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비정체형 지하대수층의 속도-대수투수계수, 속도-수두 교차공분산에 관한 연구

A Study on Velocity-Log Conductivity, Velocity-Head Cross Covariances in Aquifers with Nonstationary Conductivity Fields

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Abstract

In this study, random flow field in a nonstationary porous formation is characterized through cross covariances of the velocity with the log conductivity and the head. The hydraulic head and the velocity in saturated aquifers are found through stochastic analysis of a steady, two-dimensional flow field without recharge. Expression for these cross covariances are obtained in quasi-analytic forms all in terms of the parameters which characterize the nonstationary conductivity field and the average head gradient. The cross covariances with a Gaussian correlation function for the log conductivity are presented for two particular cases where the trend is either parallel or perpendicular to the mean head gradient and for separation distances along and across the mean flow direction. The results may be of particular importance in transport predictions and conditioning on field measurements when the log conductivity field is suspected to be nonstationary and also serve as a benchmark for testing nonstationary numerical codes.

keywords : cross covariance, nonstationary conductivity field, saturated aquifer, stochastic analysis

요 지

본 논문에서는 다공성 매질의 투수계수장이 비정체형인 경우 대수투수계수 및 수두와 속도의 교차공분산을 통하여 불규칙한 유동장을 규명하였으며, 포화대수층 내의 속도 및 수두 분포는 유입이 없는 2차원, 정상 유동문제를 추계학적으로 해석하여 구하였다. 이들 교차공분산들은 준 해석적 형태로 나타낼 수 있으며 비정체형 투수장과 수두의 평균 기울기를 나타내는 매개변수들로 표현된다. 투수계수의 상관 함수가 가우스 분포를 가지고 그 경향이 평균 수두 기울기와 평행한 경우와 수직인 두 특수한 경우의 교차공분산을 평균 유동 방향과 같은 방향이거나 수직 방향에 관하여 해석하였다. 이 교차공분산들은 투수계수장이 비정체형인 경우에 물질 이동의 예측 및 현장에서 측정 시 conditioning에 유효하게 쓰일 수 있고 비정체형 수치해석 프로그램의 검증에도 활용할 수 있다.

핵심용어 : 교차공분산, 비정체형 투수장, 포화대수층, 추계학적 해석

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1. Introduction

Movement of water and solute through natural porous formations depends not only on the subsurface flow conditions but also on the hydrogeologic properties through which the flow occurs and it is common to find these properties in earth materials highly variable. Stochastic approach regards aquifer properties such as hydraulic conductivity, K, and hydraulic head, H, as spatial random functions (SRF) characterized by probability distributions. Most of the stochastic study is based on the assumption that the conductivity field is stationary, i.e., its spatial mean is a constant. Although this assumption of stationarity greatly simplifies the mathematical analysis and may be applicable in many situations, it is by no means universal and recent field studies (Woodbury and Sudicky, 1991; Rehfeldt et al., 1992; LaVenue et al., 1995) support such a statement.

There have been theoretical studies such as Loaiciga et al. (1993) who systematically incorporated the nonstationarity of the log conductivity field in the stochastic analysis of subsurface flow and the problem of transport in a nonstationary field was treated by Rubin and Seong (1994) where they provided the first-order solution of a linear conductivity field with mean flow either parallel to or perpendicular to the trend. A more general case was studied by Indelmann and Rubin (1996) when they obtained the equivalent conductivity tensor for a general orientation of the mean flow vector.

The purpose of this study is to further explore the phenomena of subsurface flow by studying the spatial structure of the heterogeneous random velocity field through cross covariances of the velocity, U, with the log conductivity, $Y=\ln K$, and the hydraulic head, H. An exact solution of the velocity field would require a complete description of Y and H, which is impossible for a heterogeneous porous medium. A more practical method would be to solve for the statistical moments of the velocity fluctuation and these are reported in Rubin and Seong (1994).

In this paper we will further explore the coregionalized fields by obtaining expressions of the velocity cross covariances with Y and H. (The $Y-H$ cross covariance was obtained by Seong (1996).) These cross covariances can be of particular importance in modeling transport problems. Rubin (1991) outlined a procedure for conditioning transport predictions on the velocity, head and log conductivity, which requires velocity cross covariances. Furthermore, since the final results will be of quasi-analytic form requiring only quadratures, these may serve as a useful benchmark in testing of numerical codes that can handle flow and transport problems in nonstationary conductivity fields.

2. Mathematical Statement of the Problem

We will restrict our analysis to a steady, two-dimensional groundwater flow in a saturated aquifer lying horizontally without recharge and to simplify calculations we will adopt a first-order analysis.

