 & C_K^T \\ A_d^TP & -\tilde{R}_1 & 0 & 0 & C_d^T \\ B_d^TP & 0 & -\tilde{R}_2 & 0 & D_d^T \\ B_1^TP & 0 & 0 & -\gamma^2 I & D_{11}^T \\ C_K & C_d & D_d & D_{11} & -I \end{bmatrix} < 0 \tag{5}$$

holds for time delays. Here, $\tilde{R}_i=(1-\beta_i)R_i,\ i=1,2$, are positive definite symmetric matrices.

**Proof:** Firstly, we define a Lyapunov functional as

$$V(x(t)) := x(t)^TPx(t) + \int_{t-d_1(t)}^{t} x(\tau)^TR_1x(\tau)d\tau + \int_{t-d_2(t)}^{t} x(\tau)^TK^TR_2Kx(\tau)\,d\tau. \tag{6}$$

When assuming the zero input, we have

$$\begin{aligned} \dot{V}(x(t)) &= \dot{x}(t)^TPx(t)+x(t)^TP\dot{x}(t)+x(t)^TR_1x(t)+x(t)^TK^TR_2Kx(t) \\ &\quad -(1-\dot{d}_1(t))x(t-d_1(t))^TR_1x(t-d_1(t)) \\ &\quad -(1-\dot{d}_2(t))x(t-d_2(t))^TK^TR_2Kx(t-d_2(t)) \\ &\le \dot{x}(t)^TPx(t)+x(t)^TP\dot{x}(t)+x(t)^TR_1x(t)+x(t)^TK^TR_2Kx(t) \\ &\quad -(1-\beta_1)x(t-d_1(t))^TR_1x(t-d_1(t)) \\ &\quad -(1-\beta_2)x(t-d_2(t))^TK^TR_2Kx(t-d_2(t)) \\ &:= \dot{V}_a(x(t)). \end{aligned} \tag{7}$$

In the next place, assume the zero initial condition and introduce

$$J=\int_0^{\infty}[z(t)^Tz(t)-\gamma^2w(t)^Tw(t)]dt. \tag{8}$$

Noting

$$J\le \int_0^{\infty}[z(t)^Tz(t)-\gamma^2w(t)^Tw(t)+\dot{V}_a(x(t))]dt \tag{9}$$

and further substituting (7) into (9) and let $\zeta(t)=[x(t)^T\ x(t-d_1(t))^T\ x(t-d_2(t))^TK^T\ w(t)^T]^T$, then

$$J\le \int_0^{\infty}\zeta(t)^TZ\zeta(t), \tag{10}$$

where $Z$ is defined

$$Z=\begin{bmatrix} A_K^TP+PA_K+C_K^TC_K+R_1+K^TR_2K & PA_d+C_K^TC_d & PB_d+C_K^TD_d & PB_1+C_K^TD_{11} \\ A_d^TP+C_d^TC_K & C_d^TC_d-(1-\beta_1)R_1 & C_d^TD_d & C_d^TD_{11} \\ B_d^TP+D_d^TC_K & D_d^TC_d & D_d^TD_d-(1-\beta_2)R_2 & D_d^TD_{11} \\ B_1^TP+D_{11}^TC_K & D_{11}^TC_d & D_{11}^TD_d & -\gamma^2I+D_{11}^TD_{11} \end{bmatrix} \tag{11}$$

Therefore when $Z<0,\ t\ge 0$, the system (1) is quadratically stable with an $H^{\infty}$ norm bound $\gamma$. Using lemma A, Z<0 is transformed into (5). □

**Theorem 1**: Consider the continuous time delay system (1). For a given positive constant $\gamma$, if there exist a matrix $M$ and positive definite symmetric matrices $Q$, $S_1$, $S_2$ such that

$$\begin{bmatrix} U_1 & B_1 & U_2 & M^T & Q \\ B_1^T & -\gamma^2I & D_{11}^T & 0 & 0 \\ U_2^T & D_{11} & U_3 & 0 & 0 \\ M & 0 & 0 & -S_2 & 0 \\ Q & 0 & 0 & 0 & -S_1 \end{bmatrix} < 0 \tag{12}$$

holds for the time delays (2), then (1) is said to be quadratically stable with an $H^{\infty}$ norm bound $\gamma$. In here, some terms are defined as follows:

