韓國軍事運營分析學會誌
第 24 卷, 第 1 號, 1998. 6. 30

Fire Allocation and Combat Networking

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Abstract

A stochastic modeling of combat that takes more realistic situations into account has been studied with deep concern. Either the firing strategies or network formations are very important elements in the analysis of combat. The first objective of this study is to evaluate how the different strategies affect the outcomes of combat. An analytical approach has been used in an attempt to understand a small-sized battle. The results are validated and compared with existing simulation models. Extending to the moderate size of battle may be achieved with ease. Secondly, an attempt has been made to study and investigate a way to solve combat in a different fashion. We divided a two-on-two battle into two separate one-on-one battles and connected them into a network. New elements considered such as delay time of starting a firefight on a particular node or search time for the next target when a kill occurs are defined and used as the input parameters. The discussions are made to validate the hypothesized model and ask if the results are meaningful and useful in the analysis of combat operations or not.

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1. Introduction

A combat is a set of realizations which contain unpredictable factors that follow the laws of probability, resulting in extremely complex random behaviors. The methodologies adapted in modeling combat, no matter how detailed or realistic are ad hoc in nature. Specially, the analytical tools that have been applied for decades by researchers to validate, verify, or compare existing models are scattered.

An attempt has been made by introducing the renewal process especially to describe the stochastic phenomena of a combat situation. The renewal process is a counting process for which the times between successive events are independent and identically distributed random variables. Since Ancker's voluntary work[1], the gradual interest in stochastic combat led analysts to the development of small to moderate sized stochastic analytical models. The homogeneous small sized models are presented by some researchers and the results are shown in the literatures which are written by Gafarian and Ancker[10], Kress[18,19], Gafarian and Manion[11], and Hong[14]. And two-on-two heterogeneous model and nonhomogeneous Poisson approximation are presented by Hong[14]. Yang and Gafarian[24] have presented a fast way of approximating the homogeneous combat situation. Parkhideh and Gafarian[21] provided an analytical way of solving a many-on-many heterogeneous stochastic combat. Ancker and Gafarian[3] provided an excellent survey of the works done by themselves and others on the validity of assumptions of the Lanchester attrition rates.

Some other features of studying the SL(Stochastic Lanchester) model included the battle with limited ammunition supply [2], the battle termination time as a random variable [16], or the kill probability varies over time, ... etc. Most of the efforts in SL model analysis were concerned with the cases of the Lanchester square law combat environment. A brief description of the square law assumption can be found in reference [9].

The three firing strategies described in the first part of this paper differ in what preassigned target selection a combatant follows. A larger sized battle may not be possible or may need a tremendous amount of effort to get the output results desired. The preliminary study on a many-on-many combat simulation model which takes the fire allocation strategy into account is developed[18]. The only real problem in this simulation model is how to use the model since it really does not provide us with a "solution" but rather independent, detailed replications of the combat. The statistical method is also introduced to provide meaningful answers for the output parameters of interest with an appropriate number of simulation replications. The results, based on the experiments in the simulation study, indicated that the difference of the realizations obtained from the three proposed strategies taken by one side, assuming the other side holds a specific one may be significant. The relative differences of interest between some combat measures were shown. The current status of this study is an intermediate stage for the purpose of finding a new methodology of modeling a real world combat situation so called "a series of mini battles" or "a network of several small combats." All subjects encountered in this study are an extension of the SL model, not the EL(Exponential Lanchester) or DL(deterministic Lanchester) models. The first steps of these preliminary works are attempted with caution. Two separate one-on-one duels are connected to each other to form a network. The analytic solutions that are given in this particular battle are compared to the existing results.

2. Fire Allocation

Basically, we try to preserve the assumptions that apply to the Lanchester square law and some of them are modified to describe the models developed in this study. The three firing options considered are random selection (RS), concentrated power (CP), and evenly distributed power (EP). The brief descriptions of these options are as follows:

Random Selection : When two sides are engaged in combat under the original Lanchester square law assumptions, we say the model uses the random selection strategy for both sides.

Concentrated Power : Every member in the side A aims at a same target, which means that they pick an opponent B at random (all are visible and in range) so they aim at the same target and start firing with a fixed kill probability. Each marksman fires until he is killed or the target is killed by one of his colleagues including himself, at which time the killing A and remaining survivors immediately shift to a new target picked at random if any in his opponent side and resumes firing.

Evenly Distributed Power : Every member in the side A may be partitioned into several small groups each with preassigned number of combatants and one of A picks an opponent in the corresponding small group in the side B. All are visible and in range within a small combat. Each marksman fires with a fixed kill probability on every round fired. Each marksman fires until he is killed or makes a kill, at which time he immediately shifts to a new target picked at random if any are in his opponent's small group, and resumes firing. On the other hand, if there is no survivor in his opponent's small group, the killing small group takes side with their colleagues in the remaining small groups to assist them. To do so they distribute into other small groups so that the small groups that are reformed have almost same number of combatants.

One of the purposes in the previous study was to evaluate how the different strategies affect the outcomes of a combat. The large-sized stochastic simulation models are developed and have been discussed in the early version of this study[18]. The implementation of an analytical approach in the analysis of the fire allocation problem for a small-sized battle is the first part in this study. The models allow for an arbitrary interfiring time random variable for each of the sides, thus providing for more realism than the classical stochastic Lanchester square law models, such as Exponential Lanchester.

The readers who may want to understand more or are interested in knowing about the aiming configuration of the three strategies can refer to the Figure 1 in the literature [18].

The two modes of resuming firing on a survivor are defined as :

Reselect On : Consider a given marksman fires at a target. Whether his target is killed by him or by another marksman on his side, he resumes afresh the interfiring process on his next available target.

Reselect Off : If the marksman's target is killed by him, he start afresh the interfiring process on his next target, whereas, if his target is killed by another member of his side, his remaining time to fire is carried over to his next target.

2.1 Two-on-Two Analytic Model

There are two combatants on each side, A and B. The combatants on side A fire continuously and each combatant follows his own preassigned firing process which is one of the three fire allocation strategies, and have identical random interfiring times X with a probability density function $f_X(x)$ and identical kill probabilities P_A on each launch. Similar assumptions apply for side B, therefore define the random interfiring time as Y with pdf $g_Y(y)$ and single shot hit probability as P_B. And both T_A and T_B are the interkilling time random variables for side A and B, respectively.

Our solution technique has three principal features. We first considered each combatant separately as if a combatant is firing at a passive target. And secondly, we put together every firing event in time sequence. Finally, we used the forward recurrence time technique based on each firing sequence to write the state probability equations. The use of either backward recurrence time or forward recurrence time should not provide any different results, which convinces us to introduce the forward recurrence time in this section.

2.2 Finding Number of Aiming Configurations

Suppose there are m combatants on side A and n combatants on side B, and they are ready for the engagement. Let k be the number of combatants on side A aiming at the same target on side B. There are 36 possible combinations of firing strategies in a two-on-two battle. Table 1 briefly shows this and also where we consider the fire resume options 'on' or 'off'.

To write the proper state probability equations it is necessary for us to figure out all possible aiming configuration combinations that can happen between two opposing groups. For m-on-n battle with the random selection(RS) option for side A it can be obtained as follows:

$a = \binom{m}{k}$: number of ways to get k combatants out of total m members.

$b = \binom{k}{1}$: number of ways to get a killer out of k combatants.

$c = \binom{n}{1}$: number of ways to get a target out of n opponents.

$d = \left(\frac{1}{n}\right)^{k}$: probability that k combatants aim at a particular target on the enemy side.

$e = \left(1-\frac{1}{n}\right)^{m-k}$: probability that $(m-k)$ combatants aim at some other targets on the enemy side.

A value of abc becomes the number of aiming configurations for the given situation and the amount of de is the probability of aiming at their corresponding targets. In two other cases either concentrated power(CP) or evenly distributed power (EP) strategy, similarity holds except for calculating the probability of selecting a target which is just $\frac{1}{n}$. For CP case $a,b,$ and c are replaced with f,g, and h, respectively. In the EP, the probability of selecting a target which is just $\frac{1}{n}$ should be considered instead of de.