Flux, $\boldsymbol{q}$, and the hydraulic head, H, follow the continuity equation and the Darcy's law:

$$\nabla \cdot \boldsymbol{q}(\boldsymbol{x})=0 \tag{1a}$$

$$\boldsymbol{q}(\boldsymbol{x})=-K(\boldsymbol{x})\nabla H(\boldsymbol{x}) \tag{1b}$$

where $\boldsymbol{x}$ is the space coordinate. Here and subsequently boldface letters denote vectors. Following field studies (Hoeksema and Kitanidis, 1985) as well as works summarized by Gelhar (1993), the log conductivity, $Y(\boldsymbol{x})$, is treated as a lognormal SRF.

In order to investigate the effects of

nonstationarity, Y is assumed to be comprised of a spatially varying mean and a small-scale local fluctuation and in this study we assume that the expected value of Y is a linear function of space coordinate:

$$\begin{aligned} Y(\boldsymbol{x}) &= \langle Y(\boldsymbol{x})\rangle + Y'(\boldsymbol{x}) \\ &= m_0 + \boldsymbol{a}\cdot\boldsymbol{x} + Y'(\boldsymbol{x}) \end{aligned} \tag{2}$$

where m_0 and $\boldsymbol{a}$ are constants and $Y' = O[\sigma_Y]$, where σ_Y is the standard deviation of Y. Here and subsequently, angle brackets denote expected value operators and $O[\ ]$ represents the customary Landau order symbol (see Nayfeh (1973)). The local fluctuation Y' is stationary, i.e., has a zero mean and a covariance C_Y that depends only on the separation vector:

$$\begin{aligned} C_Y(\boldsymbol{x},\boldsymbol{y}) &= \langle Y'(\boldsymbol{x})Y'(\boldsymbol{y})\rangle \\ &= C_Y(|\boldsymbol{r}|) = \sigma_Y^2\rho_Y(|\boldsymbol{r}|) \end{aligned} \tag{3}$$

where $\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{y}$, σ_Y^2 the variance and ρ_Y is the correlation function of Y.

Eliminating $\boldsymbol{q}$ from Eqn. (1) and combining Eqn. (2) result in the following stochastic PDE:

$$\nabla^2 H(\boldsymbol{x}) + \boldsymbol{a}\cdot\nabla H(\boldsymbol{x}) = -\nabla Y'\cdot\nabla H(\boldsymbol{x}) \tag{4}$$

For the boundary condition we assume that the head gradient at some point $\boldsymbol{\zeta}$ in the flow domain is given as $\langle\nabla H(\boldsymbol{\zeta})\rangle = -\boldsymbol{J} = (-J_0, 0)$, implying that our coordinate be set up such that x_1-axis is aligned with the mean flow direction. To simplify calculations we limit our analysis to two particular cases: the case of $\boldsymbol{a}$ parallel to $\boldsymbol{J}$ and the case where $\boldsymbol{a}$ is orthogonal to $\boldsymbol{J}$. With our coordinate system set up as explained earlier, the former case results in $\boldsymbol{a} = (a_1, 0)$ which we will refer to as the a_1-case and the latter $\boldsymbol{a} = (0, a_2)$ which will be referred to as the a_2-case.

When the head is expressed as the sum of a mean and a local fluctuation as $H = \langle H\rangle + h$, where $h = O[\sigma_Y]$, Eqn. (4) leads to a set of equations of $O[1]$ for $\langle H\rangle$ and of $O[\sigma_Y]$ for h:

$$\nabla^2\langle H\rangle + \boldsymbol{a}\cdot\nabla\langle H\rangle = 0 \tag{5a}$$

$$\nabla^2 h + \boldsymbol{a}\cdot\nabla h = -\nabla Y'\cdot\nabla\langle H\rangle \tag{5b}$$

Solutions to the above set of equations have been obtained by Rubin and Seong (1994) for two particular cases mentioned above.

The flow velocity, $\boldsymbol{U} = \langle\boldsymbol{U}\rangle + \boldsymbol{u}$, is related to the flux, $\boldsymbol{q}$, by $\boldsymbol{U} = \boldsymbol{q}/n$ where n is the effective porosity taken to be constant. Using Eqns. (1b) and (2), the mean and the fluctuation are found as the following:

$$\langle\boldsymbol{U}\rangle = -\frac{e^{m_0}}{n}e^{\boldsymbol{a}\cdot\boldsymbol{x}}\nabla\langle H\rangle \tag{6a}$$

$$\boldsymbol{u} = -\frac{e^{m_0}}{n}e^{\boldsymbol{a}\cdot\boldsymbol{x}}(\nabla h + Y'\nabla\langle H\rangle) \tag{6b}$$

Using above results of first order solutions, we pursue derivation of various cross covariances of the velocity:

$$C_{u_iY} = \langle u_i(\boldsymbol{x})Y'(\boldsymbol{y})\rangle \tag{7a}$$

$$C_{u_iH} = \langle u_i(\boldsymbol{x})h(\boldsymbol{y})\rangle \tag{7b}$$

for $i = 1$ and 2, denoting different directions of a cartesian coordinate system. Here we choose to use the spectral method in deriving the statistical moments of Eqn. (7) and a brief note regarding Fourier transforms is given in the appendix

