Comm. Korean Math. Soc. **13** (1998), No. 1, pp. 159–170

A SUBMARTINGALE INEQUALITY

YOUNG-HO KIM AND BYUNG-IL KIM

ABSTRACT. In this paper, we extend Burkholder's exponential inequality for a strong subordinate of a nonnegative bounded submartingale.

1. Introduction

Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $(\mathcal{F}_n)_{n\geq 0}$ and let $\{\emptyset, \Omega\} = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_\infty \subset \mathcal{F}$ be a sequence of σ-algebras with $\mathcal{F}_\infty = \sigma(\cup_1^\infty \mathcal{F}_n)$. We consider an adapted process $f = (f_n)$ with difference sequence $d = (d_n); f_n = d_0 + \cdots + d_n$ for each $n \geq 0$. For $1 < p < \infty$, we set $\|f\|_p = \sup_{n\geq 0} \|f_n\|_p$. Moreover, f will be called a submartingale or martingale whenever $\{f_n, \mathcal{F}_n, n \geq 1\}$ is such. We also consider another adapted process $g = (g_n)$ and denote its difference sequence by $e = (e_n)$. For example, concider the following conditions introduced by Burkholder;

$$|e_n| \leq |d_n| \quad for \quad n \geq 0, \tag{1.1}$$

$$|\mathbb{E}(e_n|\mathcal{F}_{n-1})| \leq |\mathbb{E}(d_n|\mathcal{F}_{n-1})| \quad for \quad n \geq 1. \tag{1.2}$$

It is enough for these inequalities to hold with probability 1. Assume that the random variables d_n and e_n are integrable so their conditional expectations exist. The process g *is called differentially subordinate to* f if (1.1) holds. If (1.2) is satisfied, then g *is conditionally differentially subordinate to* f. If both of the conditions (1.1) and (1.2) are satisfied, then g *is strongly differentially subordinate to* f, or more simply, g *is*

Received September 2, 1997. Revised November 22, 1997.
1991 Mathematics Subject Classification: Primary 60G42; Seondary 60E15.
Key words and phrases: Submartingale, differential subordination, conditional differential subordination.

strongly subordinate to f. Of course, if f and g are martingales, then both sides of (1.2) vanish and (1.2) is trivially satisfied. It will be convenient to allow g to have its values in a space of possibly more than one dimension. So we assume throughout this thesis that g has its values in $\mathbb{R}^\nu$ where ν is a positive integer. The Euclidean norm of $y \in \mathbb{R}^\nu$ is denote by $|y|$ and inner product of y and $k \in \mathbb{R}^\nu$ by $y \cdot k$. Note that the conditions (1.1) and (1.2) have meaning for Banach space valued f and g.

Suppose that Φ is a nondecreasing convex function on $[0,\infty)$ with $\Phi(0) = 0$ and $\int_0^\infty \Phi(t)e^{-t}\,dt$ finite and positive. Suppose also that Φ is twice differentiable on $(0,\infty)$ and Φ' is convex on this open interval with $\Phi'(0^+) = 0$.

THEOREM 1.1. *Let* $f = (f_n)_{n\geq 0}$ *and* $g = (g_n)_{n\geq 0}$ *are adapted to a filtration* $(\mathcal{F}_n)_{n\geq 0}$ *of a probability space* $(\Omega, \mathcal{F}, P)$. *Here* f *is a nonnegative submartingale bounded from above by 1, and* g *is* $\mathbb{R}^\nu$*-valued, where* ν *is a positive integer. With* $f_n = d_0 + \cdots + d_n$ *and* $g_n = e_0 + \cdots + e_n (n \geq 0)$ *and let* $0 \leq c \leq 1$. *If*

$$|e_n| \leq |d_n| \quad (n \geq 0) \tag{1.3}$$

$$|\mathbb{E}(e_n|\mathcal{F}_{n-1})| \leq c\mathbb{E}(d_n|\mathcal{F}_{n-1}) \quad (n \geq 1), \tag{1.4}$$

then

$$\sup_{n\geq 0} \mathbb{E}\Phi\left(\frac{|g_n|}{1+c}\right) < \frac{1+c}{2+c}\int_0^\infty \Phi(t)e^{-t}\,dt. \tag{1.5}$$