$f = \binom{m}{m}$: number of ways to get m combatants out of total m members, which is one.

$g = \binom{m}{1}$: number of ways to get a combatant as a killer out of m combatants.

$h = \binom{n}{1}$: number of ways to get a combatant as a target from n opponents.

		Side A					
	Strategies	RS		CP		EP	
Side B	RS	on / on	off / on	on / on	off / on	on / on	off / on
		on / off	off / off	on / off	off / off	on / off	off / off
	CP	**on / on**	off / on	on / on	off / on	on / on	off / on
		on / off	off / off	on / off	off / off	on / off	off / off
	EP	on / on	off / on	on / on	off / on	on / on	off / on
		on / off	off / off	on / off	off / off	on / off	off / off

Table 1. Two-on-two combat with various fire strategies and reselect options.

In two-on-two battle as an example, we shall just take one of the 36 cases and show how the solution can be obtained analytically and see if any other possible ways can be found to extend the battle size. The case with RS for side A versus CP for side B both with 'reselect on' option is chosen as a pilot study. The state space diagram is given in Figure 4 below where the states (2,0), (0,2), (1,0), and (0,1) are called absorbing states and the remaining states are called transient states.

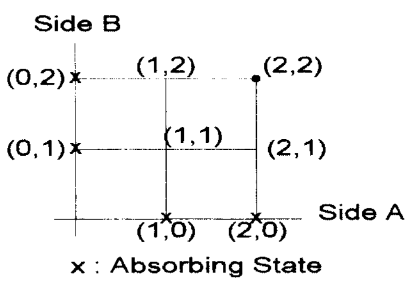
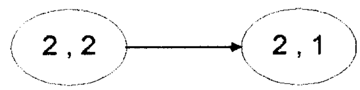


Figure 1. State space for two-on-two combat

2.3 Calculating State Probabilities

To calculate eight different state probabilities the forward recurrence time is measured at different time points during a combat. The input parameters such as interfiring random variables, single shot kill probabilities, and their corresponding interkilling *pdf*s and *cdf*s are put together. Of course the firing strategies and firing configurations for both sides are essential elements to enumerate all the possible aiming and killing processes.



The notations we will use throughout the remainder of this study will be explained when it is necessary to create and define as we make progress.

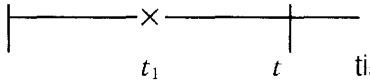
a) $P_{22}(t)$

No killing event has occurred by time t, so the state probability is simply the product of two squared interkilling complementary distribution functions. Then

$$P_{22}(t) = F(t)^2 G(t)^2 ,$$

where both $F(t)$ and $G(t)$ are the complementary distribution functions of the interkilling time random variable for side A and B, respectively.

b) $P_{21}(t)$



There are two distinct ways to reach the state (2,1); one is that two As aim at the same B and one of the As kills the target, the other case is that each of two As aims at a different B and one of the Bs is killed. The notation $\{2A^1\}$ means that the two As have changed their target since the first kill was made by one of them, and this may be a meaningful way to manipulate the system behavior in case the 'reselect on' option is employed. Some other notations used throughout this study are

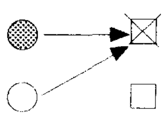
$\{2A^0\}$: two As are alive and have not switched to other target yet,

$A^0[A^0]B^0$: two As aim at the same B and one of them makes a kill,

$\{B^0\}$: one B is still alive and has not killed yet,

$\{A^1A^0\}$: two As are alive and one of them has killed a B,

A^0B^0 : nonkilling A aims at the surviving B.



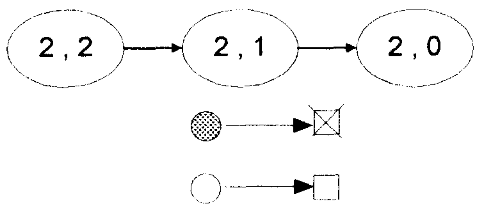
Side A	$\{2A^0\}$	$\{2A^1\}$
Aims and Killing	$A^0[A^0]B^0$	
Side B	$\{2B^0\}$	$\{B^0\}$

P(two As aim at the same B and one of As kills the target in time between t_1 and t_1+dt, and no further killing by time t)

$$= \binom{2}{2}\binom{2}{1}\binom{2}{1}\left(\frac{1}{2}\right)^2 f(t_1)dt_1 F(t_1)G^2(t_1)F(t-t_1)\frac{F(t)}{F(t_1)}\frac{G(t)}{G(t_1)}$$

$$= f(t_1)dt_1 F(t)G(t_1)F^2(t-t_1)G(t),$$

where $f(\cdot)$ is the pdf of the interkilling time random variable for side A .



Side A	$\{2A^0\}$	$\{A^1A^0\}$
Aims and Killing	A^0B^0	
Side B	$\{2B^0\}$	$\{B^0\}$

P(two As aim at the different B and one of As kills the target in time between t_1 and t_1+dt, and no further killing by time t)

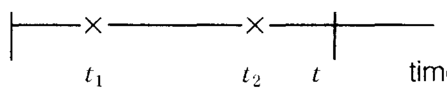
$$= \binom{2}{1}\binom{1}{1}\binom{2}{1}\left(\frac{1}{2}\right)^2 f(t_1)dt_1 F(t_1)G^2(t_1)F(t-t_1)\frac{F(t)}{F(t_1)}\frac{G(t)}{G(t_1)}$$

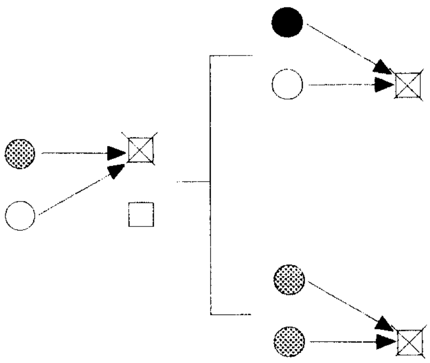
$$= f(t_1)dt_1 G(t_1)F(t-t_1)G(t).$$

Hence, $$P_{21}(t) = \int_0^t f(t_1)F(t)G(t_1)F^2(t-t_1)G(t)dt_1 + \int_0^t f(t)G(t_1)F(t-t_1)F(t)G(t)dt_1.$$

c) $P_{20}(t)$

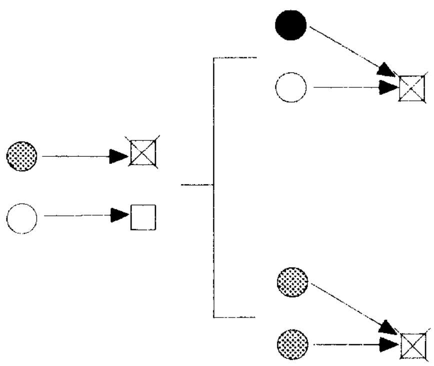
There are four distinct ways to reach this state; no matter whether two As aim at the same B or different B, the second kill may be occurred by either killer or nonkiller.





Side A	$\{2A^0\}$	$\{2A^1\}$
Aims and Killing	$A^0[A^0]B^0$	$A^1[A^1]B^0$
Side B	$\{2B^0\}$	$\{B^0\}$

Side A	$\{2A^0\}$	$\{2A^1\}$
Aims and Killing	$A^0[A^0]B^0$	$A^1[A^1]B^0$
Side B	$\{2B^0\}$	$\{B^0\}$



Side A	$\{2A^0\}$	$\{A^1A^0\}$
Aims and Killing	A^0B^0	$A^1[A^0]B^0$
Side B	$\{2B^0\}$	$\{B^0\}$

Side A	$\{2A^0\}$	$\{A^1A^0\}$
Aims and Killing	A^0B^0	$A^0[A^1]B^0$
Side B	$\{2B^0\}$	$\{B^0\}$

P(two As aim at the same B and one of As kills the target in time (t_1, t_1+dt), and the second kill is made by one of As in (t_2, t_2+dt))

$$= \binom{2}{2}\binom{2}{1}\binom{2}{1}\left(\frac{1}{2}\right)^2 f(t_1)dt_1F(t_1)G^2(t_1)f(t_2-t_1)dt_2F(t_2-t_1) \times \frac{G(t_2)}{G(t_1)} \times 2$$

Then, $P_{20}^{(1)} = \int_0^t \int_0^{t_2} 2f(t_1)F(t_1)G(t_1)f(t_2-t_1)F(t_2-t_1)G(t_2)dt_1dt_2$.

