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CONVERGENCE OF NONLINEAR SEMIGROUPS IN A HYPERBOLIC SPACE

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ABSTRACT. In this paper, we establish Trotter-Kato type convergence theorem for nonlinear semigroups generated by coaccretive operators in a hyperbolic space.

1. Introduction

Let $(X,\ \rho)$ be a metric space, and let R be a real line. A mapping $c : R \to X$ is said to be a metric embedding of R into X if

$$\rho(c(s),\ c(t)) = |s - t| \quad \text{for all} \quad s,\ t \in R.$$

The image of R under a metric embedding is called a metric line. The image of a real interval $[a,\ b]$ under such a mapping is called a metric segment.

Assume that $(X,\ \rho)$ contains a family M of metric lines, such that for each pair of distinct points $x,\ y \in X$ there is a unique metric line in M which passes through x and y. This metric line determines a unique metric segment joining x and y. We denote this segment by $[x,\ y]$. For each $0 \le t \le 1$, there is a unique point $z \in [x,\ y]$ such that

$$\rho(x,\ z) = t\rho(x,\ y) \quad \text{and} \quad \rho(z,\ y) = (1-t)\rho(x,\ y).$$

This point will be denoted by $(1-t)x \oplus ty$.

We shall say that X, or more precisely $(X,\ \rho,\ M)$, is a hyperbolic space if

$$\rho(\frac{1}{2}x \oplus \frac{1}{2}y,\ \frac{1}{2}x \oplus \frac{1}{2}z) \le \frac{1}{2}\rho(y,\ z)$$

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for all $x, y, z \in X$. It is clear that all normed spaces are hyperbolic. So are all Hadamard manifolds, that is, all finite-dimensional connected, simply connected, complete Riemannian manifolds of nonpositive curvature [2]. For more examples of hyperbolic spaces, see [4].

Let (X, ρ, M) be a hyperbolic space. For $x, y \in X$ with $x \neq y$, there is a unique metric line in M passing through x and y. For $r \geq 0$ let $z = (1+r)x \ominus ry$ be the unique point on this metric line satisfying

$$\rho(z, x) = r\rho(x, y) \quad \text{and} \quad \rho(z, y) = (1+r)\rho(x, y).$$

Note that $z = (1+r)x \ominus ry$ if and only if $x = 1/(1+r)z \oplus r/(1+r)y$.

A set-valued operator $T \subset X \times X$ with domain $D(T)$ and range $R(T)$ is said to be coaccretive if

$$\rho(x_1, x_2) \leq \rho((1+r)x_1 \ominus y_1, (1+r)x_2 \ominus y_2)$$

for all $[x_i, y_i] \in T$, $i = 1, 2$, and $r > 0$.

If X is a normed linear space, $A = I - T$ is an accretive operator, that is, $|x_1 - x_2| \leq |x_1 - x_2 + r(z_1 - z_2)|$ for all $[x_i, z_i] \in A$, $i = 1, 2$ and $r > 0$.

Let D be a subset of X. A mapping $T : D \to X$ is said to be nonexpansive if $\rho(Tx, Ty) \leq \rho(x, y)$ for all $x, y \in D$. It is not difficult to show that all nonexpansive mappings are coaccretive.

A semigroup (of nonlinear nonexpansive mappings) on a subset D of a hyperbolic space X is a family of operators $S(t) : D \to D$, $0 \leq t < \infty$, satisfying the following conditions;

(1) $S(t+s)x = S(t)S(s)x$ for all $s, t \geq 0$ and $x \in D$;
(2) $S(0)x = x$ for all $x \in D$;
(3) $S(t)x$ is continuous in $t \geq 0$ for each $x \in D$;
(4) $\rho(S(t)x, S(t)y) \leq \rho(x, y)$ for all $x, y \in D$.

In this paper, we first show that such semigroups are generated by coaccretive operators satisfying the range condition via the exponential formula. And then we establish convergence of nonlinear semigroups generated by coaccretive operators. In Banach space cases, our convergence result includes convergence of nonlinear semigroups in [1, 5].

