Comm. Korean Math. Soc. **13** (1998), No. 1, pp. 151–158

ON THE STRONG LAW OF LARGE NUMBERS FOR LINEARLY POSITIVE QUADRANT DEPENDENT RANDOM VARIABLES

TAE-SUNG KIM AND HYE-YOUNG SEO

ABSTRACT. In this note we derive inequalities of linearly positive quadrant dependent random variables and obtain a strong law of large numbers for linearly positive quardant dependent random variables. Our results imply an extension of Birkel's strong law of large numbers for associated random variables to the linear positive quadrant dependence case.

1. Introduction

Lehmann(1966) introduced a simple and natural definition of positive dependence: A sequence $\{X_j : j \geq 1\}$ of random variables is said to be pairwise positive quadrant dependent(pairwise PQD) if for any real r_i, r_j and $i \neq j$

$$P\{X_i > r_i, X_j > r_j\} \geq P\{X_i > r_i\}P\{X_j > r_j\}.$$

A much stronger concept than PQD was considered by Esary, Proschan, and Walkup(1967): A sequence $\{X_j : j \geq 1\}$ of random variables is said to be associated if for any finite collection $\{X_{j(1)}, \cdots, X_{j(n)}\}$ and any real coordinatwise nondecreasing functions f, g on R^n

$$\mathrm{Cov}(f(X_{j(1)}, \cdots, X_{j(n)}), g(X_{j(1)}, \cdots, X_{j(n)})) \geq 0$$

whenever the covariance is defined. A new concept of positive dependence between pairwise PQD and association was studied by Newman(1984): A sequence $\{X_j : j \geq 1\}$ of random variables is said to be

Received July 3, 1997. Revised October 7, 1997.

1991 Mathematics Subject Classification: 60E15, 60F05.

Key words and phrases: Positive quadrant dependent, linearly positive quadrant dependent, associated, strong law of large numbers, demimartingale inequality.

This work was supported by KOSEF grant 971-0105-025-1

linearly positive quadrant dependent(LPQD) if for any disjoint A, B and positive r_j's

$$\sum_{i\in A} r_i X_i \text{ and } \sum_{j\in B} r_j X_j \text{ are PQD.}$$

Newman and Wright(1982) derived a maximal inequality of associated random variables and Birkel(1989) applied it to prove a strong law of large numbers(SLLN) for associated sequences.

In this paper, we investigate some inequalities of LPQD random variables and extend Birkel's(1989) SLLN for associated random variables to the LPQD case.

2. Inequalities of LPQD sequences

The following theorem and proof are based on Newman and Wrights' demimartingale inequality [5]. Let

$$S_n^* = \max(S_1, \cdots, S_n), \tag{2.1}$$

$$\begin{aligned} S_{n,j} =& j\text{th largest of } (S_1, \cdots, S_n), \text{ if } j \le n, \\ &\min(S_1, \cdots, S_n) = S_{n,n}, \text{ if } j > n, \end{aligned} \tag{2.2}$$

so that $S_{n,1} = S_n^*$, where $S_n = X_1 + \cdots + X_n$.

Theorem 2.1. *Let $\{X_j : j \ge 1\}$ be an LPQD sequence with $EX_j = 0$. Then for any n and j,*

$$E\left(\int_0^{S_{n,j}} u\,du\right) \le E(S_n S_{n,j}). \tag{2.3}$$

Proof. For fixed n and j, let $Y_k = S_{k,j}$ and $Y_0 = 0$, then

$$S_n Y_n = \sum_{k=0}^{n-1} S_{k+1}(Y_{k+1} - Y_k) + \sum_{k=1}^{n-1}(S_{k+1} - S_k)Y_k. \tag{2.4}$$

Note from the definition of $S_{n,j}$, that

$$\text{(2.5)} \qquad \text{for } k < j \text{ either } Y_k = Y_{k+1} \text{ or } S_{k+1} = Y_{k+1},$$

$$\text{(2.6)} \qquad \text{for } k \geq j \text{ either } Y_k = Y_{k+1} \text{ or } S_{k+1} \geq Y_{k+1}.$$