3. Velocity-Log Conductivity Cross Covariance

3.1 The α_1-case : $\boldsymbol{\alpha} = (\alpha_1, 0)$

When $\alpha_2 = 0$, the mean head gradient from Eqn. (5a) can be found as $\partial\langle H(\boldsymbol{x})\rangle / \partial x_i = -J_0\, e^{-\alpha_1(x_1-\zeta_1)}\delta_{i1}$, where δ is the Kronecker delta, and hence Eqn. (5b) reduces to:

$$\nabla^2 h + \alpha_1 \frac{\partial h}{\partial x_1} = J_0\, e^{-\alpha_1(x_1-\zeta_1)} \frac{\partial Y'}{\partial x_1} \tag{8}$$

from which the Fourier transformed solution of h is found as:

$$\hat{h}(\boldsymbol{k}) = J_0 e^{\alpha_1\zeta_1} \frac{\mathrm{i}k_1 - \alpha_1}{k^2 + \mathrm{i}\alpha_1 k_1}\, \widehat{Y'}(k_1 + \mathrm{i}\alpha_1, k_2) \tag{9}$$

where i is the imaginary unit and $\boldsymbol{k} = (k_1, k_2)$ is the wave number vector in Fourier space. Using above results and Eqn. (6b) for $\alpha_2 = 0$, the Fourier transform of u_i can be found as:

$$\widehat{u_i}(\boldsymbol{k}) = f_i(\boldsymbol{k})\, \widehat{Y'}(\boldsymbol{k}) \tag{10}$$

where $f_1(\boldsymbol{k}) = k_2^2/(k^2 - \mathrm{i}\alpha_1 k_1)$ and $f_2(\boldsymbol{k}) = -k_1 k_2/(k^2 - \mathrm{i}\alpha_1 k_1)$. The k^2 is the modulus of the wave number vector and u_i has been non-dimensionalized with a reference velocity of $e^{m_0 + \alpha_1\zeta_1} J_0 / n$.

We can now derive the velocity-log conductivity cross covariance $C_{u_iY}(\boldsymbol{x}, \boldsymbol{y})$ by taking the inverse Fourier transform:

$$C_{u_iY}(\boldsymbol{x}, \boldsymbol{y}) = \left\langle \frac{1}{2\pi} \int \widehat{u_i}(\boldsymbol{k}) e^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} d\boldsymbol{k}\; \frac{1}{2\pi} \int \widehat{Y'}(\boldsymbol{k}') e^{-\mathrm{i}\boldsymbol{k}'\cdot\boldsymbol{y}} d\boldsymbol{k}' \right\rangle \tag{11}$$

and when Eqn. (10) and the correlation function of Eqn. (3) is used, Eqn. (11) reduces to the following:

$$C_{u_iY}(\boldsymbol{x}, \boldsymbol{y}) = \frac{\sigma_Y^2}{2\pi} \int \hat{\rho}_Y(\boldsymbol{k}) f_i(-\boldsymbol{k}) e^{-\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{y}-\boldsymbol{x})} d\boldsymbol{k} \tag{12}$$

Integrations regarding Fourier transforms are performed over the entire 2-dimensional plane. It can be seen that the velocity-log conductivity of α_1-case is stationary, i.e., it only depends on the separation distance as $C_{u_iY}(\boldsymbol{x}, \boldsymbol{y}) = C_{u_iY}(0, \boldsymbol{y} - \boldsymbol{x})$.

3.2 The α_2-case : $\boldsymbol{\alpha} = (0, \alpha_2)$

When $\alpha_1 = 0$, the mean head gradient from Eqn. (5a) can be found as $\partial\langle H(\boldsymbol{x})\rangle / \partial x_i = -\delta_{i1} J_0$ and hence the velocity fluctuation of Eqn. (6b) can be expressed as:

$$u_i = e^{\alpha_2 x_2}\left(Y'\delta_{i1} - \frac{1}{J_0}\frac{\partial h}{\partial x_i} \right) \tag{13}$$

which again has been non-dimensionalized with a reference velocity of $e^{m_0} J_0 / n$. Thus the velocity-log conductivity cross covariance can be expressed as:

$$C_{u_iY}(\boldsymbol{x}, \boldsymbol{y}) = e^{\alpha_2 x_2}\left(\delta_{i1}\langle Y'(\boldsymbol{x}) Y'(\boldsymbol{y})\rangle - \frac{1}{J_0}\left\langle \frac{\partial h(\boldsymbol{x})}{\partial x_i} Y'(\boldsymbol{y})\right\rangle\right) \tag{14}$$

As for the head fluctuation, Eqn. (5b) reduces to

$$\nabla^2 h + \alpha_2 \frac{\partial h}{\partial x_2} = J_0 \frac{\partial Y'}{\partial x_1} \tag{15}$$

from which the Fourier transformed solution is found as:

$$\hat{h}(\boldsymbol{k}) = J_0 \frac{\mathrm{i} k_1}{k^2 + \mathrm{i}\alpha_2 k_2} \widehat{Y'}(\boldsymbol{k}) \tag{16}$$