2. Convergence of Nonlinear Semigroups

For a coaccretive operator $T \subset X \times X$ we can define, for each $r > 0$, a single-valued nonexpansive mapping $J_r^T : R((I + r)I \ominus rT) \to D(T)$ by $J_r^T((1+r)x \ominus ry) = x$, where $x \in D(T)$ and $[x,\ y] \in T$. In normed linear spaces, these mappings are the resolvent of the accretive operator $A = I - T$.

The following lemmas collect some facts about J_r^T.

LEMMA 1. *Let X be a hyperbolic space. Suppose that T is a coaccretive operator. Then*

(i) *for $s > 0$ and $x \in D(J_s^T) \cap D(T)$*

$$\rho(x,\ J_s^T x) \leq s|Tx|,$$

where $|Tx| = \inf\{\rho(x,\ y) : y \in Tx\}$.

(ii) *For all $t > s > 0$ and $x \in D(J_t^T)$*

$$J_t^T x = J_s^T(\frac{s}{t}x \oplus (1 - \frac{s}{t})J_t^T x).$$

PROOF. If $y \in Tx$, then $J_s^T((1+s)x \ominus sy) = x$. Hence $\rho(x,\ J_s^T x) \leq \rho((1+s)x \ominus sy,\ x) = s\rho(x,\ y)$, and so (i) is proved. If $x \in D(T)$, there exists $[x_0,\ y_0] \in T$ such that $x = (1+t)x_0 \ominus ry_0$, that is, $J_t^T x = x_0$. It is not difficult to see that $(1+s)x_0 \ominus sy_0 = \frac{s}{t}x \oplus (1 - \frac{s}{t})J_t^T x$. □

LEMMA 2. *Let $r > s > 0$, and $x \in D((J_r^T)^m) \cap D((J_s^T)^n)$, where m and n are positive integers and $n \geq m$. Then*

$$\begin{aligned}\rho((J_s^T)^n x,(J_r^T)^m x) &\\ &\leq \sum_{j=0}^{m-1} \alpha^j \beta^{n-j} \binom{n}{j} \rho((J_r^T)^{m-j} x,\ x) \\ &+ \sum_{j=m}^{n} \alpha^m \beta^{j-m} \binom{j-1}{m-1} \rho((J_s^T)^{n-j} x,\ x),\end{aligned}$$

where $\alpha = s/r$ and $\beta = (r-s)/r$.

PROOF. For integers j and k satisfying $0 \leq j \leq n$ and $0 \leq k \leq m$, let $a_{k,j} = \rho((J_s^T)^j x,\ (J_r^T)^k x)$. Then

$$\begin{aligned} a_{k,j} &= \rho((J_s^T)^j x,\ (J_r^T)^k x) \\ &\leq \rho((J_s^T)^j x,\ J_s^T(\frac{s}{r}(J_r^T)^{k-1}x \oplus (1-\frac{s}{r})(J_r^T)^k x)) \\ &\leq \rho((J_s^T)^{j-1}x,\ \frac{r}{s}(J_r^T)^{k-1}x \oplus (1-\frac{s}{r})(J_r^T)^k x) \\ &= \rho(\frac{s}{r}(J_s^T)^{j-1} \oplus (1-\frac{s}{r})(J_s^T)^{j-1}x,\ \frac{s}{r}(J_r^T)^{k-1}x \oplus (1-\frac{r}{s})(J_r^T)^k x) \\ &\leq \frac{s}{r}\rho((J_s^T)^{j-1}x,\ (J_r^T)^{k-1}x) + (1-\frac{s}{r})\rho((J_s^T)^{j-1}x,\ (J_r^T)^k x). \end{aligned}$$

With $\alpha = s/r$ and $\beta = (r-s)/r$,

$$a_{k,j} \leq \alpha a_{k-1,j-1} + \beta a_{k,j-1}.$$

By solving these inequalities in term of $a_{k.0}$ and $a_{0,j}$, the result follows.□

LEMMA 3. *Let $n \geq m > 0$ be integers, and let α and β be positive numbers such that $\alpha + \beta = 1$. Then*

(i) $$\sum_{j=0}^{m} \binom{n}{j} \alpha^j \beta^{n-j}(m-j) \leq \sqrt{(n\alpha - m)^2 + n\alpha\beta}.$$

(ii) $$\sum_{j=m}^{n} \binom{j-1}{m-1} \alpha^m \beta^{j-m}(n-j) \leq \sqrt{\frac{m\beta}{\alpha^2} + (\frac{m\beta}{\alpha} + m - n)^2}.$$

For a proof of this lemma, see [3].