$$\begin{aligned} U_1 &= QA^T+AQ+M^TB_2^T+B_2M+(1-\beta_1)^{-1}A_dS_1A_d^T+(1-\beta_2)^{-1}B_dS_2B_d^T, \\ U_2 &= M^TD_{12}^T+QC^T+(1-\beta_1)^{-1}A_dS_1C_d^T+(1-\beta_2)^{-1}B_dS_2D_d^T, \\ U_3 &= -I+(1-\beta_1)^{-1}C_dS_1C_d^T+(1-\beta_2)^{-1}D_dS_2D_d^T, \\ M &= KP^{-1}, \\ Q &= P^{-1}, \\ S_i &= R_i^{-1},\ i=1,2. \end{aligned} \tag{13}$$

**Proof:** Using lemma A of appendix and some changes of variables, the proof is completed. The inequality (5) is equivalent to

$$\begin{bmatrix} A_K^TP+PA_K+R_1 & PA_d & PB_d & PB_1 & C_K^T & K^T \\ * & -\tilde{R}_1 & 0 & 0 & C_d^T & 0 \\ * & * & -\tilde{R}_2 & 0 & D_d^T & 0 \\ * & * & * & -\gamma^2 I & D_{11}^T & 0 \\ * & * & * & * & -I & 0 \\ * & * & * & * & * & -R_2^{-1} \end{bmatrix} < 0 \quad (14)$$

$$\Leftrightarrow \begin{bmatrix} A_K^TP+PA_K & PA_d & PB_d & PB_1 & C_K^T & K^T & I \\ * & -\tilde{R}_1 & 0 & 0 & C_d^T & 0 & 0 \\ * & * & -\tilde{R}_2 & 0 & D_d^T & 0 & 0 \\ * & * & * & -\gamma^2 I & D_{11}^T & 0 & 0 \\ * & * & * & * & -I & 0 & 0 \\ * & * & * & * & * & -R_2^{-1} & 0 \\ * & * & * & * & * & * & -R_1^{-1} \end{bmatrix} < 0 \quad (15)$$

$$\Leftrightarrow \begin{bmatrix} A_K^TP+PA_K+PA_d\tilde{R}_1^{-1}A_d^TP & PB_d & PB_1 & C_K^T+PA_d\tilde{R}_1^{-1}C_d^T & K^T & I \\ * & -\tilde{R}_2 & 0 & D_d^T & 0 & 0 \\ * & * & -\gamma^2 I & D_{11}^T & 0 & 0 \\ * & * & * & -I+C_d\tilde{R}_1^{-1}C_d^T & 0 & 0 \\ * & * & * & * & -R_2^{-1} & 0 \\ * & * & * & * & * & -R_1^{-1} \end{bmatrix} < 0 \quad (16)$$

$$\Leftrightarrow \begin{bmatrix} A_K^TP+PA_K+PA_d\tilde{R}_1^{-1}A_d^TP+PB_d\tilde{R}_2^{-1}B_d^TP & PB_1 \\ * & -\gamma^2 I \\ * & * \\ * & * \\ * & * \end{bmatrix} \quad (17)$$

$$\begin{matrix} C_K^T+PA_d\tilde{R}_1^{-1}C_d^T+PB_d\tilde{R}_2^{-1}D_d^T & K^T & I \\ D_{11}^T & 0 & 0 \\ -I+C_d\tilde{R}_1^{-1}C_d^T+D_d\tilde{R}_2^{-1}D_d^T & 0 & 0 \\ * & -R_2^{-1} & 0 \\ * & * & -R_1^{-1} \end{matrix} \Bigg] < 0$$

$$\Leftrightarrow \begin{bmatrix} P^{-1}(A+B_2K)^T+(A+B_2K)P^{-1}+A_d\tilde{R}_1^{-1}A_d^T+B_d\tilde{R}_2^{-1}B_d^T & B_1 \\ * & -\gamma^2 I \\ * & * \\ * & * \\ * & * \end{bmatrix} \quad (18)$$

$$\begin{matrix} P^{-1}(C+D_{12}K)^T+A_d\tilde{R}_1^{-1}C_d^T+B_d\tilde{R}_2^{-1}D_d^T & P^{-1}K^T & P^{-1} \\ D_{11}^T & 0 & 0 \\ -I+C_d\tilde{R}_1^{-1}C_d^T+D_d\tilde{R}_2^{-1}D_d^T & 0 & 0 \\ * & -R_2^{-1} & 0 \\ * & * & -R_1^{-1} \end{matrix} \Bigg] < 0$$

where, * means symmetric terms. Using some changes of variables, $M=KP^{-1}$, $Q=P^{-1}$, and $S_i=R_i^{-1}$, $i=1,2$, (18) is changed to (12). □