Thus the derivative term in Eqn. (14) can be expressed in terms of its Fourier transform as the following:

$$\begin{aligned} \frac{\partial h}{\partial x_i} &= \frac{\partial}{\partial x_i} \frac{1}{2\pi} \int \hat{h}(\boldsymbol{k}) e^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} d\boldsymbol{k} \\ &= \frac{J_0}{2\pi} \int \frac{k_1 k_i}{k^2 + \mathrm{i}\alpha_2 k_2} \widehat{Y'}(\boldsymbol{k}) e^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} d\boldsymbol{k} \end{aligned} \tag{17}$$

and when used in Eqn. (14), $C_{u_iY}(\boldsymbol{x}, \boldsymbol{y})$ can finally be obtained as:

$$\begin{aligned} C_{u_iY}(\boldsymbol{x}, \boldsymbol{y}) &= e^{\alpha_2 x_2} \frac{\sigma_Y^2}{2\pi} \int \hat{\rho}_Y(\boldsymbol{k}) \\ &\left(\delta_{i1} - \frac{k_1 k_i}{k^2 - \mathrm{i}\alpha_2 k_2}\right) e^{-\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{y}-\boldsymbol{x})} d\boldsymbol{k} \end{aligned} \tag{18}$$

Unlike the α_1-case, C_{u_iY} is nonstationary depending not only on the separation vector but also on the location as expressed in $e^{\alpha_2 x_2}$. However applications can be simplified by noting that $C_{u_iY}(\boldsymbol{x}, \boldsymbol{y}) = e^{\alpha_2 x_2} C_{u_iY}(0, \boldsymbol{y}-\boldsymbol{x})$. Similar results were observed for u_{ij} (Rubin and Seong, 1994) and both can be explained by the fact that flow situation in the α_2-case is dominated by the shearing action at higher conductivity region as x_2 increases.

4. Velocity-Head Cross Covariance

4.1 The α_1-case : $\boldsymbol{\alpha} = (\alpha_1, 0)$

When $\alpha_2 = 0$ the mean head gradient is $\partial\langle H(\boldsymbol{x})\rangle / \partial x_i = -J_0 e^{-\alpha_1(x_1-\zeta_1)} \delta_{i1}$ and u_i from Eqn. (6b) is:

$$u_i = \sigma_Y \left(Y' \delta_{i1} - \frac{e^{\alpha_1 \zeta_1}}{J_0} e^{\alpha_1 x_1} \frac{\partial h}{\partial x_i} \right) \tag{19}$$

where the head fluctuation was found as the following (Rubin and Seong, 1994):

$$\begin{aligned} h(\boldsymbol{x}) &= J_0 e^{\alpha_1 \zeta_1} e^{-\alpha_1 x_1/2} \\ &\int \frac{\partial Y'(\boldsymbol{y})}{\partial y_1} e^{-\alpha_1 y_1/2} G(\boldsymbol{x}-\boldsymbol{y}) d\boldsymbol{y} \end{aligned} \tag{20}$$

Here G is the Green's function of the modified Helmholtz equation. (see Arfken, 1985) Therefore we can express the head fluctuation derivative in terms of Y' as:

$$\begin{aligned} \frac{\partial h(\boldsymbol{x})}{\partial x_i} &= J_0 e^{\alpha_1 \zeta_1} e^{-\alpha_1 x_1/2} \int \frac{\partial Y'(\boldsymbol{y})}{\partial y_1} e^{-\alpha_1 y_1/2} \\ &\left\{ \frac{\partial G(\boldsymbol{x}-\boldsymbol{y})}{\partial x_i} - \delta_{i1} \alpha_1 G(\boldsymbol{x}-\boldsymbol{y}) \right\} d\boldsymbol{y} \end{aligned} \tag{21}$$

Having obtained the necessary variables in terms of Y', we can derive the cross covariance $C_{u_iH}(\boldsymbol{x}, \boldsymbol{y}) = \langle u_i(\boldsymbol{x}) h(\boldsymbol{y}) \rangle$ as:

$$C_{u_iH}(\boldsymbol{x}, \boldsymbol{y}) = \delta_{i1} C_{YH}(\boldsymbol{x}, \boldsymbol{y}) - \sigma_Y^2 e^{\alpha_1(x_1-y_1)} I(\boldsymbol{x}, \boldsymbol{y}) \tag{22}$$

where the integral I is

$$\begin{aligned} I &= \int\int \frac{\partial^2 \rho_Y(\boldsymbol{y}-\boldsymbol{z})}{\partial y_i \partial z_i} e^{-\alpha_1(y_1+z_1)/2} \\ &\left\{ \frac{\partial G(\boldsymbol{x}-\boldsymbol{y})}{\partial x_1} - \delta_{i1} \frac{\alpha_1}{2} G(\boldsymbol{x}-\boldsymbol{y}) \right\} G(\boldsymbol{y}-\boldsymbol{z}) d\boldsymbol{y} d\boldsymbol{z} \end{aligned} \tag{23}$$