COROLLARY 1.2. (Burkholder [2]) *Suppose that* $f = (f_n)_{n\geq 0}$ *and* $g = (g_n)_{n\geq 0}$ *are adapted to filtration* $(\mathcal{F}_n)_{n\geq 0}$ *of probability space* $(\Omega, \mathcal{F}, P)$. *Here* f *is a nonnegative submartingale bounded from above by 1, and* g *is* $\mathbb{R}^\nu$*-valued, where* ν *is positive integer. With* $f_n = d_0 + \cdots + d_n$ *and* $g_n = e_0 + \cdots + e_n (n \geq 0)$ *we assume that*

$$|e_n| \leq |d_n| \quad (n \geq 0),$$

$$|\mathbb{E}(e_n|\mathcal{F}_{n-1})| \leq \mathbb{E}(d_n|\mathcal{F}_{n-1}) \quad (n \geq 1),$$

then

$$\sup_{n\geq 0} \mathbb{E}\Phi\left(\frac{|g_n|}{2}\right) < \frac{2}{3}\int_0^\infty \Phi(t)e^{-t}\,dt.$$

EXAMPLE 1.3. If $\Phi(t) = (1+c)^p t^p e^{(1+c)\beta t}$ where $0 \le \beta < 1/(1+c)$ and $2 \le p < \infty$, then

$$\sup_{n\ge 0} \mathbb{E}|g_n|^p \exp(\beta|g_n|) < \frac{(1+c)^{p+1}\Gamma(p+1)}{2+c}\frac{1}{(1-(1+c)\beta)^{p+1}}$$

2. Preliminaries

Let $0 \le c \le 1$ be a constant. Put $S = \{(x,y) : 0 < x < 1 \quad \text{and} \quad y \in \mathbb{R}^\nu \quad \text{with} \quad y \neq 0\}$, where ν is a positive integer. We define two functions U and V on S by

$$V(x,y) = \Phi\Big(\frac{|y|}{1+c}\Big)$$

and

$$\begin{aligned} U(x,y) &= \frac{(1+c)\alpha}{2+c} + \frac{\alpha}{2+c}\Big(|y| - (1+c)x\Big)\Big(x+|y|\Big)^{\frac{1}{1+c}}, \quad 0 < x+|y| \le 1, \\ &= (1-x)A\Big(\frac{x+|y|+c}{1+c}\Big) + xB\Big(\frac{x+|y|+c}{1+c}\Big), \qquad x+|y| > 1. \end{aligned}$$

where $\alpha = \int_0^\infty \Phi(t)e^{-t}\,dt$ and, for all $t \ge 1$,

$$A(t) = e^t \int_t^\infty B(s)e^{-s}\,ds \quad \text{and} \quad B(t) = \Phi(t-1).$$

Since $A(1) = \alpha$, U is continuous. Let $F(t) = A(t) - B(t)$. Then for $t \in (1,\infty)$, $A'(t) = A(t) - B(t) = F(t)$ and $F'(t) = A'(t) - B'(t) = F(t) - B'(t)$. If $t > 1$, then $F(t) > 0$, $F'(t) > 0$, $F''(t) \ge 0$, and $tF'(t) \ge F(t)$.