(12) is a LMI form in terms of $Q$, $M$, $S_1$, and $S_2$. Therefore the continuous time $H^\infty$ state feedback controller $K$ can be calculated from the $M=KP^{-1}$ after finding the LMI solutions, $Q$, $M$, $S_1$, and $S_2$, from the (12). Using LMI Toolbox[6], the solutions can be easily obtained at a time because (12) is a LMI form in terms of variables.

**Corollary 1**: The continuous parameter uncertain system with time-varying delay in states and control inputs

$$\begin{aligned} \dot{x}(t) &= [A+\Delta A(t)]x(t)+[A_d+\Delta A_d(t)]x(t-d_1(t)) \\ &\quad +[B_u+\Delta B_u(t)]u(t)+[B_d+\Delta B_d(t)]u(t-d_2(t)) \\ &\quad +[B_w+\Delta B_w(t)]w(t) \\ z(t) &= [C_z+\Delta C_z(t)]x(t)+[C_{zd}+\Delta C_{zd}(t)]x(t-d_1(t)) \\ &\quad +[D_{zu}+\Delta D_{zu}(t)]u(t)+[D_{zd}+\Delta D_{zd}(t)]u(t-d_2(t)) \\ &\quad +[D_{zw}+\Delta D_{zw}(t)]w(t) \end{aligned} \quad (19)$$

$$\begin{bmatrix} \Delta A(t) & \Delta B_u(t) & \Delta B_w(t) & \Delta A_d(t) & \Delta B_d(t) \\ \Delta C_z(t) & \Delta D_{zu}(t) & \Delta D_{zw}(t) & \Delta C_{zd}(t) & \Delta D_{zd}(t) \end{bmatrix} = \begin{bmatrix} H_x \\ H_z \end{bmatrix} F(t)[E_x \; E_u \; E_w \; E_{dx} \; E_{du}] \quad (20)$$

can be transformed into the form of (1) without the parameter uncertainties under preserving quadratic stability and $H^\infty$ norm bound through some manipulations[13]. In here, unknown matrix is defined as

$$F(t) \in \Omega := \{ F(t) : F(t)^T F(t) \le I, \text{ the elements of } F(t) \text{ are Lebesgue measurable} \}. \quad (21)$$

Therefore the robust $H^\infty$ state feedback controller design problem for parameter uncertain delay system can be solvable using the proposed method.

**Example 1:** Consider a generalized continuous time delay system

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} x(t)+\begin{bmatrix} 0.2 & 0.1 \\ 0 & 0.1 \end{bmatrix} x(t-d_1(t))+\begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix} w(t) \\ &\quad +\begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)+\begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix} u(t-d_2(t)) \\ z(t) &= [1 \; 1]x(t)+[0.1 \; 0.1]x(t-d_1(t))+0.1w(t) \\ &\quad +u(t)+0.1u(t-d_2(t)) \end{aligned} \quad (22)$$

$$\gamma=1, \; d_1(t)=2+0.2\cos t, \; d_2(t)=5+0.2\sin(3t).$$

From the solutions satisfying (12) and changes of variables (13), all solutions are obtained at a time as follows:

$$\begin{aligned} P &= \begin{bmatrix} 25.8838 & -8.7399 \\ -8.7399 & 4.2486 \end{bmatrix}, \\ R_1 &= \begin{bmatrix} 0.2676 & -0.0082 \\ -0.0082 & 0.2200 \end{bmatrix}, \\ M &= [-1.1255 \; -1.4670], \\ R_2 &= 0.2177. \end{aligned} \quad (23)$$

Therefore the continuous time state feedback gain is

$$K = [-16.3114 \quad 3.6044]. \quad (24)$$

The obtained controller guarantees the stability for time-varying delays and satisfies $H^\infty$ norm bound of the closed loop system.