Similarly, when two As aim at the different B the state probability equations are written as

$$P_{20}^{(2,1)} = \binom{2}{1}\binom{1}{1}\binom{2}{1}\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)f(t_1)dt_1F(t_1)G^2(t_1)f(t_2-t_1)dt_2 \times \frac{F(t_2)}{F(t_1)}\frac{G(t_2)}{G(t_1)} ,$$

and $P_{20}^{(2,2)} = \binom{2}{1}\binom{1}{1}\binom{2}{1}\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)f(t_1)dt_1F(t_1)G^2(t_1)\frac{f(t_2)}{F(t_1)}dt_2 \times \frac{F(t_2)}{F(t_1)}\frac{G(t_2)}{G(t_1)}$.

Now then

$$P_{20}^{(2)}(t) = \int_0^t \int_0^{t_2} f(t_1)G(t_1)f(t_2-t_1)F(t_2)G(t_2)dt_1dt_2 + \int_0^t \int_0^{t_2} f(t_1)G(t_1)f(t_2)F(t_2-t_1)G(t_2)dt_1dt_2 .$$

Therefore, $P_{20}(t) = P_{20}^{(1)} + P_{20}^{(2)}$.

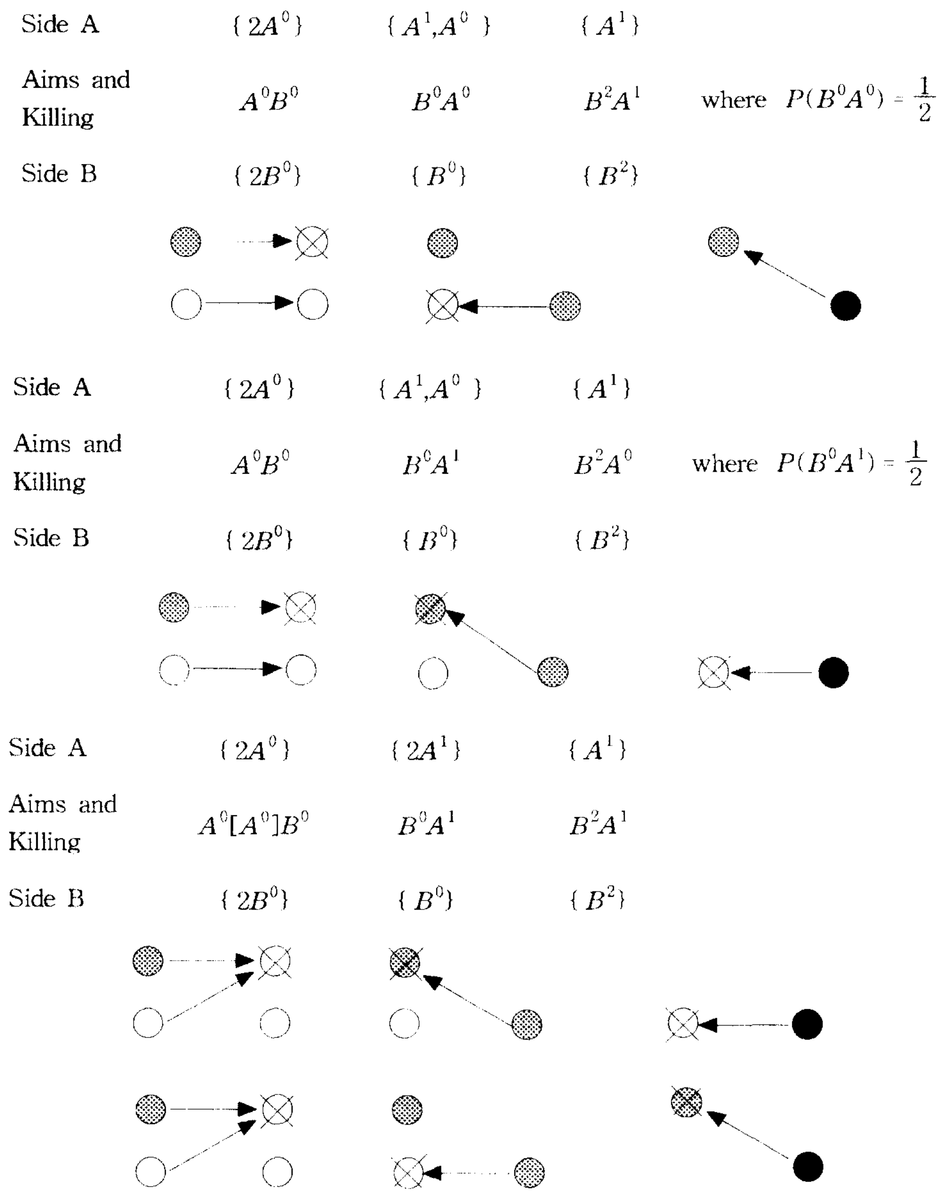


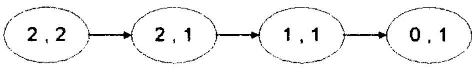
Figure 2. Sequence for aiming and killing events in computing the state $(0,1)^{(1)}$.

d) $P_{02}(t)$

This state can be reached by two killing events occurred by side B. Since B side employed CP strategy, the state probability can be obtained simply as

$$P_{02}(t) = 4\int_0^t \int_0^{t_2} g(t_1)G(t_1)g(t_2-t_1)G(t_2-t_1)F(t_1)F(t_2)dt_1dt_2 \quad .$$

e) $P_{01}(t)$



Let us define two state probabilities such as $P_{01}^{(1)}(t)$ and $P_{01}^{(2)}(t)$, where $P_{01}^{(1)}(t)$ is the probability that one of As kills a B, then the surviving B kills two As consecutively in $(0,t)$, and similarly $P_{01}^{(2)}(t)$ is the probability that one of Bs kills an A first, next the surviving A kills a B second, then the surviving B kills the remaining A in $(0,t)$. Figure 2 shows the aiming and killing events sequence that can happen during a combat to reach state (0,1) if $P_{01}^{(1)}(t)$ is to be computed.

Using similar way to calculate the state probability as others above, we can write it as

$$P_{01}^{(1)}(t)=\frac{1}{2}\int_0^t\int_0^{t_3}\int_0^{t_2} f(t_1)F(t_2)F(t_3\ \ t_1)G(t_1)g(t_2)g(t_3\cdot t_2)dt_1dt_2dt_3$$
$$+\frac{1}{2}\int_0^t\int_0^{t_3}\int_0^{t_2} f(t_1)F(t_2-t_1)F(t_3)G(t_1)g(t_2)g(t_3\ \ t_2)dt_1dt_2dt_3$$
$$+\int_0^t\int_0^{t_3}\int_0^{t_2} f(t_1)F(t_1)F(t_2-t_1)F(t_3-t_1)G(t_1)g(t_2)g(t_3-t_2)dt_1dt_2dt_3,$$

and

$$P_{01}^{(2)}(t)=2\int_0^t\int_0^{t_3}\int_0^{t_2} g(t_1)G(t_1)G(t_2-t_1)g(t_3-t_1)F(t_1)f(t_2)F(t_3-t_2)dt_1dt_2dt_3.$$

Then $P_{01}(t) = P_{01}^{(1)} + P_{01}^{(2)}$.

