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DETERMINATION OF NORMAL VECTORS FOR BOUNDARIES OF PLASMAS BASED UPON RANKINE-HUGONIOT RELATIONS ESTIMATED WITH A SINGLE SPACECRAFT

J. Seon
Satellite Business Divison, Hyundai Electronics Inc.
I-Chon, Kyoung-Ki-Do, Rep. of Korea, 467-701
e-mail: jhseon@hei.co.kr

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ABSTRACT

A method to determine normal vectors for boundaries of plasmas with a series of data acquired from a single spacecraft is investigated. The determination of the normal vector is possible through a set of Rankine-Hugoniot (R-H) relations that are conservation relations of plasmas across a boundary. It is assumed that the boundary is planar and that the structure of the boundary is not varing in the rest frame of plasmas. The present method utilizes a complete set of R-H relations and provides self-consistent predictions of the plasma densities, bulk velocities, and temperatures as well as magnetic fields. It is expected that the present method provides a more accurate normal vector than the previous methods which employ only subsets of the available R-H relations.

1. INTRODUCTION

A great deal of physically interesting phenomena in space occasionally occur at the boundaries of plasmas. For example, the thermalization of supersonic solar wind flow streaming away from the Sun is accomplished at the planetary bow shock (Kennel *et al.* 1985 and references therein) and the separation of magnetic fields of the solar wind from those of Earth occurs at the Earth's magnetopause (Ness *et al.* 1964). A proper interpretation of the data for such boundaries of space plasmas requires an accurate determination of the normal vector for the boundaries and relative motion of the spacecraft with respect to the boundaries. The direct determination of these quantities requires at least three-point measurements in space.

On the other hand, most of the previous in-situ measurements in space have been made with a single spacecraft. This trend is expected to remain for future spacecraft missions. Therefore, the method to determine the normal vector for plasma boundaries will be a subject of great interest as long as the necessity to investigate the measurements from a single spacecraft exists.

2. R-H RELATIONS

As was previously mentioned the direct determination of the normal vector with measurements from a single spacecraft is not generally possible. However, if a series of observations across a boundary is available it is possible to infer the normal vector with the conservation relations of plasma parameters. In this section a set of R-H relations that explains the conserved quantities across a boundary between two physically distinct plasmas will be presented.

A set of ideal MagnetoHydroDynamics (MHD) equations combined with Maxwell's equation can be expressed in conservation forms as follows (Hudson 1970). $P_{ij} = P_{\perp}\delta_{ij} + (P_{\parallel} - P_{\perp})B_iB_j/B^2$ were derived by Hudson (1970). Here δ_{ij}=1 if i=j and 0 otherwise. B_i and B_j are Cartesian components of magnetic fields. The R-H relations derived by Hudson are

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0 \tag{1}$$

$$\frac{\partial(\rho\mathbf{v})}{\partial t} + \nabla\cdot(\rho\mathbf{v}\mathbf{v} + \overleftrightarrow{P} + \frac{B^2}{8\pi}\overleftrightarrow{I} - \frac{1}{4\pi}\mathbf{B}\mathbf{B}) = 0 \tag{2}$$

$$\frac{\partial}{\partial t}(\rho v^2) + \frac{1}{2}\mathrm{Tr}(\overleftrightarrow{P})) + \nabla\cdot(\frac{\rho v^2}{2}\mathbf{v} + \frac{5}{6}\overleftrightarrow{P}\cdot\mathbf{v} + \frac{1}{4\pi}\mathbf{B}\times(\mathbf{v}\times\mathbf{B})) = \mathbf{0} \tag{3}$$

$$\nabla\times(\mathbf{v}\times\mathbf{B}) = 0 \tag{4}$$

$$\nabla\cdot(\mathbf{B}) = 0 \tag{5}$$

Here ρ, $\mathbf{v}$, $\overleftrightarrow{P}$, and $\mathbf{B}$ are plasma mass density, bulk flow velocity, plasma pressure tensor, and magnetic fields, respectively in the rest frame of plasmas. $\overleftrightarrow{P}$ is the trace of the pressure tensor. In the present study a gyrotropic form of the plasma pressure tensor

$$\overleftrightarrow{P}_{ij} = P_{\perp}\delta_{ij} + (P_{\parallel} - P_{\perp})B_iB_j/B^2$$

will be used. Here $P_{\parallel}$ and $P_{\perp}$ are the elements of the pressure tensor estimated along and perpendicular to the magnetic field **B**. B_i and B_j are the Cartesian components of the field. In equation (3) the heat flux of plasmas is ignored. The specific heat ratio of plasma is assumed to be 5/3.