The following theorem is given in [6] without the proof. We present a complete proof of theorem 1. In order to show our main theorem, we need the inequalities appeared in the proof of theorem 1.

THEOREM 1. *Let X be a complete hyperbolic space. If $T \subset X \times X$ is a coaccretive operator satisfying*

$$cl(D(T)) \subset R((I+r)I \ominus rT) \quad \text{for} \quad r > 0,$$

then T generates a (nonlinear) semigroup on $cl(D(T))$ via the exponential formula

$$S(t)x = \lim_{n\to\infty} (J_{t/n}^T)^n x,$$

where $t \geq 0$ and $x \in cl(D(T))$.

PROOF. Let $x \in D(T)$, $r \geq s \geq 0$ and let $n \geq m$ be positive integers. Then $x \in D((J_s^T)^m) \cap D((J_r^T)^n)$. By Lemma 1, 2 and 3,

$$
\begin{aligned}
&\rho((J_s^T)^n x,\ (J_s^T)^m) \\
&\qquad \leq \sum_{j=0}^{m} \alpha^j \beta^{n-j} \binom{n}{j} \rho((J_r^T)^{m-j} x,\ x) \\
&\qquad + \sum_{j=0}^{n} \alpha^m \beta^{j-m} \binom{j-1}{m-1} \rho((J_s^T)^{n-j} x,\ x) \\
&\qquad \leq \sum_{j=0}^{m} \alpha^j \beta^{n-j} \binom{n}{j} (m-j) \rho(J_r^T x,\ x) \\
&\qquad\quad + \sum_{j=m}^{n} \alpha^m \beta^{j-m} \binom{j-1}{m-1} (n-j) \rho(J_s^T x,\ x) \\
&\qquad \leq r \sum_{j=0}^{m} \alpha^j \beta^{n-j} \binom{n}{j} (m-j) |Tx| \\
&\qquad\quad + s \sum_{j=m}^{n} \alpha^m \beta^{j-m} \binom{j-1}{m-1} (n-j) |Tx| \\
&\qquad \leq (\sqrt{(n\alpha - m)^2 + n\alpha\beta}\, r + \sqrt{\frac{m\beta}{\alpha^2} + (\frac{m\beta}{\alpha} + m - n)^2 s})|Tx| \\
&\qquad \leq (\sqrt{(ns - mr)^2 + ns(r-s)} + \sqrt{mr(r-s) + (mr - ns)^2})|Tx|.
\end{aligned}
$$

Taking $s = t/n$ and $r = t/m$, we obtain

$$
\rho((J_{t/n}^T)^n x,\ (J_{t/m}^T)^m x) \leq 2\sqrt{\frac{1}{m} - \frac{1}{n}}\, t|Tx|.
$$

Since X is complete, $\lim_{n\to\infty}(J_{t/n}^T)^n x$ exists. Since $(J_{t/n}^T)^n$ is nonexpansive, $S(t)x = \lim_{n\to\infty}(J_{t/n}^T)^n x$ exists for $x \in cl(D(T))$. Clearly, $S(t) : cl(D(T)) \to cl(D(T))$.

If $x \in D(T)$ and $\tau > t \geq 0$, by taking $r = \tau/n$ and $s = t/n$, we have

$$
\begin{aligned}
&\rho((J_{t/n}^T)^n x, (J_{\tau/n}^T)^n x) \\
&\qquad\qquad \leq (\sqrt{(t-\tau)^2 + \frac{t(\tau - t)}{n}} + \sqrt{\frac{\tau(\tau - t)}{n} + (t-\tau)^2}.
\end{aligned}
$$

Hence

$$\rho(S(t)x,\ S(\tau)x) \le 2|t-\tau||Tx|.$$

Since $S(t)x$ is Lipschitz continuous in t, $S(t)x$ is continuous in t for $x \in cl(D(T))$.