## III. Discrete Time Controller Design

Consider a discrete time linear system with time delays

$$\begin{aligned} x(k+1) &= Ax(k)+A_d x(k-d_1)+B_1 w(k)+B_2 u(k)+B_d u(k-d_2) \\ z(k) &= Cx(k)+C_d x(k-d_1)+D_{11}w(k)+D_{12}u(k)+D_d u(k-d_2) \\ x(k) &= 0,\ k\le 0 \end{aligned} \tag{25}$$

where $x(k)\in R^n$ is the state, $u(k)\in R^m$ is the control input, $w(k)\in R^l$ is the exogenous input, and $z(k)\in R^p$ is the controlled signal output, and all matrices are constant matrices with appropriate dimensions. And we assume that all states are measurable. Here, time delays are satisfied with

$$0\le d_i < \infty,\ i=1,2. \tag{26}$$

As an $H^\infty$ controller of delayed system (25), we propose a state feedback law

$$u(k) = Kx(k). \tag{27}$$

When we apply the control (27) to the delayed system (25), the closed loop system from $w(k)$ to $z(k)$ is given by

$$\begin{aligned} x(k+1) &= A_K x(k)+A_d x(k-d_1)+B_1 w(k)+B_d Kx(k-d_2) \\ z(k) &= C_K x(k)+C_d x(k-d_1)+D_{11}w(k)+D_d Kx(k-d_2) \end{aligned} \tag{28}$$

where, $A_K = A+B_2K$ and $C_K = C+D_{12}K$.

**Lemma 2:** For a given $\gamma>0$, the system described (25) is said to be quadratically stable with an $H^\infty$ norm bound $\gamma$ with the controller (27) if there exist positive definite symmetric matrices $P$, $R_1$, and $R_2$ such that

$$\begin{bmatrix} -P^{-1} & A_K & A_d & B_d & B_1 & 0 \\ A_K^T & -P+R_1+K^TR_2K & 0 & 0 & 0 & C_K^T \\ A_d^T & 0 & -R_1 & 0 & 0 & C_d^T \\ B_d^T & 0 & 0 & -R_2 & 0 & D_d^T \\ B_1^T & 0 & 0 & 0 & -\gamma^2 I & D_{11}^T \\ 0 & C_K & C_d & D_d & D_{11} & -I \end{bmatrix} < 0 \tag{29}$$

holds for the time delays (26).

**Proof:** Firstly, we define a Lyapunov functional as

$$\begin{aligned} V(x(k)) := x(k)^TPx(k) &+ \sum_{i=k-d_1}^{k-1} x(i)^TR_1x(i) \\ &+ \sum_{i=k-d_2}^{k-1} x(i)^TK^TR_2Kx(i). \end{aligned} \tag{30}$$

When assuming the zero input, we have

$$\begin{aligned} \triangle V_k &= V(x(k+1)) - V(x(k)) \\ &= x(k+1)^TPx(k+1)-x(k)^T(P-R_1-K^TR_2K)x(k) \\ &\quad -x(k-d_1)^TR_1x(k-d_1)-x(k-d_2)^TK^TR_2Kx(k-d_2). \end{aligned} \tag{31}$$

In the next place, assume the zero initial condition and introduce

$$J = \sum_{k=0}^{\infty}[z(k)^Tz(k)-\gamma^2 w(k)^Tw(k)]. \tag{32}$$

Noting

$$J \le \sum_{k=0}^{\infty}[z(k)^Tz(k)-\gamma^2 w(k)^Tw(k)+\triangle V_k] \tag{33}$$

and further substituting (31) into (33) and let $\delta(k)=[x(k)^T\ x(k-d_1)^T\ x(k-d_2)^TK^T\ w(k)^T]^T$, then

$$J \le \sum_{k=0}^{\infty} \delta(k)^TZ\delta(k), \tag{34}$$

where $Z$ is defined

$$Z = \begin{bmatrix} A_K^TPA_K-P+R_1+K^TR_2K+C_K^TC_K & A_K^TPA_d+C_K^TC_d & A_K^TPB_d+C_K^TD_d & A_K^TPB_1+C_K^TD_{11} \\ * & -R_1+A_d^TPA_d+C_d^TC_d & A_d^TPB_d+C_d^TD_d & A_d^TPB_1+C_d^TD_{11} \\ * & * & -R_2+B_d^TPB_d+D_d^TD_d & B_d^TPB_1+D_d^TD_{11} \\ * & * & * & -\gamma^2I+B_1^TPB_1+D_{11}^TD_{11} \end{bmatrix} \tag{35}$$

where * mean symmetric terms. Therefore when $Z<0$, $k\ge 0$, the system (25) is quadratically stable with an $H^\infty$ norm bound $\gamma$. Using lemma 1, (35) is transformed into (29).□