In general the boundaries of plasma have relative velocities with respect to the spacecraft. If we denote the relative velocity as vsh the plasma velocities measured in the rest frame of the boundary is written as . If the structure of the boundary is not changing with respect to time in the rest frame of the boundary, *i.e.*, and the boundary is planar, the above relation can be rewritten as below. Here all the plasma velocities are expressed in the rest frame of the boundary and the differential operator is replaced with the normal vector of the boundary n according to the stokes's theorem. The subscripts n and t in the following equations represent the normal and tangential components with respect to the plasma boundary. The brackets [ ] mean that a quantity within the brackets should be evaluated on either side of the boundary and then subtracted from the quantity evaluated on the other side. Therefore each equation states that the bracketed quantity should be constant across the boundary.

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Hudson are

$$[\rho \tilde{V}_n] = 0 \tag{6}$$

$$\left[\rho \tilde{V}_n \tilde{\mathbf{V}}_t - \frac{B_n \mathbf{B}_t}{4\pi}\left(1 - \frac{4\pi(P_\parallel - P_\perp)}{B^2}\right)\right] = 0 \tag{7}$$

$$\left[\rho \tilde{V}_n^2 + P_\perp + (P_\parallel - P_\perp)\frac{B_n^2}{B^2} + \frac{B_t^2}{8\pi}\right] = 0 \tag{8}$$

$$\left[\rho \tilde{V}_n \frac{(\tilde{V}_n^2 + \tilde{V}_t^2)}{2} + \tilde{V}_n \frac{(P_\parallel + 4P_\perp)}{2} + (P_\parallel - P_\perp)\frac{(\tilde{\mathbf{V}} \cdot \mathbf{B})B_n}{B^2} + \frac{B^2}{4\pi}\tilde{V}_n - \frac{(\tilde{\mathbf{V}} \cdot \mathbf{B})B_n}{4\pi}\right] = 0 \tag{9}$$

$$\left[B_n \tilde{\mathbf{V}}_t - \tilde{V}_n \mathbf{B}_t\right] = 0 \tag{10}$$

$$[B_n] = 0 \tag{11}$$

3. PREVIOUS METHODS

Several methods that utilize the R-H relations have been developed to determine the normal vector to the boundary. Brief discussions of these methods will be made in order to describe the present method for determining the shock normal. The methods are generally categorized according to the number of R-H relations that are employed in the methods.

The minimum variance analysis developed by Sonnerup & Cahill (1967) determines the normal vector based on equation (11), the conservation of the normal component of magnetic fields. The normal vector to be determined with the analysis is obtained by minimizing

$$\sigma^2 = \frac{1}{N}\sum_{i=1}^{N}(\mathbf{B}_i \cdot \mathbf{n} - \bar{\mathbf{B}} \cdot \mathbf{n}) \tag{12}$$

subject to the constraint $\mathbf{n}$=1. Here B_i denotes individual measurements of magnetic fields and the overhead bar denotes an average over the N measurements. It can be shown that the minimization of (12) is equivalent to finding the smallest eigenvalue of the matrix defined as

$$M_{\alpha\beta} = \overline{B_\alpha B_\beta} - \overline{B}_\alpha \overline{B}_\beta \qquad \alpha, \beta = 1, 2, 3, \tag{13}$$

where B_α and B_β are the components of each measurement. The normal vector to the boundary is the eigenvector of the smallest eigenvalue. If the boundary of interest is a shock, it is well known that the asymptotic magnetic field for the downstream of the shock is located in a plane defined with the asymptotic upstream magnetic field and the shock normal (see, for example, Colburn & Sonett 1966). This property of magnetic coplanarity (Figure 1) for the shocks yields an expression for the shock normal vector

$$\mathbf{n} = \pm \frac{(\mathbf{B}_1 \times \mathbf{B}_2) \times (\mathbf{B}_1 - \mathbf{B}_2)}{|(\mathbf{B}_1 \times \mathbf{B}_2) \times (\mathbf{B}_1 - \mathbf{B}_2)|} \tag{14}$$