f) $P_{11}(t)$

State (1,1) can be reached through five different aiming and killing sequence and Table 2 presents these in detail.

$$P_{11}^{(1)}(t_1,t_2,t) = g(t_1)G(t_1)F(t_1)f(t_2)G(t_2-t_1)F(t-t_2)G(t-t_1)dt_1dt_2,$$

$$P_{11}^{(2)}(t_1,t_2,t) = g(t_1)G(t_1)F(t_1)f(t_2)G(t_2-t_1)F(t-t_2)G(t-t_1)dt_1dt_2,$$

$$P_{11}^{(3)}(t_1,t_2,t) = f(t_1)F(t_1)G(t_1)g(t_2)F(t_2-t_1)F(t-t_2)G(t-t_2)dt_1dt_2,$$

$$P_{11}^{(4)}(t_1,t_2,t) = \frac{1}{2}f(t_1)G(t_1)g(t_2)F(t_2-t_1)G(t-t_2)F(t)dt_1dt_2,$$

$$P_{11}^{(5)}(t_1,t_2,t) = \frac{1}{2}f(t_1)G(t_1)g(t_2)F(t_2)G(t-t_2)F(t-t_1)dt_1dt_2 \quad ,$$

where $g(\cdot)$ is the pdf of the interkilling time random variable for side B .

States	Aiming and Killing Events	Time index
$(1,1)^{(1)}$	one of Bs kills an A, then surviving A kills the killing B	first kill occurred at t_1, second kill at t_2, and then no further kills by time t.
$(1,1)^{(2)}$	one of Bs kills an A, then surviving A kills the nonkilling B	
$(1,1)^{(3)}$	two As aim at the same B and one of them kills it, and surviving B kills one of As next	
$(1,1)^{(4)}$	two As aim at the different B and one of them kills it, and surviving B kills killing A next	
$(1,1)^{(5)}$	two As aim at the different B and one of them kills it, and surviving B kills nonkilling A next	

Table 2. Sequence for aiming and killing events to reach the state (1,1) . Each state probability can be computed as just we have done for other states, these are

Then the state probability $P_{11}(t)$ is the sum of five possible probability values, and it can be written as

$$P_{11}(t)=\int_0^t\int_0^{t_1} g(t_1)G(t_1)F(t_1)f(t_2)G(t_2-t_1)F(t-t_2)G(t-t_1)dt_1dt_2$$

$$+\int_0^t\int_0^{t_1} f(t_1)F(t_1)G(t_1)g(t_2)F(t_2-t_1)F(t-t_2)G(t-t_2)dt_1dt_2$$

$$+\int_0^t\int_0^{t_1} f(t_1)F(t_1)G(t_1)g(t_2)F(t_2-t_1)F(t-t_2)G(t-t_2)dt_1dt_2$$

$$+\int_0^t\int_0^{t_1} \frac{1}{2}f(t_1)G(t_1)g(t_2)F(t_2-t_1)G(t-t_2)F(t)dt_1dt_2$$

$$+\int_0^t\int_0^{t_1} \frac{1}{2}f(t_1)G(t_1)g(t_2)F(t_2)G(t-t_2)F(t-t_1)dt_1dt_2 \quad .$$

g) $P_{12}(t)$

Two Bs aim at an A simultaneously and either one of them kills it in time $(0,t_1)$ and no further killing event occurred by time t. The state probability is now obtained in simple form as

$$P_{12}(t)=2\int_0^t g(t_1)G(t_1)F(t_1)G(t-t_1)^2F(t)dt_1 \quad .$$

h) $P_{10}(t)$

State (1,0) can be reached through four different aiming and killing sequence and Table 3 presents these in detail. Each state probability can be computed as just we have done for other states, these are

$$P_{10}^{(1)}(t_1,t_2,t_3,t)=\frac{1}{2}f(t_1)F(t_2)f(t_3-t_1)G(t_1)g(t_2)G(t_3-t_2)dt_1dt_2dt_3,$$

$$P_{10}^{(2)}(t_1,t_2,t_3,t)=\frac{1}{2}f(t_1)F(t_2-t_1)f(t_3)G(t_1)g(t_2)G(t_3-t_2)dt_1dt_2dt_3,$$

$$P_{10}^{(3)}(t_1,t_2,t_3,t)=f(t_1)F(t_1)F(t_2-t_1)f(t_3-t_1)G(t_1)\times g(t_2)G(t_3-t_2)dt_1dt_2dt_3,$$

$$P_{10}^{(4)}(t_1,t_2,t_3,t)=2g(t_1)G(t_1)G(t_2-t_1)G(t_3-t_1)F(t_1)\times f(t_2)f(t_3-t_2)dt_1dt_2dt_3.$$

Then the state probability $P_{10}(t)$ is the sum of five possible probability values, and it can be written as

$$P_{10}(t)=\int_0^t\int_0^{t_1}\int_0^{t_2}\frac{1}{2}f(t_1)F(t_2)f(t_3-t_1)G(t_1)g(t_2)G(t_3-t_2)dt_1dt_2dt_3$$

$$+\int_0^t\int_0^{t_1}\int_0^{t_2}\frac{1}{2}f(t_1)F(t_2-t_1)f(t_3)G(t_1)g(t_2)G(t_3-t_2)dt_1dt_2dt_3$$

$$+\int_0^t\int_0^{t_1}\int_0^{t_2}f(t_1)F(t_1)F(t_2-t_1)f(t_3-t_1)G(t_1)g(t_2)G(t_3-t_2)dt_1dt_2dt_3$$

$$+\int_0^t\int_0^{t_1}\int_0^{t_2} 2g(t_1)G(t_1)G(t_2-t_1)G(t_3-t_1)\times F(t_1)f(t_2)f(t_3-t_2)dt_1dt_2dt_3 \quad .$$

<table>
<tr><th>States</th><th>Aiming and Killing Events</th><th>Time index</th></tr>
<tr><td>$(1,0)^{(1)}$</td><td>one of As kills a B, and surviving B kills the nonkilling A, then remaining A kills killing B</td><td rowspan="4">first kill occurred at t_1, second kill at t_2, and then third kill at time t_3.</td></tr>
<tr><td>$(1,0)^{(2)}$</td><td>one of As kills a B, and surviving B kills the killing A, then remaining A kills killing B</td></tr>
<tr><td>$(1,0)^{(3)}$</td><td>two As aim at the same B and one of them kills it, and surviving B kills one of As next, and then the remaining A kills killing B</td></tr>
<tr><td>$(1,0)^{(4)}$</td><td>two Bs aim at the same A and surviving A kills two Bs consecutively</td></tr>
</table>

Table 3. Sequence for aiming and killing events to reach the state (1,0).

2.4 Verification and Results

The verification for the models in this study was done in two phases. First, each model was checked manually to ensure its proper functioning. Both transient and absorbing state probabilities are observed at different times. The transient state probabilities have to converge to zero at time infinity and the absorbing state probabilities should have nonzero probabilities at the end of the battle. Furthermore, the sum of all the state probabilities must be equal to one at a given time.

The second part of model verification involves comparing the output of the hypothesized models with the simulation results provided by the early version of this study. Simulations were run for ten thousand replications each to ensure that the estimates of all eight overall combat figures of merit were reliable. The data is shown in Tables [4], [5], and [6], also in Figures 3 and 4. We could ascertain the truth of the results.

Models \ Measures	Analytic	Simulation (10,000 Replications)
E[T]	2.1495	2.1775
σ[T]	1.4249	1.4709
E[A]	0.9311	0.9187
σ[A]	0.8951	0.8984
E[B]	0.7234	0.7316
σ[B]	0.8728	0.8744
P(A)	0.5600	0.5525
P(B)	0.4400	0.4475

Table 4. Analytic Solution VS Simulation , two-on-two, RS for side A and CP for side B, Erlang-2 interfiring time distribution for both sides, $\mu_A = .5$ and $\mu_B = .3$, $p_A = .2$ and $p_B = .1$.