**Theorem 2**: Consider the discrete time delay system (25). For a given $\gamma$, if there exist positive definite symmetric matrices $Q$, $S_1$, $S_2$, and a matrix $M$ such that

$$\begin{bmatrix} -Q+B_dS_2B_d^T+A_dS_1A_d^T & AQ+B_2M & B_1 & B_dS_2D_d^T+A_dS_1C_d^T & 0 & 0 \\ QA^T+M^TB_2^T & -Q & 0 & QC^T+M^TD_{12}^T & M^T & Q \\ B_1^T & 0 & -\gamma^2I & D_{11}^T & 0 & 0 \\ D_dS_2B_d^T+C_dS_1A_d^T & CQ+D_{12}M & D_{11} & -I+C_dS_1C_d^T+D_dS_2D_d^T & 0 & 0 \\ 0 & M & 0 & 0 & -S_2 & 0 \\ 0 & Q & 0 & 0 & 0 & -S_1 \end{bmatrix} < 0 \tag{36}$$

holds for the time delays (26), then (25) is quadratically stable with an $H^\infty$ norm bound $\gamma$. Here, some variables are defined as follows:

$$\begin{aligned} M &= KP^{-1} \\ Q &= P^{-1} \\ S_i &= R_i^{-1},\ i=1,2. \end{aligned} \tag{37}$$

**Proof:** Using lemma A of appendix and some changes of variables, the proof is completed. (29) is equivalent to

$$\begin{bmatrix} -P^{-1} & A_K & A_d & B_d & B_1 & 0 & 0 \\ * & -P+R_1 & 0 & 0 & 0 & C_K^T & K^T \\ * & * & -R_1 & 0 & 0 & C_d^T & 0 \\ * & * & * & -R_2 & 0 & D_d^T & 0 \\ * & * & * & * & -\gamma^2I & D_{11}^T & 0 \\ * & * & * & * & * & -I & 0 \\ * & * & * & * & * & * & -R_2^{-1} \end{bmatrix} < 0 \tag{38}$$

$$\Leftrightarrow \begin{bmatrix} -P^{-1}+B_dR_2^{-1}B_d^T & A_K & A_d & B_1 & B_dR_2^{-1}D_d^T & 0 \\ * & -P+R_1 & 0 & 0 & C_K^T & K^T \\ * & * & -R_1 & 0 & C_d^T & 0 \\ * & * & * & -\gamma^2 I & D_{11}^T & 0 \\ * & * & * & * & -I+D_dR_2^{-1}D_d^T & 0 \\ * & * & * & * & * & -R_2^{-1} \end{bmatrix} < 0 \quad (39)$$

$$\Leftrightarrow \begin{bmatrix} -P^{-1}+B_dR_2^{-1}B_d^T & A_K & A_d & B_1 & B_dR_2^{-1}D_d^T & 0 & 0 \\ * & -P & 0 & 0 & C_K^T & K^T & I \\ * & * & -R_1 & 0 & C_d^T & 0 & 0 \\ * & * & * & -\gamma^2 I & D_{11}^T & 0 & 0 \\ * & * & * & * & -I+D_dR_2^{-1}D_d^T & 0 & 0 \\ * & * & * & * & * & -R_2^{-1} & 0 \\ * & * & * & * & * & * & -R_1^{-1} \end{bmatrix} < 0 \quad (40)$$

$$\Leftrightarrow \begin{bmatrix} -P^{-1}+B_dR_2^{-1}B_d^T+A_dR_1^{-1}A_d^T & A_K & B_1 & B_dR_2^{-1}D_d^T+A_dR_1^{-1}C_d^T & 0 & 0 \\ * & -P & 0 & C_K^T & K^T & I \\ * & * & -\gamma^2 I & D_{11}^T & 0 & 0 \\ * & * & * & -I+C_dR_1^{-1}C_d^T+D_dR_2^{-1}D_d^T & 0 & 0 \\ * & * & * & * & -R_2^{-1} & 0 \\ * & * & * & * & * & -R_1^{-1} \end{bmatrix} < 0 \quad (41)$$