LEMMA 2.1. *$U_x + c|U_y| \le 0$ on S, where $0 \le c \le 1$. Also U_x and U_y are continuous on S.*

PROOF. We see that the function U has all partial derivatives. The definition of U implies that if $0 < x + |y| \le 1$, using the chain rule we get

$$U_x(x,y) = -\frac{\alpha}{(1+c)(2+c)}(x+|y|)^{\frac{-c}{1+c}}\Big((1+c)(2+c)x + (2c+c^2)|y|\Big)$$

$$U_y(x,y) = \frac{\alpha}{(1+c)(2+c)}(x+|y|)^{\frac{-c}{1+c}}\Big((2+c)|y|\Big)\frac{y}{|y|}.$$

Hence, if $0 \le c \le 1$, then, $c|U_y| \le -U_x$ because

$$c|((2+c)|y|)| \le x(1+c)(2+c) + (2c+c^2)|y|.$$

And if $x + |y| > 1$, then

$$U_x(x,y) = \frac{-1}{1+c}\left[cA'\left(\frac{x+|y|+c}{1+c}\right) + xF'\left(\frac{x+|y|+c}{1+c}\right)\right]$$
$$U_y(x,y) = \frac{1}{1+c}\left[A'\left(\frac{x+|y|+c}{1+c}\right) - xF'\left(\frac{x+|y|+c}{1+c}\right)\right]\frac{y}{|y|}.$$

Hence if $0 \le c \le 1$, then $c|U_y| + U_x \le 0$ because

$$\begin{aligned} c\left|A'\left(\frac{x+|y|+c}{1+c}\right) - xF'\left(\frac{x+|y|+c}{1+c}\right)\right| \\ \le cA'\left(\frac{x+|y|+c}{1+c}\right) + xF'\left(\frac{x+|y|+c}{1+c}\right). \end{aligned}$$

And, if $|y| + x = 1$, then

$$U_x = -\frac{\alpha(x+c)}{1+c}, \quad U_y = \frac{\alpha y}{1+c}.$$

That is, U_x and U_y are continuous. □

LEMMA 2.2. *Let $(x,y), (x+h, y+k) \in S$ with $|h| \ge |k|$ and $|y+tk| > 0$ for all $t \in \mathbb{R}$, then*

$$U(x+h, y+k) \le U(x,y) + U_x(x,y)h + U_y(x,y) \cdot k. \tag{2.1}$$

PROOF. Since U has all partial derivatives on S and $(x+th, y+tk) \in S$, by the continuity of U, it is enough to prove the above for $|k| < |h|$ with x and $x+h$ in (0,1). Let $G(t) = U(x+ht, y+kt)$. Then G has a continuous first derivative on open interval containing [0, 1]. Using the chain rule we have

$$G'(t) = U_x(x+th, y+tk)h + U_y(x+th, y+tk) \cdot k$$

and

$$G'(0) = U_x(x,y)h + U_y(x,y) \cdot k.$$

Hence, (2.1) is equivalent to

$$G(1) \le G(0) + G'(0). \tag{2.2}$$

But (2.2) follows from fact that G' is nonincreasing on (0,1). To see that G' is nonincreasing, consider the functions r, N and m given by

$$r(t) = m(t) + N(t), \quad m(t) = x + ht \quad \text{and} \quad N(t) = |y + kt|.$$

We will write m for $m(t)$, etc. Differentiation gives $NN' = k \cdot (y + tk)$ and $NN'' = k^2 - (N')^2$. The Cauchy-Schwarz inequality gives $N|N'| = |NN'| \le |k||y + tk| = |k|N$. Hence $|N'| \le |k|$. Therefore,

$$r'' = (m + N)'' = m'' + N'' = 0 + N'' \ge 0.$$

Put

$$I = \{t \in (0,1) : r(t) < 1\} \quad \text{and} \quad J = \{t \in (0,1) : r(t) > 1\}.$$

A number $t \in (0,1)$ satisfies $r(t) = 1$ only if it is a zero of the polynomial

$$[(1 - x) - ht]^2 - |y + kt|^2 = (h^2 - |k|^2)t^2 + \cdots .$$

Here $h^2 - |k|^2 > 0$, so the complement of $I \cup J$ with respect to (0,1) is finite and it will be enough to show that G' is nonincreasing on each component of $I \cup J$. On I, the second derivative G'' exists, which, of course also follows from the convexity of the Euclidean norm.