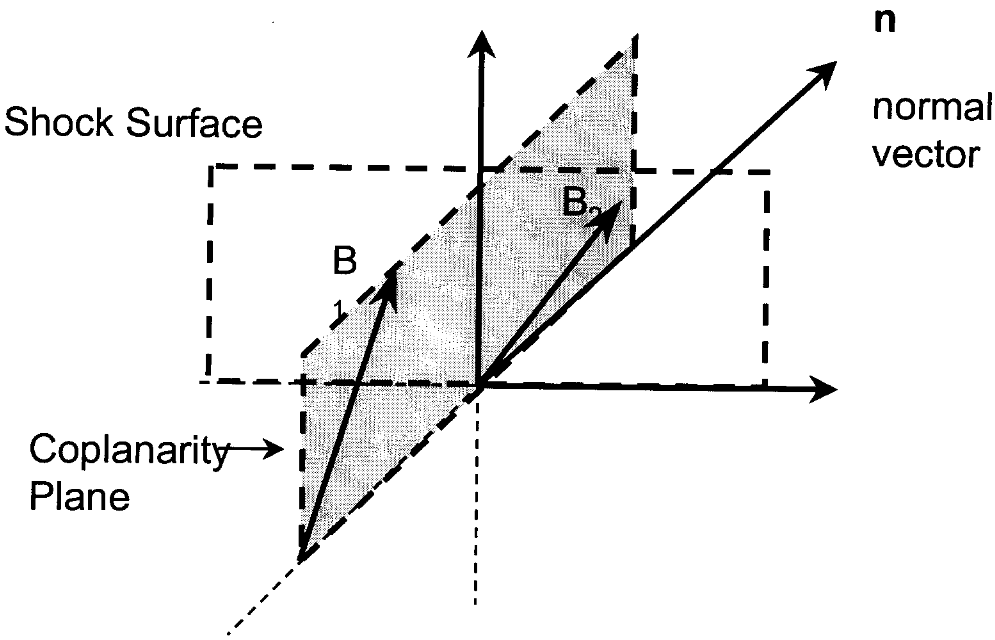

Figure 1. Illustration of the magnetic coplanarity when the plasma boundary is a shock.

Generalization of the above method of magnetic coplanarity for the velocity vectors was made with an inclusion of the conservation of tangential electric field and of tangential momentum flux, together with the conservation of mass flux (Abraham-Shrauner 1972). It was shown that the normal vector to the shock can be alternatively expressed as

$$\mathbf{n} = \pm \frac{[(\mathbf{V}_2 - \mathbf{V}_1) \times \mathbf{B}_1] \times (\rho_2 \mathbf{V}_2/\rho_1 - \mathbf{V}_1)}{|[(\mathbf{V}_2 - \mathbf{V}_1) \times \mathbf{B}_1] \times (\rho_2 \mathbf{V}_2/\rho_1 - \mathbf{V}_1)|} \tag{15}$$

Previous methods such as the methods of magnetic coplanarity or Abraham-Shrauner do not take into account the uncertainties for the measurements and the fluctuations of plasma parameters and magnetic fields. As noted earlier, the estimates for the R-H relations as determined from the plasma parameters and magnetic fields are critical for the identification of the boundary. Therefore, a careful consideration of the uncertainties is required in order to achieve a convincing identification.

Lepping & Argentiero (1971) developed a method of nonlinear least-square fit of plasma parameters for a shock based on the R-H relations without plasma pressure terms. The method of the least-square fitted is undertaken in the 11-parameter space $\mathbf{B}_1, \mathbf{B}_2, \mathbf{V}_1 - \mathbf{V}_2$. They also showed that their procedure which employs both the magnetic field and plasma moments provides a better estimate for the shock normal than the method of magnetic coplanarity. Viñas & Scudder (1986) reduced the procedure of Lepping & Argentiero (1971) into two successive applications of a least-square fit that is undertaken in a smaller parameter space. Viñas & Scudder used the same R-H relations as Lepping & Argentiero (1971), but the procedure of Viñas & Scudder (1986) first determines the shock normal that coincides with the best predictions for the velocities and magnetic fields for both upstream and downstream regions of the shock. These velocities and magnetic fields, which are expressed as functions of the densities for each side of the shock, are then fit with respect to the density by the method of another least-square fit. A similar procedure, but with the inclusion of the R-H relations for the conservation of normal momentum and of energy, was developed by Szabo (1994).