and C_{YH} was found by Seong (1996). After much tedious calculation including integration by parts and using the convolution theorem, the velocity-head cross covariance for the α_1-case is found as the following:

$$C_{u_iH}(\boldsymbol{x}, \boldsymbol{y}) = e^{-\alpha_1 y_1} \frac{\sigma_Y^2}{2\pi} \int \hat{\rho}_Y(\boldsymbol{k})\, g_i(\boldsymbol{k})\, e^{-\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{y}-\boldsymbol{x})} d\boldsymbol{k} \tag{24}$$

where $g_1 = \mathrm{i}k_1k_2^2/(k^4+\alpha_1k_1^2)$, $g_2 = -\mathrm{i}k_1^2k_2/(k^4+\alpha_1^2k_1^2)$ and k^4 in g_i is the square of the wave number modulus, i.e., $k^4=(k_1^2+k_2^2)^2$. Unlike C_{u_iY} of the previous section, it is seen that the α_1-case C_{u_iH} is nonstationary decaying exponentially as the log conductivity increases in y_1-direction. This stems from the exponential decay in the head fluctuation with a positive $\alpha_1 x_1$ as seen in Eqn. (20). However applications can again be simplified when we take advantage of the relationship $C_{u_iH}(\boldsymbol{x}, \boldsymbol{y}) = e^{-\alpha_1 x_1} C_{u_iH}(0, \boldsymbol{y}-\boldsymbol{x})$.

4.2 The α_2-case : $\boldsymbol{\alpha} = (0, \alpha_2)$

Using the velocity fluctuation for the case of $\alpha_1 = 0$ previously found (Eqn. (13)), the cross covariance can be expressed as:

$$C_{u_iH} = e^{\alpha_2 x_2}(\delta_{i1}\langle Y'(\boldsymbol{x})h(\boldsymbol{y})\rangle - \frac{1}{J_0}\langle \frac{\partial h(\boldsymbol{x})}{\partial x_i} h(\boldsymbol{y})\rangle) \tag{25}$$

The first term in Eqn. (25) was found by Seong (1996) as:

$$\langle Y'(\boldsymbol{x})h(\boldsymbol{y})\rangle = \frac{\sigma_Y^2}{2\pi}\int \frac{\mathrm{i}k_1}{k^2+\mathrm{i}\alpha_2 k_2} \hat{\rho}_Y(\boldsymbol{k}) e^{-\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{y}-\boldsymbol{x})} d\boldsymbol{k} \tag{26}$$

and the second term can be found using Eqn. (17) and the inverse Fourier transform of $\hat{h}(\boldsymbol{k})$ expressed in Eqn. (16). When these are used in Eqn. (25), final expression for the velocity-head cross covariance becomes:

$$C_{u_iH}(\boldsymbol{x}, \boldsymbol{y}) = e^{\alpha_2 x_2} \frac{\sigma_Y^2}{2\pi} \int \hat{\rho}(\boldsymbol{k}) \frac{\delta_{i1}(\alpha_2 k_1 k_2 + \mathrm{i}k_1 k^2) - \mathrm{i}k_i k_1^2}{k^4 + \alpha_2^2 k_2^2} e^{-\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{y}-\boldsymbol{x})} d\boldsymbol{k} \tag{27}$$

This is also nonstationary which again is due to exponential decay of head fluctuation similar to the α_1-case as found by Rubin and Seong (1994) and it also satisfies the simplifying relationship of $C_{u_iH}(\boldsymbol{x}, \boldsymbol{y}) = e^{\alpha_2 x_2} C_{u_iH}(0, \boldsymbol{y}-\boldsymbol{x})$.

5. Results and Discussions for a Gaussian Covariance

To study the effects of a linear trend in the log conductivity on the velocity cross covariances, a two-dimensional Gaussian correlation function will be used :

$$\rho_Y(\boldsymbol{r}) = \exp[-\frac{r^2}{a^2}] \quad ; \quad a/I_Y = 2/\sqrt{\pi} \tag{28}$$