$$G(t) = \frac{(1+c)\alpha}{2+c} + \frac{\alpha}{2+c}\Big(N(t) - (1+c)m(t)\Big)\Big(r(t)\Big)^{\frac{1}{1+c}}$$

We compute

$$
\begin{aligned}
G'(t) &= \frac{\alpha}{2+c}\left[(N' - (1+c)h)r^{\frac{1}{1+c}} + \frac{1}{1+c}(N - (1+c)m)r^{\frac{-c}{1+c}}r'\right] \\
G''(t) &= \frac{\alpha}{(1+c)(2+c)}\Bigg[(1+c)r''r^{\frac{1}{1+c}} + r^{\frac{-c}{1+c}}r'(2N' - 2(1+c)h) \\
&\quad + \frac{-c}{1+c}(N - (1+c)m)(r')^2 r^{\frac{-1-2c}{1+c}} + (N - (1+c)m)r^{\frac{-c}{1+c}}r''\Bigg] \\
&= \frac{\alpha}{(1+c)(2+c)}r^{\frac{-c}{1+c}}r^{-1}\Bigg[r''r[r(1+c) + N - (1+c)m] \\
&\quad + \frac{-c}{1+c}(N - (1+c)m)(r')^2 + (2N' - 2(1+c)h)rr'\Bigg] \\
&= \frac{\alpha}{(1+c)(2+c)}r^{\frac{-(1+2c)}{1+c}}\left[(2+c)r[Nr'' - 2hr + (r')^2]\right] + Y
\end{aligned}
$$

where

$$
Y = \frac{\alpha}{(1+c)(2+c)}r^{\frac{-(1+2c)}{1+c}}\left[\left(\frac{-c}{1+c}N + cm - cr\right)(r')^2\right].
$$

Now,

$$
\begin{aligned}
Nr'' - 2hr' + (r')^2 &= NN'' - 2hr' + (r')^2 \\
&= |k|^2 - (N')^2 + (r' - h)^2 - h^2 \\
&= |k|^2 - (N')^2 + (N')^2 - h^2 \\
&= |k|^2 - h^2,
\end{aligned}
$$

and

$$
\begin{aligned}
\left[\frac{-c}{1+c}N + cm - cr\right](r')^2 &= \left[\frac{-c}{1+c}N + c(r - N) - cr\right](r')^2 \\
&= \left[\frac{-c}{1+c}N + cr - cN - cr\right](r')^2 \\
&= \left[-\left(\frac{c}{1+c} + c\right)N\right](r')^2 \\
&\le 0.
\end{aligned}
$$

Hence we get

$$G'' \le [|k|^2 - h^2]\frac{\alpha}{1+c} r^{\frac{-c}{1+c}}.$$

By the assumption $x + h > 0$ and $|k| \le |h|$. Hence $G''(t) \le 0$ on I. The second derivative G'' exists on J also since $N(t) > 1 - m(t) > 0$ for all $t \in J$. On J

$$G(t) = A\left(\frac{r+c}{1+c}\right) - mF\left(\frac{r+c}{1+c}\right).$$

We compute

$$G'(t) = F\left(\frac{r+c}{1+c}\right)\left(\frac{r'}{1+c} - h\right) - mF'\left(\frac{r+c}{1+c}\right)\frac{r'}{1+c}.$$