$$\Leftrightarrow \begin{bmatrix} -P^{-1}+B_dR_2^{-1}B_d^T+A_dR_1^{-1}A_d^T & (A+B_2K)P^{-1} & B_1 & B_dR_2^{-1}D_d^T+A_dR_1^{-1}C_d^T & 0 & 0 \\ * & -P^{-1} & 0 & P^{-1}(C+D_{12}K)^T & P^{-1}K^T & P^{-1} \\ * & * & -\gamma^2 I & D_{11}^T & 0 & 0 \\ * & * & * & -I+C_dR_1^{-1}C_d^T+D_dR_2^{-1}D_d^T & 0 & 0 \\ * & * & * & * & -R_2^{-1} & 0 \\ * & * & * & * & * & -R_1^{-1} \end{bmatrix} < 0$$

**(42)**

Using some changes of variables, $M=KP^{-1}$, $Q=P^{-1}$, and $S_i=R_i^{-1}$, $i=1,2$, (42) is changed to (36). □

(36) is also a LMI form in terms of $Q$, $M$, $S_1$, and $S_2$. In similar to the continuous time case, state feedback controller $K$ can be calculated from the $M=KP^{-1}$ after finding the LMI solutions, $Q$, $M$, $S_1$, and $S_2$, from the (36). Using LMI Toolbox[6], the solutions can be easily obtained at a time because (36) is a LMI form in terms of variables.

**Corollary 2**: The discrete parameter uncertain system with time delay in states and control inputs

$$\begin{aligned} x(k+1) = {} & [A+\Delta A(k)]x(k)+[A_d+\Delta A_d(k)]x(k-d_1) \\ & +[B_u+\Delta B_u(k)]u(k)+[B_d+\Delta B_d(k)]u(k-d_2) \\ & +[B_w+\Delta B_w(k)]w(k) \\ z(k) = {} & [C_z+\Delta C_z(k)]x(k)+[C_{zd}+\Delta C_{zd}(k)]x(k-d_1) \\ & +[D_{zu}+\Delta D_{zu}(k)]u(k) \\ & +[D_{zd}+\Delta D_{zd}(k)]u(k-d_2)+[D_{zw}+\Delta D_{zw}(k)]w(k) \end{aligned} \quad (43)$$

$$\begin{aligned} & \begin{bmatrix} \Delta A(k) & \Delta B_u(k) & \Delta B_w(k) & \Delta A_d(k) & \Delta B_d(k) \\ \Delta C_z(k) & \Delta D_{zu}(k) & \Delta D_{zw}(k) & \Delta C_{zd}(k) & \Delta D_{zd}(k) \end{bmatrix} \\ & = \begin{bmatrix} H_x \\ H_z \end{bmatrix} F(k)[E_x \; E_u \; E_w \; E_{dx} \; E_{du}] \end{aligned} \quad (44)$$

can be transformed into the form of (25) without the parameter uncertainties under preserving quadratic stability and $H^\infty$ norm bound through some manipulations[7,15]. In here, unknown matrix is defined as

$$\begin{aligned} & F(k) \in \Omega := \{ F(k) : F(k)^T F(k) \le I, \\ & \text{the elements of } F(k) \text{ are Lebesgue measurable} \}. \end{aligned} \quad (45)$$

Therefore the robust $H^\infty$ state feedback controller design problem in discrete time case for parameter uncertain delay system can be solvable using the proposed method.

**Example 2:** Consider a generalized discrete time delay system

$$\begin{aligned} x(k+1) = {} & \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} x(k)+\begin{bmatrix} 0.2 & 0.1 \\ 0 & 0.1 \end{bmatrix} x(k-d_1) \\ & +\begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix} w(k)+\begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k)+\begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix} u(k-d_2) \\ z(k) = {} & [1 \;\; 1]x(k)+[0.1 \;\; 0.1]x(k-d_1) \\ & +0.1w(k)+u(k)+0.1u(k-d_2). \end{aligned} \quad (46)$$

From the solutions satisfying (36) and changes of variables (37), all solutions are obtained as follows:

$$\begin{aligned} P & = \begin{bmatrix} 0.7261 & 0.1781 \\ 0.1781 & 0.1535 \end{bmatrix}, \\ R_1 & = \begin{bmatrix} 0.1149 & 0.0567 \\ 0.0567 & 0.0610 \end{bmatrix}, \\ M & = [-0.2279 \;\; -6.2240], \\ R_2 & = 0.0623. \end{aligned} \quad (47)$$

Therefore the discrete time state feedback gain is

$$K = [-1.2741 \;\; -0.9961]. \quad (48)$$

The obtained controller guarantees the stability for time delays and satisfies $H^\infty$ norm bound of the closed loop system. Here, we take $\gamma=1$.