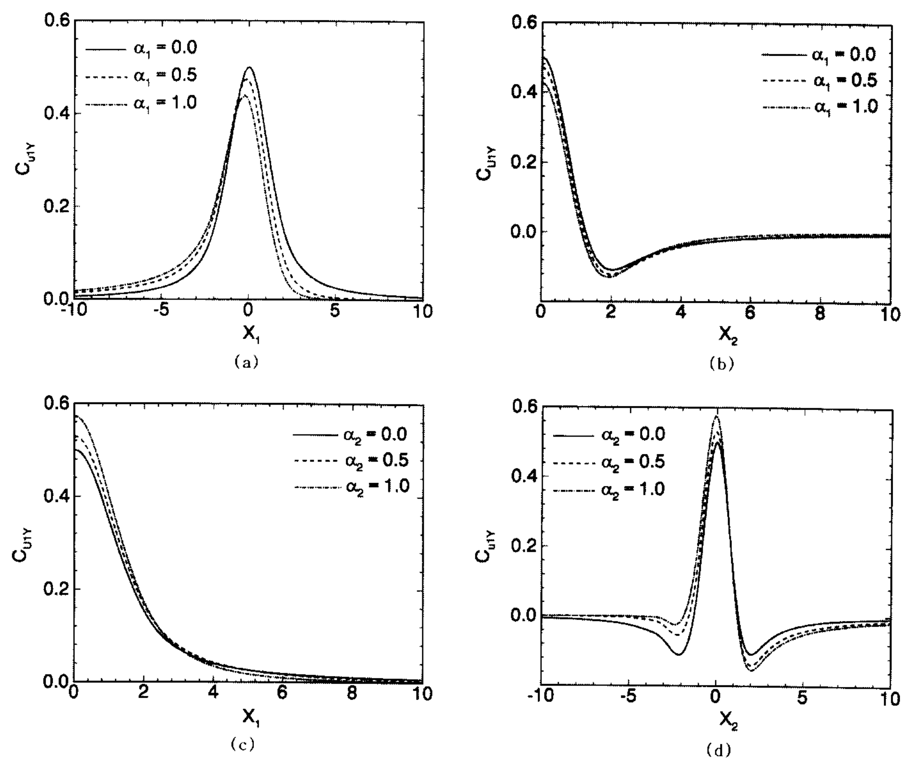
where I_Y is the log conductivity integral scale. We will concentrate on the cross covariances along separation distances in the mean flow direction, along x_1-axis, and one perpendicular to it, along x_2-axis. Furthermore when the covariances are nonstationary, only separation distances between the origin will be presented and more general cases can be obtained through the simple relationships previously found.

The physical plausibility of the four velocity cross covariances was confirmed numerically by checking the divergence free condition through control volume analysis with several different control volume shapes and sizes. In this analysis, a unit deterministic perturbation of Y or H is introduced at a given point and the resulting velocity excitations are computed using the cross covariances and integrated over the control surface. For example, when a unit deterministic perturbation in Y is introduced at the origin C_{u_iY} represents the deviation in the velocity components. Therefore, C_{u_iY} along $x_1 = L_x/2$ would be the deviation in u_1 along the right surface and C_{u_iY} along $x_2 = L_y/2$ the deviation in u_2 along the upper surface of a rectangular control volume of L_x by L_y centered at the origin. Hence it is possible to check if these velocities satisfy the continuity equation. Control volumes of various shapes and sizes were evaluated successfully with both C_{u_iY} and C_{u_iH} within numerical error bounds.

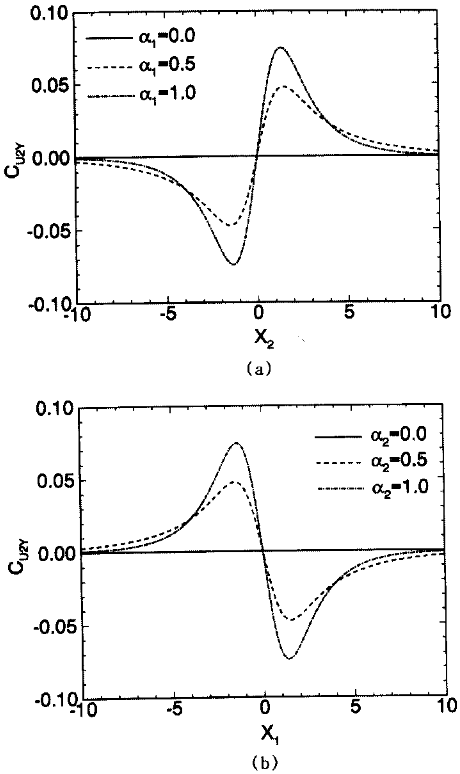
5.1 Velocity-Log Conductivity Cross Covariance

Cross covariances with a Gaussian ρ_Y along x_1 and x_2-axis are obtained through numerical quadratures of Eqns. (12) and (18) and are given in Figs. 1 and 2. It is clear that C_{u_iY} is anisotropic and our model successfully depicts the symmetric profile of C_{u_1Y} of the stationary model

C_{u_1Y} is positive along all of x_1-axis (see Figs. 1(a) and 1(c)), whereas it is positive near the origin becoming negative for large distances along x_2-axis for both α_1 and α_2-cases (see Figs. 1(b) and 1(d)). This general structure can be explained as follows. Larger value in the



(a) α_1-case along x_1-axis, (b) α_1-case along x_2-axis
(c) α_2-case along x_1-axis, (d) α_2-case along x_2-axis.
Fig. 1. Velocity-Log Conductivity Cross Covariances C_{u_1Y}



(a) α_1-case along x_2-axis
(b) α_2-case along x_1-axis.
Fig. 2. Velocity-Log Conductivity Cross Covariances C_{u_2Y}

conductivity leads to an increase in the local velocity diminishing with distance. However, normal to the mean flow, i.e. along x_2-direction, a positive deviation in the near vicinity has to be counterbalanced by a negative velocity deviation to satisfy continuity.