$$\begin{aligned} G''(t) &= F'\left(\frac{r+c}{1+c}\right)\left[\left(\frac{r'}{1+c}\right)^2 - h\frac{r'}{1+c} - h\frac{r'}{1+c} - m\frac{r''}{1+c}\right] \\ &\quad + F\left(\frac{r+c}{1+c}\right)\left(\frac{r''}{1+c}\right) - mF''\left(\frac{r+c}{1+c}\right)\left(\frac{r'}{1+c}\right)^2 \\ &= \frac{1}{1+c}\left[F'\left(\frac{r+c}{1+c}\right)\left((r')^2 - 2hr' + Nr''\right) - c\frac{(r')^2}{(1+c)}F'\left(\frac{r+c}{1+c}\right)\right. \\ &\quad \left. - N''\left[rF'\left(\frac{r+c}{1+c}\right) - F\left(\frac{r+c}{1+c}\right)\right] - mF''\left(\frac{r+c}{1+c}\right)\frac{(r')^2}{(1+c)}\right]. \end{aligned}$$

The first term on the right is nonpositive since $F' \ge 0$ and $|k| \le |h|$. The second term is also nonpositive. The last term is nonpositive since $F'' \ge 0$ and $m(t) \in (0,1)$ for all $t \in (0,1)$. Also, the third term is also nonpositive; by the triangle inequality, N is convex so $N'' \ge 0$ on J and

$$F\left(\frac{r+c}{1+c}\right) \le \frac{r+c}{1+c}F'\left(\frac{r+c}{1+c}\right) \le \frac{r+rc}{1+c}F'\left(\frac{r+c}{1+c}\right) = rF'\left(\frac{r+c}{1+c}\right).$$

Therefore, on each component of $I \cup J$, the derivative G'' is nonpositive and G' is nonincreasing. So, by Mean Value theorem we have

$$G(1) - G(0) = G'(\tau) \quad (0 < \tau < 1)$$

and $G'(\tau) \le G'(0)$, therefore $G(1) - G(0) \le G'(0)$. □

3. Proof of The Theorem

We may assume that

$$(3.1) \qquad 1 > f_{n-1} > 0 \quad \text{and} \quad |g_{n-1}| > 0 \quad \text{for} \quad n \geq 1, \quad t \in \mathbb{R}.$$

Indeed, for each $0 < \epsilon < 1$, consider a new pair of processes f^ϵ and g^ϵ defined by $f_n^\epsilon = \frac{f_n + \epsilon}{1+2\epsilon}$ and $g_n^\epsilon = \frac{1}{1+2\epsilon}(g_n, \epsilon)$. Then g^ϵ is $\mathbb{R}^{\nu+1}$-valued. Observe that

$$d_0^\epsilon = \frac{d_0 + \epsilon}{1+2\epsilon}, \quad e_0^\epsilon = \frac{(e_0, \epsilon)}{1+2\epsilon}, \quad d_n^\epsilon = \frac{d_n}{1+2\epsilon}, \quad e_n^\epsilon = \frac{(e_n, 0)}{1+2\epsilon} \quad \text{for} \quad n > 1.$$

Thus (3.1) is satisfied with f^ϵ and g^ϵ. Clearly f^ϵ is a submartingale and (1.3) holds for $n \geq 1$. Also we have

$$|e_0^\epsilon| \leq \frac{|e_0| + \epsilon}{1+2\epsilon} \leq \frac{|d_0| + \epsilon}{1+2\epsilon} = |d_0^\epsilon|.$$

For $n \geq 1$, we have

$$|\mathbb{E}(e_n^\epsilon|\mathcal{F}_{n-1})| = \frac{1}{1+2\epsilon}|[\mathbb{E}(e_n|\mathcal{F}_{n-1}), 0]| = \frac{1}{1+2\epsilon}|\mathbb{E}(e_n|\mathcal{F}_{n-1})|$$

and

$$|\mathbb{E}(d_n^\epsilon|\mathcal{F}_{n-1})| = \frac{1}{1+2\epsilon}|\mathbb{E}(d_n|\mathcal{F}_{n-1})|.$$

Thus (1.4) holds for the fair f^ϵ and g^ϵ. So the inequality (1.5) for f^ϵ and g^ϵ gives

$$(3.2) \qquad \mathbb{E}\Phi\left(\frac{|g_n^\epsilon|}{1+c}\right) < \frac{1+c}{2+c}\int_0^\infty \Phi(t)e^{-t}\,dt.$$