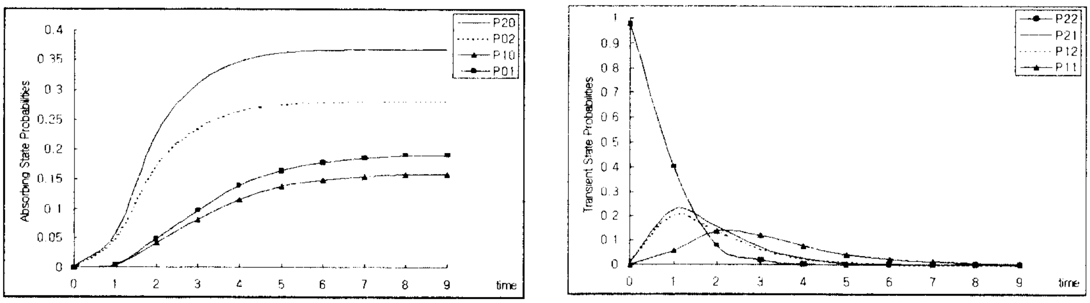


Figure 3. Both transient and absorbing state probabilities, RS for side A and CP for side A CP for side B, Erlang-2

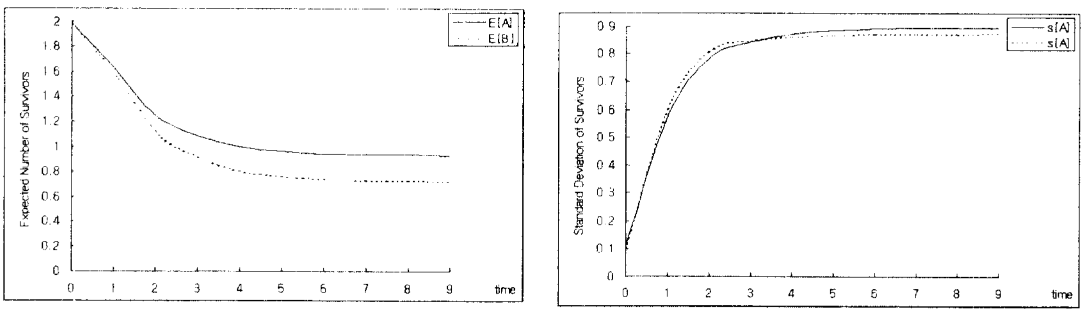


Figure 4. Both means and standard deviations of survivors, RS for side A and CP for side B, Erlang-2 interfiring time distribution for both sides, $\mu_A = .5$ and $\mu_B = .3$, $p_A = .2$ and $p_B = .1$.

Measures / Models	Analytic	Simulation (10,000 Replication)
E[T]	1.0683	1.0733
σ[T]	0.7248	0.7255
E[A]	0.3859	0.3933
σ[A]	0.7282	0.7339
E[B]	1.3050	1.2984
σ[B]	0.8302	0.8343
P(A)	0.2390	0.2433
P(B)	0.7610	0.7567

Table 5. Analytic Solution VS Simulation, two-on-two, RS for side A and CP for side B, exponential for side A and Erlang-2 interfiring time distribution for side B, $\mu_A = .7$ and $\mu_B = .3$, $p_A = .3$ and $p_B = .3$.

3. Combat Networking

It has been asserted by many combat analysts that combat is a complicated hierarchy network of small firefights. The firefight is simply regarded as the basic building block. A field experiment which has clearly shown the network trait and stochasticity of a combat process in a proper scientific manner is the work reported by Rowland[22]. Ancker[4] has emphasized the importance of modeling a combat as a series or a network of several small firefights. He explains why the networking is a closely represented way to formulate the real situation. The formation of networks at the lower levels of aggregation depends primarily on the environmental and weapon systems factors, the tactical decisions of commanders, and the firefight outcomes; Any changes can terminate firefights or start new ones.

There is early fundamental research in progress on network formation by Bathe[6]. However, we do not have any solid guidance on how firefights aggregate into larger engagements. The fundamental elements for network formation may include environmental factors, routing channels, mobility, weather, weapon characteristics, vehicle capabilities, human capabilities, and etc. And furthermore, the question is what factors or combinations of factors are important?

3.1 Series of two one-on-one battles

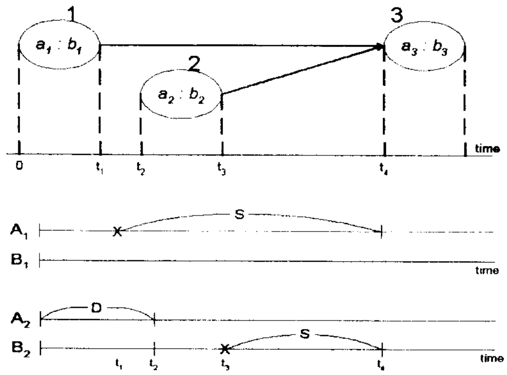
This study is a first step in identifying how the networking can be achieved as a series of distinct nodes. A combat is begun with two separate nodes at the different time as shown in Figure 5. In other words, two-on-two combat is divided into two separate small battles each with one-on-one at the beginning. Either one of the two starts the conflict at time zero and the other node begins after a random amount of time delay (D), and let this time point be t_2. Suppose the battle of the first node is over at time t_1, and then the killer starts to search for the next target in the second node and found it at $t_4 = t_1 + S$. S is defined to be a search time random variable. Conversely, the killer in the second node may find

					Strategy		Distribution	m	p
		Side A		RS		ER-2		0.5	0.2
		Side B		CP		ER-2		0.3	0.1

Time	P22	P21	P12	P11	P20	P02	P10	P01	Total	E[A]	E[B]	s[A]	s[B]
0.06	0.978	0.010	0.012	0.000	0.000	0.000	0.000	0.000	1.000	1.988	0.110	1.990	0.098
0.31	0.732	0.122	0.123	0.010	0.007	0.007	0.000	0.000	1.000	1.853	0.372	1.855	0.371
0.71	0.398	0.228	0.206	0.057	0.056	0.047	0.004	0.004	1.000	1.631	0.579	1.591	0.601
1.23	0.182	0.218	0.192	0.112	0.144	0.113	0.018	0.020	1.000	1.413	0.711	1.322	0.740
1.78	0.078	0.157	0.140	0.136	0.227	0.173	0.041	0.048	1.000	1.248	0.784	1.117	0.807
2.29	0.036	0.104	0.094	0.133	0.281	0.213	0.064	0.076	1.000	1.144	0.823	0.987	0.836
2.70	0.019	0.072	0.066	0.120	0.310	0.235	0.082	0.097	1.000	1.085	0.842	0.914	0.848
2.94	0.013	0.057	0.053	0.111	0.323	0.244	0.091	0.108	1.001	1.058	0.851	0.880	0.852
3.06	0.011	0.051	0.047	0.106	0.328	0.248	0.096	0.114	1.001	1.046	0.855	0.866	0.854
3.31	0.008	0.040	0.037	0.095	0.337	0.255	0.104	0.124	1.001	1.026	0.862	0.840	0.857
3.71	0.004	0.026	0.025	0.079	0.348	0.264	0.116	0.139	1.001	0.999	0.870	0.807	0.861
4.23	0.002	0.015	0.015	0.060	0.357	0.271	0.128	0.153	1.001	0.976	0.878	0.779	0.865
4.78	0.001	0.008	0.009	0.043	0.362	0.275	0.138	0.165	1.002	0.960	0.884	0.759	0.867
5.29	0.000	0.005	0.005	0.031	0.365	0.277	0.145	0.173	1.002	0.950	0.887	0.746	0.869
5.70	0.000	0.003	0.003	0.024	0.366	0.279	0.149	0.178	1.002	0.944	0.890	0.740	0.870
5.94	0.000	0.002	0.002	0.020	0.367	0.279	0.150	0.181	1.002	0.942	0.891	0.737	0.870
6.06	0.000	0.002	0.002	0.019	0.367	0.279	0.151	0.182	1.002	0.941	0.891	0.735	0.870
6.31	0.000	0.002	0.002	0.016	0.367	0.280	0.153	0.183	1.002	0.939	0.892	0.733	0.871
6.71	0.000	0.001	0.001	0.012	0.368	0.280	0.155	0.186	1.002	0.936	0.893	0.730	0.871
7.23	0.000	0.001	0.001	0.008	0.368	0.280	0.157	0.188	1.002	0.934	0.894	0.727	0.872
7.78	0.000	0.000	0.000	0.006	0.368	0.280	0.158	0.190	1.003	0.933	0.894	0.725	0.872
8.29	0.000	0.000	0.000	0.004	0.368	0.281	0.159	0.191	1.003	0.932	0.895	0.724	0.872
8.70	0.000	0.000	0.000	0.003	0.368	0.281	0.159	0.191	1.003	0.931	0.895	0.724	0.873
8.94	0.000	0.000	0.000	0.002	0.368	0.281	0.159	0.192	1.003	0.931	0.895	0.723	0.873