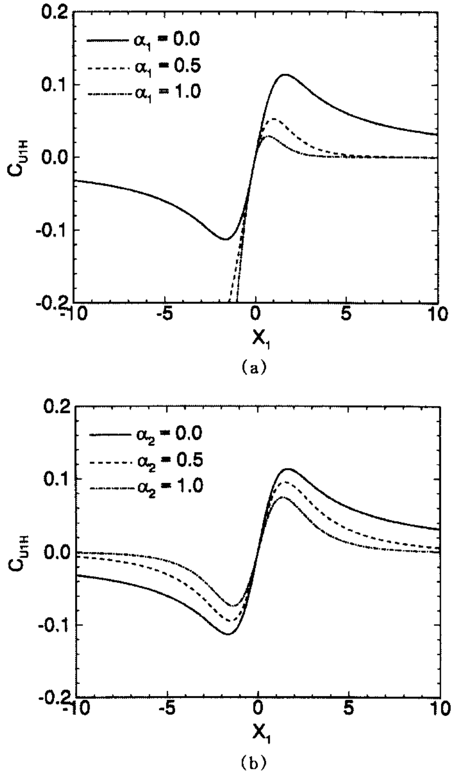
C_{u_1Y} of α_1-case along x_1-axis (Fig. 1(a)) becomes skewed as α_1 increases since same amount of deviation in the lower conductivity region has greater effect on the velocity than in a region of higher conductivity. Same can be said regarding C_{u_1Y} of α_2-case along x_2-axis (Fig. 1(d)). Finally we see that C_{u_1Y} of α_1-case along x_2-direction (Fig. 1(b)) and C_{u_1Y} of α_2-case along x_1-direction (Fig. 1(c)) remain symmetric as can be expected from geometry of the Y field.

As to C_{u_2Y}, which are zero along both axis for a stationary field, they are nonzero along x_2-axis for α_1-case (Fig. 2(a)) and along x_1-axis for α_2-case (Fig. 2(b)). Deviations in the normal flow component are antisymmetric and much smaller than those of the mean flow component.

5.2 Velocity-Head Cross Covariance

Cross covariances with a Gaussian ρ_Y along x_1 and x_2-axis are obtained through numerical quadratures of Eqns. (24) and (27) and are given in Figs. 3 and 4. The general characteristics of C_{u_iH} are that they are anisotropic and antisymmetric except for C_{u_1H} along x_1-axis for α_1-case as seen in Fig. 3(a). This is due to an exponential decay of the head fluctuation with a positive $\alpha_1 x_1$ or exponential amplification with a negative $\alpha_1 x_1$ (see Eqn. (21)). The antisymmetry is equivalent to stating that a velocity deviation resulting from a positive head fluctuation upstream is same as one resulting from a negative head fluctuation downstream. C_{u_1H} along x_2-axis and C_{u_2H} along x_1-axis are identically zero for both α_1 and α_2-cases, which means that u_1 along x_2 and u_2 along x_1-axis are not sensitive to head fluctuations.

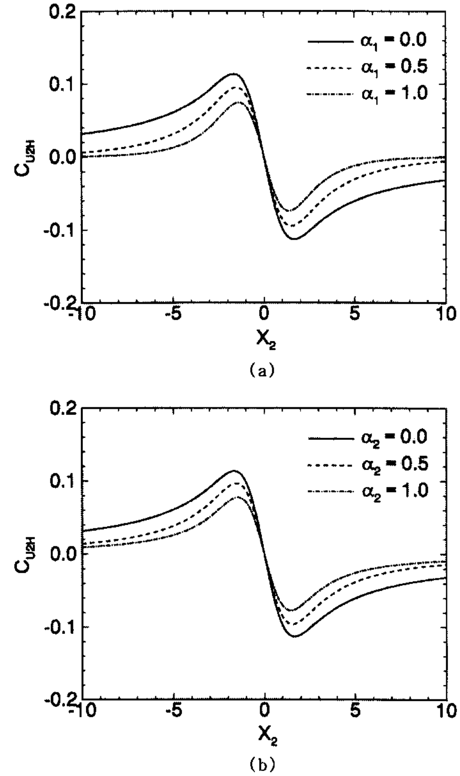
From these results, we can observe not only the quantitative effects of the log conductivity trend on the velocity cross covariances but also different qualitative characteristics compared to the case where the conductivity field is stationary.



(a) α_1-case along x_1-axis

(b) α_2-case along x_1-axis.

Fig. 3. Velocity-Head Cross Covariances C_{u_1H}



(a) α_1-case along x_2-axis

(b) α_2-case along x_2-axis.