## IV. Conclusion

In this paper, we proposed $H^\infty$ state feedback controller design methods for the generalized continuous time system with time-varying delays in states and control inputs, and the generalized discrete time system with time delays in states and control inputs. The existence conditions and design methods of $H^\infty$ state feedback controllers were given. Through some changes of variables and Schur complement, the obtained sufficient condition was changed into a LMI form in terms of $Q$, $M$, $S_1$, and $S_2$. An $H^\infty$ state feedback controller could be easily obtained at a time using LMI toolbox because the transformed sufficient condition was a LMI form in terms of variables. The state feedback $H^\infty$ controller guarantees not only the quadratic stability of the closed loop system but also the $H^\infty$ norm bound within a $\gamma$. Also, the proposed controller design method can be extended into the problem of robust $H^\infty$ state feedback controller design method for parameter uncertain time delay system through some manipulations.

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## Appendix

In this appendix, we introduce some lemmas of Schur complement and quadratic stability which are useful in the procedure of proof. The definition of Schur complement[1] is summarized in lemma A. In lemma B and lemma C, the meaning of quadratic stability is explained in continuous time delay system and discrete time delay system, respectively.

**Lemma A:** For any symmetric matrix $L=\begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix}$, the following

$$\begin{aligned} &(\text{i})\ L<0 \\ &(\text{ii})\ L_{11}<0,\ L_{22}-L_{12}^TL_{11}^{-1}L_{12}<0 \\ &(\text{iii})\ L_{22}<0,\ L_{11}-L_{12}L_{22}^{-1}L_{12}^T<0 \end{aligned} \tag{A.1}$$

are equivalent.

**Proof:** (i)⇔(ii):

$$\begin{aligned} & L<0 \\ \Leftrightarrow & \begin{bmatrix} I & 0 \\ -L_{12}^TL_{11}^{-1} & I \end{bmatrix}\begin{bmatrix} L_{11} & L_{12} \\ L_{12}^T & L_{22} \end{bmatrix}\begin{bmatrix} I & -L_{11}^{-1}L_{12} \\ 0 & I \end{bmatrix}<0 \\ \Leftrightarrow & \begin{bmatrix} L_{11} & 0 \\ 0 & L_{22}-L_{12}^TL_{11}^{-1}L_{12} \end{bmatrix}<0 \\ \Leftrightarrow & (\text{ii}) \end{aligned} \tag{A.2}$$

(i)⇔(iii):

$$\begin{aligned} & L<0 \\ \Leftrightarrow & \begin{bmatrix} I & -L_{12}L_{22}^{-1} \\ 0 & I \end{bmatrix}\begin{bmatrix} L_{11} & L_{12} \\ L_{12}^T & L_{22} \end{bmatrix}\begin{bmatrix} I & 0 \\ -L_{22}^{-1}L_{12}^T & I \end{bmatrix}<0 \\ \Leftrightarrow & \begin{bmatrix} L_{11}-L_{12}L_{22}^{-1}L_{12}^T & 0 \\ 0 & L_{22} \end{bmatrix}<0 \\ \Leftrightarrow & (\text{iii}) \end{aligned} \tag{A.3}$$

**Lemma B:** The continuous linear time delay system

$$\dot{x}(t) = Ax(t)+A_{d_1}x(t-d_1(t))+A_{d_2}x(t-d_2(t)) \tag{B.1}$$

is quadratically stable if there exist positive definite symmetric matrices $P$, $R_1$, and $R_2$ such that

$$\begin{bmatrix} A^TP+PA+R_1+R_2 & PA_{d_1} & PA_{d_2} \\ A_{d_1}^TP & -(1-\dot{d}_1(t))R_1 & 0 \\ A_{d_2}^TP & 0 & -(1-\dot{d}_2(t))R_2 \end{bmatrix}<0 \tag{B.2}$$

holds for the time varying delays.