Table 6. Numerical results, state probabilities, means and standard deviations for survivors on both sides, RS for side A and CP for side B, Erlang-2 interfiring time distribution for both sides, $\mu_A = .5$ and $\mu_B = .3$, $p_A = .2$ and $p_B = .1$.

the next target at time $t_4 = t_3 + S$. Figure 6 shows six possible ways to reach an absorbing state when the battle continues until one side is annihilated.

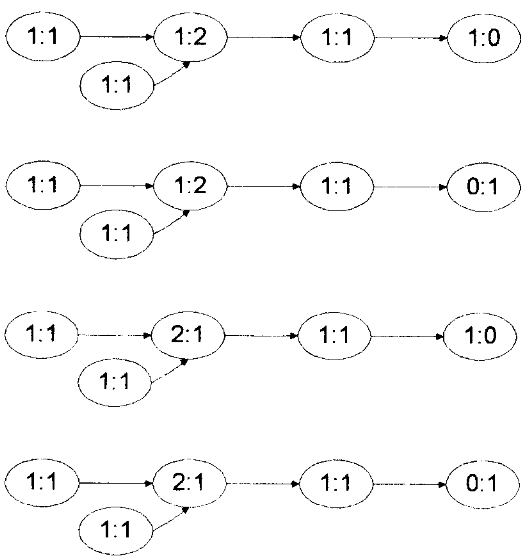


$t_4 - t_1$ ~ Search time r.v. after Node 1 (S)

$t_4 - t_3$ ~ Search time r.v. after Node 2 (S)

t_2 ~ Delay time r.v. (D)

Figure 5. Design configuration of two 1-on-1 battles.



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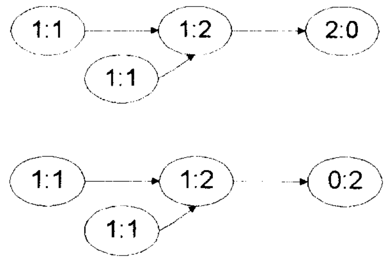


Figure 6. Possible networks of two 1-on-1 battles.

3.2 Random variables and their pdfs and cdfs

We will maintain the notations that are used up to this point and a few more new notations are to be defined for the networking problem. These new notations are

p : probability that a combatant aims keeps engaging with his own his original target given that a new target from other node is appeared, and $q = 1 - p$,

$d(\cdot)$: pdf of the random variable D,

$s(\cdot)$: pdf of the random variable S,

$V = D + T_A$: side A's time to a kill including a delay, and its pdf is $f_D(v) = f * d\,(v)$,

$W = D + T_B$: side B's time to a kill including a delay, and its pdf is $g_D(w) = g * d\,(w)$,

$\Phi = S + T_A$: side A's time to a kill including a search, and its pdf is $f_S(\Phi) = f * s\,(\Phi)$,

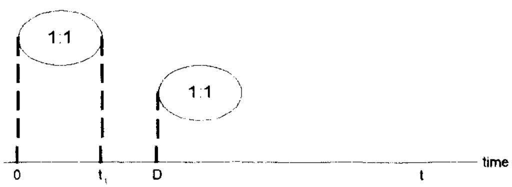
$\Psi = S + T_B$: side B's time to a kill including a search, and its pdf is $g_S(\Psi) = g * s\,(\Psi)$, where $*$ denote the convolution of two random variables, and the corresponding complementary cdfs are $\bar{F}_D(\upsilon), \bar{G}_D(w), \bar{F}(\Phi)$, and $\bar{G}_S(\Psi)$, respectively.

3.3 Calculating State Probabilities

The forward recurrence time technique is applied here again to calculate the state probabilities.

a) $P_{22}(t)$

No killing event has occurred at any node by time t, so the state probability is simply the product of four new interkilling complementary cdfs.

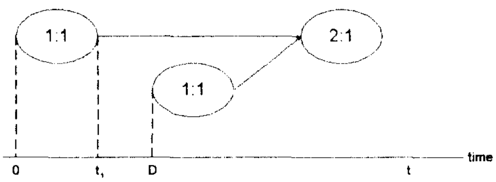


Then $P_{22}(t) = F(t)G(t)F_D(t)G_D(t)$.

b) $P_{21}(t)$ and $P_{12}(t)$

There are two distinct ways to reach the state (2,1); a kill is made by side A either at first or second node. We define an event $\{A_iB_j\}$ and this represents that an A in node i kills a B in node j in a given time period.

Now considering all conditions for each of four combatants at any time point in $(0,t)$, the



state probability can be written as

$$\begin{aligned} P_{21}(t) &= P(A_1B_1) + P(A_2B_2) \\ &= \int_0^t f(t_1)G(t)F_S(t-t_1)F_D(t)G_D(t)dt_1 \\ &+ \int_0^t f_D(t_1)G_D(t_1)F_S(t-t_1)F(t)G(t)dt_1. \end{aligned}$$

The state probability $P_{12}(t)$ can be obtained by an interchange of $f(\cdot)$ and $F_*(\cdot)$ with $g(\cdot)$ and $G_*(\cdot)$ in $P_{21}(t)$, respectively. Hence we can write it as

$$\begin{aligned} P_{12}(t) &= P(B_1A_1) + P(B_2A_2) \\ &= \int_0^t g(t_1)F(t)G_S(t-t_1)G_D(t)F_D(t)dt_1 \\ &+ \int_0^t g_D(t_1)F_D(t_1)G_S(t-t_1)G(t)F(t)dt_1. \end{aligned}$$

c) $P_{20}(t)$ and $P_{02}(t)$

Two consecutive kills are made by the same side to reach these states. There are four possible ways of happening this.

$$\begin{aligned} P_{20}(t) &= P[(A_1B_1)\cap(A_2B_2)] + P[(A_1B_1)\cap(A_1B_2)] \\ &+ P[(A_2B_2)\cap(A_2B_1)] + P[(A_2B_2)\cap(A_1B_1)] \\ &= \int_0^t\int_0^{t_2} f(t_1)G(t_1)f_D(t_2)G_D(t_2)F_S(t_2-t_1)dt_1dt_2 \\ &+ \int_0^t\int_0^{t_2} f(t_1)G(t_1)f_S(t_2-t_1)G_D(t_2)F_D(t_2)dt_1dt_2 \\ &+ \int_0^t\int_0^{t_2} f_D(t_1)G_D(t_1)f_S(t_2-t_1)G(t_2)F(t_2)dt_1dt_2 \\ &+ \int_0^t\int_0^{t_2} f_D(t_1)G_D(t_1)f(t_2)G(t_2)F_S(t_2-t_1)dt_1dt_2. \end{aligned}$$