As $\epsilon \downarrow 0$ Fatou's lemma and the continuity of Φ enable us to derive from the inequality (3.2) the desired inequality

$$\sup_{n\geq 0} \mathbb{E}\Phi\left(\frac{|g_n|}{1+c}\right) < \frac{1+c}{2+c}\int_0^\infty \Phi(t)e^{-t}\,dt.$$

Let the functions U and V be as in the previous section. From now on we assume that (3.1) holds with f and g. By Lemma 2.2 we have that if (x, y) and $(x+h, y+k)$ are in S, and $|k| \le |h|$, then

$$U(x+h, y+k) \le U(x,y) + U_x(x,y)h + U_y(x,y) \cdot k. \tag{3.3}$$

Fix $n \ge 1$. Then (3.3), with $x = f_{n-1}, h = d_n, y = g_{n-1}$ and $k = e_n$, gives the inequality

$$U(f_n, g_n) - U(f_{n-1}, g_{n-1}) \le U_x(f_{n-1}, g_{n-1})d_n + U_y(f_{n-1}, g_{n-1}) \cdot e_n. \tag{3.4}$$

Thus all random variable in (3.4) are integrable. The integrability follows from the boundedness of f_{n-1} and g_{n-1} although, of course, g need not be uniformly bounded. Also observe that $U(f_{n-1}, g_{n-1}), U_x(f_{n-1}, g_{n-1})$ and $U_y(f_{n-1}, g_{n-1})$ are $\mathcal{F}_{n-1}$-measurable. Thus conditioning on $\mathcal{F}_{n-1}$ we get

$$\mathbb{E}\Big(U(f_n, g_n) - U(f_{n-1}, g_{n-1})\Big|\mathcal{F}_{n-1}\Big) = \mathbb{E}\Big(U(f_n, g_n)\Big|\mathcal{F}_{n-1}\Big) - U\Big(f_{n-1}, g_{n-1}\Big),$$

$$\mathbb{E}\Big(U_x(f_{n-1}, g_{n-1})d_n\Big|\mathcal{F}_{n-1}\Big) = U_x\Big(f_{n-1}, g_{n-1}\Big)\mathbb{E}\Big(d_n\Big|\mathcal{F}_{n-1}\Big)$$

and

$$\mathbb{E}\Big(U_y(f_{n-1}, g_{n-1}) \cdot e_n\Big|\mathcal{F}_{n-1}\Big) = U_y\Big(f_{n-1}, g_{n-1}\Big) \cdot \mathbb{E}\Big(e_n\Big|\mathcal{F}_{n-1}\Big).$$

From (3.4) we get

$$\begin{aligned} &\mathbb{E}\Big(U(f_n, g_n)\Big|\mathcal{F}_{n-1}\Big) - U\Big(f_{n-1}, g_{n-1}\Big) \\ &\le U_x\Big(f_{n-1}, g_{n-1}\Big)\mathbb{E}\Big(d_n\Big|\mathcal{F}_{n-1}\Big) + U_y\Big(f_{n-1}, g_{n-1}\Big) \cdot \mathbb{E}\Big(e_n\Big|\mathcal{F}_{n-1}\Big). \end{aligned} \tag{3.5}$$

Since f is a submartingale

$$\mathbb{E}(d_n|\mathcal{F}_{n-1}) \ge 0.$$

Using the Cauchy-Schwarz inequality and the assumption (1.4) we have

$$\begin{aligned} U_y(f_{n-1}, g_{n-1}) \cdot \mathbb{E}(e_n|\mathcal{F}_{n-1}) &\le |U_y(f_{n-1}, g_{n-1})||\mathbb{E}(e_n|\mathcal{F}_{n-1})| \\ &\le |U_y(f_{n-1}, g_{n-1})|c(\mathbb{E}(d_n|\mathcal{F}_{n-1})). \end{aligned}$$