**Proof**: If we take a Lyapunov functional as

$$V(x(t)) = x(t)^T P x(t) + \int_{t-d_1(t)}^{t} x(\tau)^T R_1 x(\tau) d\tau + \int_{t-d_2(t)}^{t} x(\tau)^T R_2 x(\tau) d\tau, \quad \text{(B.3)}$$

then the time derivative of (B.3) is obtained

$$\begin{aligned} \dot V(x(t)) = {} & \dot x(t)^T P x(t) + x(t)^T P \dot x(t) + x(t)^T R_1 x(t) + x(t)^T R_2 x(t) \\ & - (1 - \dot d_1(t)) x(t - d_1(t))^T R_1 x(t - d_1(t)) \\ & - (1 - \dot d_2(t)) x(t - d_2(t))^T R_2 x(t - d_2(t)). \end{aligned} \quad \text{(B.4)}$$

Let $\zeta_c(t) = [x(t)^T \; x(t-d_1(t))^T \; x(t-d_2(t))^T]^T$, (B.4) is expressed as

$$\begin{aligned} & \zeta_c(t)^T \begin{bmatrix} A^T P + PA + R_1 + R_2 & PA_{d_1} & PA_{d_2} \\ A_{d_1}^T P & -(1-\dot d_1(t)) R_1 & 0 \\ A_{d_2}^T P & 0 & -(1-\dot d_2(t)) R_2 \end{bmatrix} \zeta_c(t) \\ & := \zeta_c(t)^T Y_c \zeta_c(t). \end{aligned} \quad \text{(B.5)}$$

Therefore when $Y_c < 0$, the system (B.1) is quadratically stable. □

**Lemma C**: The discrete linear time delay system

$$x(k+1) = Ax(k) + A_{d_1} x(k-d_1) + A_{d_2} x(k-d_2) \quad \text{(C.1)}$$

is quadratically stable if there exist positive definite symmetric matrices $P$, $R_1$, and $R_2$ such that

$$\begin{bmatrix} A^T PA - P + R_1 + R_2 & A^T PA_{d_1} & A^T PA_{d_2} \\ A_{d_1}^T PA & -R_1 + A_{d_1}^T PA_{d_1} & A_{d_1}^T PA_{d_2} \\ A_{d_2}^T PA & A_{d_2}^T PA_{d_1} & -R_2 + A_{d_2}^T PA_{d_2} \end{bmatrix} < 0 \quad \text{(C.2)}$$

holds for time delays.

**Proof**: In similar to lemma B, if we take a Lyapunov functional as

$$V(x(k)) = x(k)^T P x(k) + \sum_{i=k-d_1}^{k-1} x(i)^T R_1 x(i) + \sum_{i=k-d_2}^{k-1} x(i)^T R_2 x(i), \quad \text{(C.3)}$$

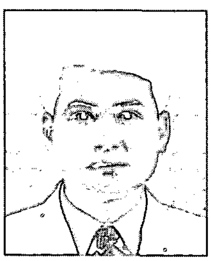
then the difference of (C.3) is obtained

$$\begin{aligned} \triangle V_k &= V(x(k+1)) - V(x(k)) \\ &= x(k+1)^T P x(k+1) - x(k)^T (P - R_1 - R_2) x(k) \\ &\quad - x(k-d_1)^T R_1 x(k-d_1) - x(k-d_2)^T R_2 x(k-d_2). \end{aligned} \quad \text{(C.4)}$$

Let $\zeta_d(t) = [x(k)^T \; x(k-d_1)^T \; x(k-d_2)^T]^T$, (C.4) is expressed as

$$\begin{aligned} & \zeta_d(t)^T \begin{bmatrix} A^T PA - P + R_1 + R_2 & A^T PA_{d_1} & A^T PA_{d_2} \\ A_{d_1}^T PA & -R_1 + A_{d_1}^T PA_{d_1} & A_{d_1}^T PA_{d_2} \\ A_{d_2}^T PA & A_{d_2}^T PA_{d_1} & -R_2 + A_{d_2}^T PA_{d_2} \end{bmatrix} \zeta_d(t) \\ & := \zeta_d(t)^T Y_d \zeta_d(t). \end{aligned} \quad \text{(C.5)}$$

Therefore when $Y_d < 0$, the system (C.1) is quadratically stable. □



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