Since $S(t)x = \lim_{k\to\infty}(J^T_{t/k})^k x = \lim_{k\to\infty}(J^T_{mt/mk})^{km}x = S(t)^m x$,

$$S(\frac{l}{k}+\frac{n}{m})x = S(\frac{1}{km})^{lm+kn}x = S(\frac{1}{km})^{lm}S(\frac{1}{km})^{kn}$$
$$= S(\frac{l}{k})S(\frac{n}{m})x.$$

That is, the semigroup property holds for rationals. By the continuity of $S(t)$ in t and the nonexpansiveness of $S(t)$ in x, the the result follows.□

Let $S(t)$ be the semigroup associated with T through the exponential formula in Theorem 1. We shall say the $S(t)$ is generated by T. In the proof of Theorem 1, we have the following inequalities.

LEMMA 4. *Let $T \subset X \times X$ be a coaccretive operator such that $cl(D(T)) \subset R((I+r)I \ominus rT)$ for $r > 0$. Let S be the semigroup generated by T. Then for each $x \in D(T)$,*

(i) $$\rho(S(t)x,\ (J^t_{t/n})^n) \le 2tn^{-1/2}|Tx|.$$

(ii) $$\rho(S(t)x,\ S(\tau)x) \le 2|t-\tau||Tx|.$$

THEOREM 2. *Let $T \subset X \times X$ be a coaccretive operator such that $cl(D(T)) \subset R((I+r)I \ominus rT)$ for $r > 0$, and let S be the semigroup generated by T. Let $T_n \subset X \times X$ be coaccretive operators such that $cl(D(T_n)) \subset R((I+r)I \ominus rT_n)$ for $r > 0$ and each n, and let S_n be the semigroups generated by T_n.*

Suppose that

$$\lim_{n\to\infty} J^{T_n}_r x = J^T_r x \quad \text{for} \quad x \in cl(D) \quad \text{and} \quad r > 0,$$

where $D = \cap D(T_n) \cap D(T)$. Then

$$\lim_{n\to\infty} S_n(t)x = S(t)x$$

for $x \in cl(D)$ and the convergence is uniform on bounded t-intervals.

PROOF. Let $x \in D$. Since $\lim_{n\to\infty} J_r^{T_n}x = J_r^T x$, there exists n_0 depending on x and $r > 0$ such that

$$\frac{1}{r}\rho(x,\ J_r^{T_n}x) \le \frac{1}{r}\rho(x,\ J_r^T x) \le |Tx| + 1 \quad \text{for} \quad n \ge n_0.$$

Consider $\rho(S_n(t)x,\ S(t)x) \le \rho(S_n(t)x,\ S_n(t)J_r^{T_n}x) + \rho(S_n(t)J_r^{T_n}x,\ S(t)x)$. Since $S_n(t)$ is nonexpansive, $\rho(S_n(t)x,\ S_n(t)J_r^{T_n}x) \le \rho(x,\ J_r^{T_n}x)$.

Next, we will consider the second term.

$$\begin{aligned}
&\rho(S_n(t)J_r^{T_n}x,\ S(t)x)\\
&\qquad \le \rho(S_n(t)J_r^{T_n}x,\ (J_{t/k}^{T_n})^k J_r^{T_n}x) + \rho((J_{t/k}^{T_n})^k J_r^{T_n}x,\ (J_{t/k}^{T_n})^k x)\\
&\qquad\quad + \rho((J_{t/n}^{T_n})^k x,\ (J_{t/k}^{T})^k x) + \rho((J_{t/k}^{T})^k x,\ S(t)x).
\end{aligned}$$

Note that by Lemma 4 and nonexpansiveness of J_r^T, we have

$$\rho(S_n(t)J_r^{T_n}x,\ (J_{t/k}^{T_n})^k J_r^{T_n}x) \le 2t\sqrt{\frac{1}{k}}|T_n J_r^{T_n}x|,$$

$$\rho((J_{t/k}^{T_n})^k J_r^{T_n}x,\ (J_{t/k}^{T_n})^k x) \le \rho(J_r^{T_n}x,\ x)$$