Comprehensive observations of plasmas and magnetic fields from recent spacecraft now allow inference of the shock normal from the entire R-H relations. Previous investigations utilized only a subset of the above R-H relations because of the limited availability of the plasma measurements. For example, the direct measurements of plasma pressure tensors for both the ions and electrons were not available until the recent advent of sophisticated plasma instrumentation. This utilization of the entire R-H relations is expected to provide a more accurate shock normal than those from the methods of magnetic coplanarity of Abraham-Shrauner. Further, self-consistent predictions for all the downstream parameters are possible from the R-H relations. It will be demonstrated below how such determinations of the shock normal and prediction of the downstream parameters are possible in the present study. The present method takes into account the uncertainties for downstream plasma parameters and magnetic fields.

4. PRESENT METHOD

The R-H relations (6)-(11) are defined in a reference frame where the boundary is time-stationary. Note that of equations (6)-(12) is related to v measured in spacecraft frame through . If equation (6) is solved for , then

$$\mathbf{v}_{sh} \cdot \mathbf{n} = \frac{\rho_2 \mathbf{v}_2 \cdot \mathbf{n} - \rho_1 \mathbf{v}_1 \cdot \mathbf{n}}{\rho_2 - \rho_1} \tag{16}$$

Therefore equation (16) gives an estimate for if n is given. The shock is assumed to be propagating only along the direction of the shock normal, *i.e.*, $\mathbf{v}_{sh} = v_{sh}\mathbf{n}$. An estimate for the shock normal is obtained through the following procedure (Seon 1996). It is convenient to express R-H relations in the normal incidence frame where the flow velocity of plasmas in the upstream region is parallel to the shock normal. The velocity in the normal incidence frame $\mathbf{v}'$, where the plasma flow is along the normal vector of the boundary, is therefore related to the velocity in spacecraft frame through $\mathbf{v}' = \mathbf{v} - \mathbf{v}_{1t} - \mathbf{v}_{sh}$ The magnetic fields remain the same after these two successive Galilean transformations of reference frame. In the equations shown below $P_{\|}$ and $P_{\perp}$ and are related to the pressure of the plasma that is defined as $P = \frac{1}{3}(P_{\|} + 2P_{\perp})$ and the anisotropy factor α through

$$P_{\|} = P + \frac{2}{3}\frac{B^2}{4\pi}(1-\alpha) \quad \text{and} \quad P_{\perp} = P - \frac{1}{3}\frac{B^2}{4\pi}(1-\alpha) \tag{17}$$

The above R-H relations are not a closed set of equations. Without an additional equation that gives a relation between and the downstream plasma parameters and magnetic fields cannot be self-consistently determined.

In the present study the downstream anisotropy factor α_2 will be left as a free parameter and is to be provided by measurement. For a given value of α_2, the above equations form a closed set and all the downstream plasma parameters and magnetic fields can be predicted. This approach was proposed by Chao (1970) and essentially the same equations can be found in his paper. The above equations can be rewritten in the normal incidence frame in terms of the following dimensionless variables $n_2 = \rho_2/\rho_1$, $u_{2x} = V'_{2x}/V'_1$, $u_{2z} = V'_{2z}/V'_1$, $b_{1,2} = \mathbf{B}_{1,2}/(\sqrt{4\pi\rho_1}V'_1)$, and $v_{1,2} = \sqrt{P_{1,2}/\rho_{1,2}}/V'_1$.

$$n_2 u_{2x} = 1 \tag{18}$$

$$b_{1z} b_x \alpha_1 = b_{2z} b_x \alpha_2 - u_{2z} \tag{19}$$

$$(1 + v_1^2) + \frac{1}{3}\left(\alpha_1 + \frac{1}{2}\right) b_{1z}^2 - \frac{2}{3}\alpha_1 b_x^2 = n_2(u_{2x}^2 + v_2^2) + \frac{1}{3}\left(\alpha_2 + \frac{1}{2}\right) b_{2z}^2 - \frac{2}{3}\alpha_2 b_x^2 \tag{20}$$

$$\begin{aligned} 1 + 5v_1^2 + 2b_{1z}^2 + (1-\alpha_1)\left(\frac{4}{3}b_x^2 - \frac{2}{3}b_{1z}^2\right) &= u_{2x}^2 + u_{2z}^2 + 5v_2^2 + 2b_{2z}b_{1z} \\ &+2b_x b_{2z} u_{2z}(1-\alpha_2) + \frac{4}{3}b_x^2 u_{2x}(1-\alpha_2) - \frac{2}{3}b_{2z}^2 u_{2x}(1-\alpha_2) \end{aligned} \tag{21}$$