Similarly, $P_{02}(t)$ can be calculated by an interchange of $f(\cdot)$ and $F_*(\cdot)$ with $g(\cdot)$ and $G_*(\cdot)$ in $P_{20}(t)$, respectively. So we can write as

$$\begin{aligned} P_{02}(t) &= P[(B_1A_1)\cap(B_2A_2)] + P[(B_1A_1)\cap(B_1A_2)] \\ &+ P[(B_2A_2)\cap(B_2A_1)] + P[(B_2A_2)\cap(B_1A_1)] \\ &= \int_0^t\int_0^{t_2} g(t_1)F(t_1)g_D(t_2)F_D(t_2)G_S(t_2-t_1)dt_1dt_2 \\ &+ \int_0^t\int_0^{t_2} g(t_1)F(t_1)g_S(t_2-t_1)F_D(t_2)G_D(t_2)dt_1dt_2 \\ &+ \int_0^t\int_0^{t_2} g_D(t_1)F_D(t_1)g_S(t_2-t_1)F(t_2)G(t_2)dt_1dt_2 \\ &+ \int_0^t\int_0^{t_2} g_D(t_1)F_D(t_1)g(t_2)F(t_2)G_S(t_2-t_1)dt_1dt_2. \end{aligned}$$

d) $P_{11}(t)$

Two kills, one by each side, occurred in time $(0,t)$. We notice the following situation. Suppose that a kill is made by A side, for example, then a killer will search for the next target and find it. The surviving combatant on side B can meet two possible occasions here; either he keeps firing at his original target or he may switch to the appearing new target from the other node. Let p be the probability that a combatant aims at his original target given that a new target is appeared from other node, and $q = 1-p$. Table 7 presents all possible combinations that make this situation in a given period.

The state probability can be obtained by summing all possible cases and we write it as

$$P_{11}(t) = P_{11}^{(1)}(t) + P_{11}^{(2)}(t) + P_{11}^{(3)}(t) + P_{11}^{(4)}(t) \quad .$$

Now $P_{11}^{(1)}(t)$ is the sum of the following two probability values since there are two nodes and the first kill can occur from either one of them. Therefore,

$$\begin{aligned} P_{11}^{(1)}(t) &= P[(A_1B_1)\cap(B_2A_2)] + P[(A_2B_2)\cap(B_1A_1)] \\ &= p\int_0^t\int_0^{t_2} f(t_1)G(t_1)g_D(t_2)F_D(t_2)G_S(t-t_2)F_S(t-t_1)\ dt_1dt_2 \\ &+ \int_0^t\int_0^{t_2} f_D(t_1)G_D(t_1)g(t_2)F(t_2)G_S(t-t_2)F_S(t-t_1)\ dt_1dt_2. \end{aligned}$$

Three other state probabilities are can be obtained similarly, these are

$$\begin{aligned} P_{11}^{(2)}(t) &= P[(A_1B_1)\cap(B_2A_1)] + P[(A_2B_2)\cap(B_1A_2)] \\ &= q\int_0^t\int_0^{t_2} f(t_1)G(t_1)g_D(t_2)F_S(t_2-t_1)G(t-t_2)F_D(t)\ dt_1dt_2 \\ &+ q\int_0^t\int_0^{t_2} f_D(t_1)G_D(t_1)g(t_2)F_S(t-t_1)G(t-t_2)F(t)\ dt_1dt_2, \end{aligned}$$

$$\begin{aligned} P_{11}^{(3)}(t) &= P[(B_1A_1)\cap(A_2B_2)] + P[(B_2A_2)\cap(A_1B_1)] \\ &= p\int_0^t\int_0^{t_2} g(t_1)F(t_1)f_D(t_2)G_D(t_2)F_S(t-t_2)G_S(t-t_1)\ dt_1dt_2 \\ &+ p\int_0^t\int_0^{t_2} g_D(t_1)F_D(t_1)f(t_2)G(t_2)F_S(t-t_2)G_S(t-t_1)\ dt_1dt_2, \end{aligned}$$

$$\begin{aligned} P_{11}^{(4)}(t) &= P[(B_1A_1)\cap(A_2B_1)] + P[(B_2A_2)\cap(A_1B_2)] \\ &= q\int_0^t\int_0^{t_2} g(t_1)F(t_1)f_D(t_2)G_S(t_2-t_1)F(t-t_2)G_D(t)\ dt_1dt_2 \\ &+ q\int_0^t\int_0^{t_2} g_D(t_1)F_D(t_1)f(t_2)G_S(t-t_1)F(t-t_2)G(t)\ dt_1dt_2. \end{aligned}$$

States	Aiming and Killing Events	Time index
$(1,1)^{(1)}$	A in a node kills B, then start to search the next target, and surviving B in other node kills A who is the original target	first kill occurred at t_1, second kill at t_2, and then no further kills by time t
$(1,1)^{(2)}$	A in a node kills B, then start to search the next target, and surviving B in other node kills A who is appeared as a new target	
$(1,1)^{(3)}$	B in a node kills A, then start to search the next target, and surviving A in other node kills B who is the original target	
$(1,1)^{(4)}$	B in a node kills A, then start to search the next target, and surviving A in other node kills B who is appeared as a new target	

Table 7. Sequence for aiming and killing events to reach the state $(1,1)$.

e) $P_{10}(t)$ and $P_{01}(t)$

These probabilities can be calculated simply from $P_{11}(t)$ by considering who is going to be the next killer. We will avoid the detailed manipulation of the procedure for setting them. We write $P_{10}(t)$ as

$$P_{10}(t) = P_{10}^{(1)}(t) + P_{10}^{(2)}(t) + P_{10}^{(3)}(t) + P_{10}^{(4)}(t) \quad ,$$

where

$$\begin{aligned} P_{10}^{(1)}(t) &= P[(A_1B_1)\cap(B_2A_2)\cap(A_1B_2)] + P[(A_2B_2)\cap(B_1A_1)\cap(A_2B_1)] \\ &= p\int_0^t\int_0^{t_3}\int_0^{t_2} f(t_1)G(t_1)g_D(t_2)F_D(t_2)f_S(t_3-t_1)G_S(t_3-t_2)dt_1dt_2dt_3 \\ &+ p\int_0^t\int_0^{t_3}\int_0^{t_2} f_D(t_1)G_D(t_1)g(t_2)F(t_2)f_S(t_3-t_1)G_S(t_3-t_2)dt_1dt_2dt_3, \end{aligned}$$

$$\begin{aligned} P_{10}^{(2)}(t) &= P[(A_1B_1)\cap(B_2A_1)\cap(A_2B_2)] + P[(A_2B_2)\cap(B_1A_2)\cap(A_1B_1)] \\ &= q\int_0^t\int_0^{t_3}\int_0^{t_2} f(t_1)G(t_1)g_D(t_2)F_S(t_2-t_1)f_D(t_3)G(t_3-t_2)\ dt_1dt_2dt_3 \\ &+ q\int_0^t\int_0^{t_3}\int_0^{t_2} f_D(t_1)G_D(t_1)g(t_2)F_S(t_2-t_1)f_S(t_3)G(t_3-t_2)\ dt_1dt_2dt_3, \end{aligned}$$

$$\begin{aligned} P_{10}^{(3)}(t) &= P[(B_1A_1)\cap(A_2B_2)\cap(A_2B_1)] + P[(B_2A_2)\cap(A_1B_1)\cap(A_1B_2)] \\ &= p\int_0^t\int_0^{t_3}\int_0^{t_2} g(t_1)F(t_1)f_D(t_2)G_D(t_2)f_S(t_3-t_2)G_S(t_3-t_1)\ dt_1dt_2dt_3 \\ &+ p\int_0^t\int_0^{t_3}\int_0^{t_2} g_D(t_1)F_D(t_1)f(t_2)G(t_2)f_S(t_3-t_2)G_S(t_3-t_1)\ dt_1dt_2dt_3, \end{aligned}$$

$$\begin{aligned} P_{10}^{(4)}(t) &= P[(B_1A_1)\cap(A_2B_1)\cap(A_2B_2)] + P[(B_2A_2)\cap(A_1B_2)\cap(A_1B_1)] \\ &= q\int_0^t\int_0^{t_3}\int_0^{t_2} g(t_1)F(t_1)f_D(t_2)G_S(t_2-t_1)f(t_3-t_2)G_D(t_3)\ dt_1dt_2dt_3 \\ &+ q\int_0^t\int_0^{t_3}\int_0^{t_2} g_D(t_1)F_D(t_1)f(t_2)G_S(t_2-t_1)f(t_3-t_2)G(t_3)\ dt_1dt_2dt_3. \end{aligned}$$

The state probability $P_{01}(t)$ can be obtained by an interchange of $f(\cdot)$ and $F_*(\cdot)$ with $g(\cdot)$ and $G_*(\cdot)$ in $P_{10}(t)$, respectively.