and

$$\rho((J_{t/k}^{T})^k x,\ S(t)x) \le 2t\sqrt{\frac{1}{k}}|Tx|.$$

So we have

$$\begin{aligned}
&\rho(S_n(t)x,\ S(t)x)\\
\le&\rho(x,\ J_r^{T_n}x) + 2t\sqrt{\frac{1}{k}}|T_n J_r^{T_n}x| + \rho(J_r^{T_n}x,\ x) + 2t\sqrt{\frac{1}{k}}|Tx|\\
&+ \rho((J_{t/k}^{T_n})^k x,\ (J_{t/k}^{T})^k x)\\
=&2\rho(x,\ J_r^{T_n}x) + 2t\sqrt{\frac{1}{k}}|T_n J_r^{T_n}x| + 2t\sqrt{\frac{1}{k}}|Tx| + \rho((J_{t/k}^{T_n})^k x,\ (J_{t/k}^{T})^k x)\\
\le&2r(|Tx| + 1) + 2t\sqrt{\frac{1}{k}}(|Tx| + 1) + 2t\sqrt{\frac{1}{k}}|Tx| + \rho((J_{t/k}^{T_n})^k x,\ (J_{t/k}^{T_n})^k x).
\end{aligned}$$

Let $\varepsilon > 0$ be given. First fix $r > 0$ such that $2r(|Tx|+1) < \varepsilon/3$. Then fix $k \geq k_0$ such that $2t\sqrt{1/k}(2|Tx|+1) < \varepsilon/3$. Finally, choose $n \geq n_0$ such that $\rho((J_{t/k}^{T_n})^k x,\ (J_{t/k}^{t})^k x) < \varepsilon/3$. So we have $\lim_{n\to\infty} S_n(t)x = S(t)x$ for $x \in D$.

Next, we will show that the convergence is uniform in $t \in [0,\ T]$. By Lemma 4, we have

$$\begin{aligned}
\rho(S(t)x,\ S_n(t)x) &\leq \rho(S(t)x,\ S_n(t)J_r^{T_n}x) + \rho(S_n(t)J_r^{T_n}x,\ S_n(t)x) \\
&\leq \rho(S(t)x,\ S_n(t)J_r^{T_n}x) + \rho(J_r^{T_n}x,\ x) \\
&\leq (S(t)x,\ S(\tau)x) + \rho(S(\tau)x,\ S_n(\tau)x) + \rho(S_n(\tau)x,\ S_n(\tau)J_r^{T_n}x) \\
&\quad + \rho(S_n(\tau)J_r^{T_n}x,\ S_n(t)J_r^{T_n}x) + \rho(J_r^{T_n}x,\ x) \\
&\leq 2|t-\tau||Tx| + \rho(S(\tau)x,\ S_n(\tau)x) + \rho(x,\ J_r^{T_n}x) \\
&\quad + 2|t-\tau||T_nJ_r^{T_n}x| + \rho(J_r^{T_n}x,\ x) \\
&\leq 2|t-\tau||Tx| + \rho(S(\tau)x,\ S_n(\tau)x) + 2|t-\tau|\frac{1}{r}\rho(J_r^{T_n}x,\ x) \\
&\quad + 2\rho(J_r^{T_n}x,\ x) \\
&\leq 2|t-\tau||Tx| + \rho(S(\tau)x,\ S_n(\tau)x) + 2|t-\tau|(|Tx|+1) \\
&\quad + 2r(|Tx|+1).
\end{aligned}$$

Then we have

$$\rho(S(t)x,\ S_n(t)x) \leq \rho(S(\tau)x,\ S_n(\tau)x) + 2|t-\tau|(2|Tx|+1) + 2r(|Tx|+1).$$

This implies the uniform convergence. Since $S_n(t)$ and $S(t)$ are nonexpansive, the result follows for $x \in cl(D)$. □

REMARK. If X is a Banach space, then $A = I - T$ and $A_n = I - T_n$ are accretive operators satisfying the range condition $R(I + rA) \supset cl(D(A))$ and $R(I + rA_n) \supset cl(D(A_n))$ for $r > 0$, and $S(t)$ and $S_n(t)$ are the semigroups generated by $-A$ and $-A_n$, respectively. Theorem 2 says that convergence of resolvent of accretive operators implies convergence of nonlinear semigroups, that is , our result extends nonlinear Trotter-Kato Theorem in [1, 5].

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