Thus (3.5) gives

$$\begin{aligned}&\mathbb{E}\Big(U(f_n,g_n)\Big|\mathcal{F}_{n-1}\Big)-U\Big(f_{n-1},g_{n-1}\Big)\\&\leq\Big(U_x(f_{n-1},g_{n-1})+c\Big|U_y(f_{n-1},g_{n-1})\Big|\Big)\Big(\mathbb{E}\Big(d_n\Big|\mathcal{F}_{n-1}\Big)\Big)\leq 0\end{aligned}$$

or

$$\mathbb{E}\Big(U(f_n,g_n)\Big|\mathcal{F}_{n-1}\Big)\leq U\Big(f_{n-1},g_{n-1}\Big). \tag{3.6}$$

In the above we used the inequality $U_x+c|U_y|\leq 0$ from Lemma 2.1. But from the definition of conditional expectation we have

$$\mathbb{E}\Big(\mathbb{E}\Big(U(f_n,g_n)\Big|\mathcal{F}_{n-1}\Big)\Big)=\mathbb{E}U\Big(f_n,g_n\Big).$$

Thus taking expectation in (3.6), we get

$$\mathbb{E}U(f_n,g_n)\leq\mathbb{E}U(f_{n-1},g_{n-1}). \tag{3.7}$$

Note that

$$V(x,y)=\Phi\Big(\frac{|y|}{1+c}\Big)=B\Big(\frac{|y|}{1+c}+1\Big)=B\Big(\frac{1+|y|+c}{1+c}\Big)=U(1,y).$$

Thus Lemma 2.2 gives

$$V(x,y)=U(1,y)=U(x+(1-x),y)\leq U(x,y)+U_x(x,y)(1-x)\leq U(x,y).$$

Hence for all $n\geq 0$

$$\mathbb{E}V(f_n,g_n)\leq\mathbb{E}U(f_n,g_n). \tag{3.8}$$

Also, Lemma 2.1, Lemma 2.2 and the definition of U imply that if $|y|\leq x$, then $(x/2)+|y/2|=(x/2)+|y|/2\leq x\leq 1$ and

$$U\Big(\frac{x}{2}+\frac{x}{2},\frac{y}{2}+\frac{y}{2}\Big)\leq U\Big(\frac{x}{2},\frac{y}{2}\Big)<\frac{(1+c)\alpha}{2+c}-\frac{c\alpha}{2+c}\Big(\frac{x}{2}\Big)^{\frac{2+c}{1+c}}$$

Hence,

$$U(f_0, g_0) < \frac{(1+c)}{2+c}\alpha - \frac{c\alpha}{2+c}(f_0)^{\frac{2+c}{1+c}} \leq \frac{(1+c)}{2+c}\alpha$$

and

$$\mathbb{E}U(f_0, g_0) < \frac{(1+c)}{2+c}\alpha. \tag{3.9}$$

Now we have

$$\mathbb{E}V(f_n, g_n) = \mathbb{E}\Phi\left(\frac{|g_n|}{1+c}\right)$$

and (1.3) follows from

$$\mathbb{E}V(f_n, g_n) < \frac{(1+c)}{2+c}\alpha.$$

Combining (3.7), (3.8) and (3.9) we get

$$\mathbb{E}V(f_n, g_n) \leq \mathbb{E}U(f_n, g_n) \leq \cdots \leq \mathbb{E}U(f_0, g_0) < \frac{(1+c)}{2+c}\alpha.$$

This completes the proof of the inequality in Theorem 1.1.

ACKNOWLEDGEMENT. The author would like to thank Professor Changsun Choi for his guidance and kindness during the research of this paper.

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Young-Ho Kim
Department of Mathematics
Changwon National University
Changwon 641-773, Korea

Byung-Il Kim
Department of Mathematics
Chung-Ang University
Seoul 256-756, Korea