Fig. 4. Velocity-Head Cross Covariances C_{u_2H}

6. Summary

In this paper, we derived expressions of the velocity-log conductivity and velocity-head cross covariances in a two-dimensional heterogeneous nonstationary conductivity field. Final expressions are quasi-analytic requiring only quadratures and expressed in terms of the parameters that characterize the conductivity trend and the mean head gradient.

The velocity cross covariances were developed through a linearization of the flow equation followed by a perturbation-like expansion up to first order. Results are presented using a Gaussian correlation function for the log conductivity and for two particular cases of the trend and for separation distances along x_1 and x_2-axis.

Investigation of the cross covariances result in the following findings.

(1) All results reveal an anisotropic structure.

(2) $C_{u_1Y}(\alpha_1;0,x_2)$ and $C_{u_1Y}(\alpha_2;0,x_1)$ are symmetric.

(3) $C_{u_2Y}(\alpha_1;0,x_2)$ and $C_{u_2Y}(\alpha_2;x_1,0)$ are nonzero and antisymmetric.

(4) $C_{u_1H}(\alpha_2;x_1,0)$ are antisymmetric.

(5) $C_{u_2H}(\alpha_1,\alpha_2;0,x_2)$ are nonzero and anti-symmetric.

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Appendix

Here we present a brief summary concerning the use of the Fourier transform method. A more complete treatment can be found in such books as Carrier et al. (1966) for a general treatise and Dagan (1989) for specific applications regarding random functions.

The Fourier integral transform of a space function $f(\boldsymbol{x})$ and its inverse are defined by:

$$\mathrm{FT}[f(\boldsymbol{x})] = \hat{f}(\boldsymbol{k}) = \frac{1}{(2\pi)^{m/2}} \int_{-\infty}^{\infty} f(\boldsymbol{x}) e^{\,i\boldsymbol{k}\cdot\boldsymbol{x}} d\boldsymbol{x} \tag{29a}$$

$$f(\boldsymbol{x}) = \frac{1}{(2\pi)^{m/2}} \int_{-\infty}^{\infty} \hat{f}(\boldsymbol{k})\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} d\boldsymbol{k} \tag{29b}$$

where m is the number of space dimensions, $\boldsymbol{x}$ and $\boldsymbol{k}$ are the coordinate and wave number vectors of m-dimensions. The definition in Eqn. (29) may not be suitable for some of the functions which appear in stochastic processes and this restriction can be surmounted if we use the extended Fourier transform that includes the following relationship regarding the Dirac delta function, δ:

$$\int_{-\infty}^{\infty} e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} d\boldsymbol{x} = \int_{-\infty}^{\infty} e^{\,i\boldsymbol{k}\cdot\boldsymbol{x}} d\boldsymbol{x} = (2\pi)^m \delta(\boldsymbol{k}) \tag{30}$$

Also used in the analysis are following useful relationships of the Fourier transform for the gradients and products as well as the convolution theorem :

$$\mathrm{FT}[\nabla f(\boldsymbol{x})] = -\,\mathrm{i}\,\boldsymbol{k}\hat{f}(\boldsymbol{k}) \tag{31a}$$

$$\mathrm{FT}[f_1(\boldsymbol{x})f_2(\boldsymbol{x})] = \frac{1}{(2\pi)^{m/2}}\int_{-\infty}^{\infty}\hat{f}_1(\boldsymbol{k}_1)\hat{f}_2(\boldsymbol{k}-\boldsymbol{k}_1)\,d\boldsymbol{k}_1 \tag{31b}$$

$$\mathrm{FT}\left[\int f_1(\boldsymbol{x}_1)f_2(\boldsymbol{x}_1+\boldsymbol{x}_2)\,d\boldsymbol{x}_1\right] = (2\pi)^{m/2}\,\hat{f}_1^{\,*}(\boldsymbol{k})\hat{f}_2(\boldsymbol{k}) \tag{31c}$$

where $\hat{f}_1^{\,*}$ is the complex conjugate of $\hat{f}_1$. And finally, the Fourier transform of a 2-point statistical moment of the stationary functions f_1 and f_2 is given by :

$$\mathrm{FT}[\langle f_1(\boldsymbol{x}_1)f_2(\boldsymbol{x}_2)\rangle] = \langle \hat{f}_1(\boldsymbol{k}_1)\hat{f}_2(\boldsymbol{k}_2)\rangle \tag{32}$$

and, since this moment is invariant under translation in space, it is a function of the distance vector $\boldsymbol{r} = \boldsymbol{x}_2 - \boldsymbol{x}_1$ and we have the following relationship.

$$\langle \hat{f}_1(\boldsymbol{k}_1)\hat{f}_2(\boldsymbol{k}_2)\rangle = (2\pi)^{m/2}\delta(\boldsymbol{k}_1+\boldsymbol{k}_2)\hat{C}(\boldsymbol{k}_2) \tag{33}$$

where $\hat{C}$ is the Fourier transform of the covariance function, $C(\boldsymbol{r}) = \langle f_1(\boldsymbol{x}_1)f_2(\boldsymbol{x}_2)\rangle$.

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