$$b_{1z} = u_{2x} b_{2z} - u_{2z} b_x \tag{22}$$

$$b_{1x} = b_{2x} \equiv b_x, \tag{23}$$

where the subscripts x and z denote the normal and tangential components in the normal incidence frame, respectively. Equations (19) and (22) give

$$b_{2z} = b_{1z}(1 - b_x^2\alpha_1)/(u_{2x} - b_x^2\alpha_2) \tag{24}$$

and

$$u_{2z} = b_{1z} b_{1x}(\alpha_2 - u_{2x}\alpha_1)/(u_{2x} - b_x^2\alpha_2). \tag{25}$$

Combination of equations (18), (20), (21), (24), and (25) yields the following equation to predict the downstream density

$$c_4 u_{2x}^4 + c_3 u_{2x}^3 + c_2 u_{2x}^2 + c_1 u_{2x} + c_0 = 0, \tag{26}$$

where

$$\begin{aligned} c_4 &= 4 \\ c_3 &= -10\alpha_2 b_x^2 + \left(\frac{10}{3}\alpha_1 - \frac{4}{3}\right) b_x^2 - (5 + 5v_1^2) - \frac{5}{6}\left(2\alpha_1 + 1\right) b_{1z}^2 \\ c_2 &= \left(8\alpha_2^2 + \frac{8}{3}\alpha_2 - \frac{20}{3}\alpha_1\alpha_2\right) b_x^4 + \left(\frac{4}{3}(1-\alpha_1) + 10(1+v_1^2)\alpha_2\right) b_x^2 \\ &+ \left(\frac{10}{3}\alpha_1\alpha_2 + \frac{5}{3}\alpha_2 - \alpha_1^2\right) b_x^2 b_{1z}^2 + (1 + 5v_1^2) + \frac{2}{3}\left(2 + \alpha_1\right) b_{1z}^2 \end{aligned}$$

$$
\begin{aligned}
c_1 = & -\frac{2}{3}\Big(3\alpha_2 - 5\alpha_1 + 2\Big)\alpha_2^2 b_x^6 - \Big((5+5v_1^2)\alpha_2^2 + \frac{8}{3}(1-\alpha_1)\alpha_2\Big)b_x^4 \\
& -\frac{1}{6}\Big(3\alpha_1^2 + 5\alpha_2^2 + 10\alpha_1\alpha_2^2 - 18\alpha_1^2\alpha_2\Big)b_x^4 b_{1z}^2 - \left(\frac{8}{3}\alpha_2 - \alpha_1 + \frac{10}{3}\alpha_1\alpha_2\right) b_x^2 b_{1z}^2 \\
& -(2+10v_1^2)\alpha_2 b_x^2 - \left(\frac{1}{2} - \alpha_2\right) b_{1z}^2
\end{aligned}
$$

and

$$
c_0 = \frac{4}{3}\Big(1-\alpha_1\Big)\alpha_2^2 b_x^6 + \frac{4}{3}\Big(1-\alpha_1\Big)\alpha_2^2 b_x^4 b_{1z}^2 + \alpha_2^2 b_x^2 b_{1z}^2 + (1+5v_1^2)\alpha_2^2 b_x^4
$$

Once the solution to (26) is found, (18), (20), (24), and (25) provide predictions for the downstream plasma parameters and magnetic fields as a function of downstream anisotropy factor. Therefore specification of completes the prediction.

6. CONCLUSION

It is shown in the present study that the extension of the previous methods to determine normal vectors for the plasma boundaries is possible with R-H relations for anisotropic, gyrotropic plasmas. The method presented in this paper advances the existing methods by 1) utilizing complete R-H relations, 2) considering anisotropy of plasmas, 3) providing self-consistency among the predicted plasma parameters and 4) considering proper accounting of uncertainties of measurements. Such considerations have not been made in the previous investigations partly due to the unavailability of the plasma measurements. Application of the present method to the identification of slow-mode shocks in Earth's distant magnetotail and comparisons with results from the previous methods are given by Seon *et al.* (1996), but the method is applicable to other types of MHD boundaries such as Earth's bow shock or rotational discontinuities in the solar wind or at the Earth's magnetopause.

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