3.4 Results and Comparisons

Five cases are implemented to verify and investigate the system we have studied. Table 8 presents the parameter values particularly used in each these cases. The Gaussian quadrature method is applied here to solve the integration equations and the results were outstanding and there are given as follows

	Case I	Case II	Case III	Case IV	Case V
X_A	Erlang-2	Erlang-2	Exp	Erlang-2	Erlang-2
X_B	Erlang-2	Erlang-2	Exp	Erlang-2	Erlang-2
μ_A	1.0	1.5	1.5	1.0	10.0
μ_B	1.0	2.0	2.0	1.0	8.0
p_A	.50	.30	.30	.50	.85
p_B	.50	.50	.50	.50	.90
p	.70	.70	.70	.70	.80
S	Exp λ=.50	Exp λ=2.0	Exp λ=2.0	U(1,3)	U(12,15)
D	Exp λ=.50	Exp λ=2.0	Exp λ=2.0	U(1,3)	U(12,15)

Table 8. Parameters that are used for five different cases.

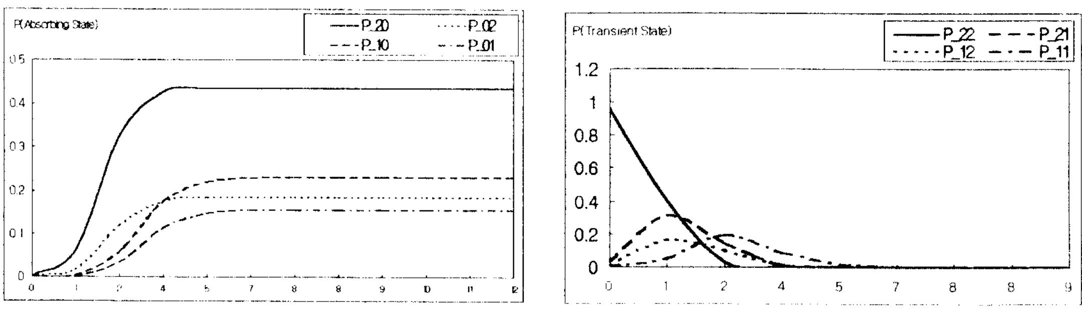


Figure 7. Absorbing and transient state probabilities in Case II.

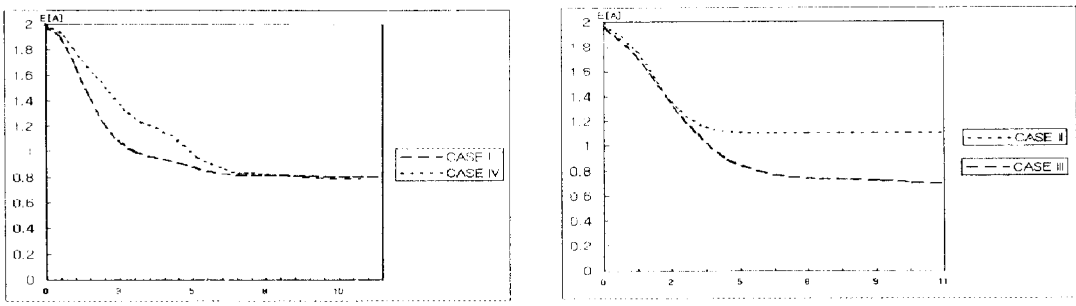
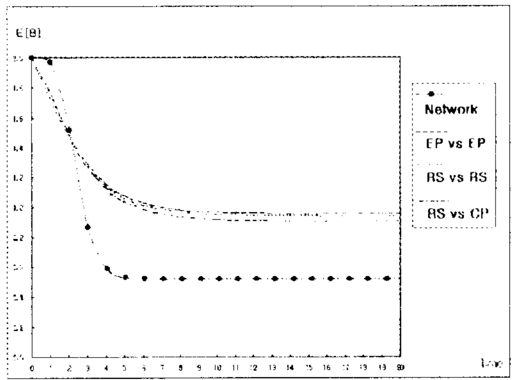
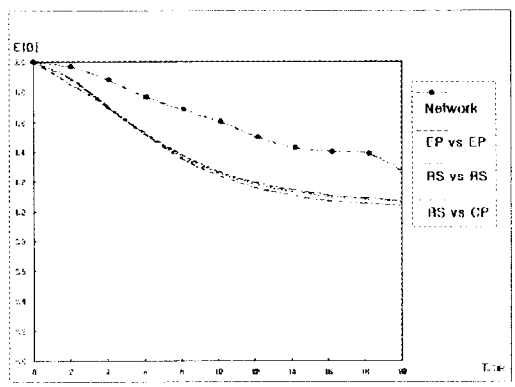


Figure 8. Mean number of survivors of side A, Case I vs Case IV(left) and Case II vs Case III(right)

Figure 7 presents the time trajectory for both transient and absorbing state probabilities in Case II, they all behave as we have intended. The mean number of survivors on side A in Case I is compared to that of Case IV where the distributions are different for both delay and search time random variables. However, we keep the true mean values of the random variables which is 2. It is shown in Figure 8-(a). In Figure 8-(b), two interfiring random variables, Erlang-2 and exponential, are adapted to check the difference in outputs where the other parameters are set equal for both case II and case III. This result supports the importance of the stochasticity in a realization of the combat analysis. Figures 9-(a) and 9-(b) present the amount of departure from the models mentioned earlier such that the firing strategy involved in it. We easily find that the network of two one-on-one duels shows significant differences in $E[B]$ when they are compared to the situations with various firing strategies.



(a)



(b)

Figure 9. Mean number of survivors of side B, Case II-(a) and Case V-(b).

4. Summary and Conclusions

Fire allocation strategies of combatants are an important element in the analysis of combat operation. Three common and reasonable strategies were defined and examined by applying a theory of stochastic combat. This study begins with an analytic approach to see if any of the general solutions for a large-sized battle could be feasible. The solution procedure for the two-on-two battle is attempted and the results are verified.

Three firing strategic options are considered, which are random selection, concentrated power, and evenly distributed power. Two out of seven Lanchester square law assumptions are to be modified in the case for either concentrated power or evenly distributed power.

Current knowledge and experience in the theory of combat inform us to assert two statements as basic laws. First, a firefight is a terminating stochastic process, and second, a combat is to be modeled as a network of several small battles. We have seen many studies, at great length, describing the importance of stochasticity of the firefight process. There is yet much to be done on the second issue. The word network is intended to imply either simultaneous or sequential or both type of events which may or may not be correlated depending on the particular circumstances being modeled.

This paper was intended to be a part of intermediate works toward building a combat situation more realistically, so called 'a series of mini battles.' A small and simple combat, two-on-two sized, was chosen to investigate a way to analyze a combat in this fashion. We divided a two-on-two combat into two separate one-on-one battles and connected them in a network. New elements considered such as delay time of starting a firefight on a particular node or search time for the next target when a kill occurs are defined and used as the input parameters.

Some combat parameters including number of survivors on both sides, their standard deviations, winning probabilities, termination time and its standard deviation are obtained based on the state probabilities and used to compare with other options of the same sized combat. The discussions are made to validate the hypothesized models and to ask if the results are meaningful and useful in the analysis of combat operations or not.

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