Thus for any k,

$$\text{(2.7)} \qquad S_{k+1}(Y_{k+1} - Y_k) \geq Y_{k+1}(Y_{k+1} - Y_k) \geq \int_{Y_k}^{Y_{k+1}} u\,du,$$

so that (2.4) yields

$$\text{(2.8)} \qquad S_n Y_n \geq \int_0^{Y_n} u\,du + \sum_{k=1}^{n-1}((S_{k+1} - S_k)Y_k).$$

Next we note that

$$\text{(2.9)} \qquad E((S_{k+1} - S_k)Y_k) = \operatorname{Cov}((S_{k+1} - S_k), Y_k) \geq 0$$

by the definition of LPQD since Y_k is a sum of some X_i's such that i is less than $k+1$ and $S_{k+1} - S_k = X_{k+1}$. By taking the expectation of (2.8) and using (2.9), (2.3) is derived since $Y_n = S_{n,j}$. □

Corollary 2.2. *Let $\{X_j : j \geq 1\}$ be an LPQD sequence with $EX_j = 0$. Then for any n and j,*

$$\text{(2.10)} \qquad E((S_{n,j} - S_n)^2) \leq E(S_n^2)$$

PROOF. It follows from (2.3) that

$$E(S_{n,j}^2) - 2E(S_n S_{n,j}) \le 0.$$

By adding $E(S_n^2)$ both sides of the above inequality, (2.10) is obtained. □

In the next corollary we show the maximal inequalities which will be used to obtain weak convergence and strong law of large numbers for LPQD sequences. Inequalities of the corollary are essentially identical to those of associated random variables(See [5, Corollary 5])

COROLLARY 2.3. *Let $\{X_j : j \ge 1\}$ be a sequence of LPQD random variables with $EX_j = 0$. Then*

$$E(S_{n,j}^2) \le E(S_n^2) = s_n^2 \tag{2.11}$$

and for $\lambda_1 > \lambda_2$,

$$\left(\frac{1 - s_n^2}{(\lambda_2 - \lambda_1)^2}\right) P(S_n^* \ge \lambda_2) \le P(S_n \ge \lambda_1), \tag{2.12}$$

so that for $\alpha_1 < \alpha_2$ with $\alpha_2 - \alpha_1 > 1$,

(2.13)
$$P(\max(|S_1|, \cdots, |S_n|) \ge \alpha_2 s_n) \le \frac{(\alpha_2 - \alpha_1)^2}{(\alpha_2 - \alpha_1)^2 - 1} P(|S_n| \ge \alpha_1 s_n).$$

PROOF. It is obvious that $X_n, X_{n-1}, \cdots, X_2, X_1$ are a sequence of LPQD random variables. Define $T_1 = 0$, and $T_k = \sum_{i=n-k+2}^{n} X_i$, for $k = 2, 3, \cdots, n+1$. Then by the similar arguments in the proof of Theorem 2.1

$$E((T_{n+1} - T_n)T_{n,n-j+1}) = \mathrm{Cov}((T_{n+1} - T_n), T_{n,n-j+1}) \ge 0 \tag{2.14}$$

according to (2.9) and hence by (2.3) we have

$$E\left(\frac{T_{n,n-j+1}^2}{2}\right) \leq E(T_n T_{n,n-j+1}). \tag{2.15}$$

Thus it follows from (2.14) and (2.15) that

$$E\left(\frac{T_{n,n-j+1}^2}{2}\right) \leq E(T_n T_{n,n-j+1}) \leq E(T_{n+1} T_{n,n-j+1}). \tag{2.16}$$

Now (2.16) implies that

$$E((T_{n+1} - T_{n,n-j+1})^2) \leq E(T_{n+1}^2),$$

and this is the same as (2.11) since $T_{n+1} = S_n$ and

$$\begin{aligned}&T_{n+1} - T_{n,n-j+1}\\ =&j\text{th largest of } (T_{n+1} - T_n, T_{n+1} - T_{n-1}, \cdots, T_{n+1} - T_1)\\ =&S_{n,j}.\end{aligned}$$

To obtain (2.12) we first note that

$$\begin{aligned} P(S_n^* \geq \lambda_2) &= P(S_n^* \geq \lambda_2, S_n \geq \lambda_1) + P(S_n^* \geq \lambda_2, S_n < \lambda_1) \\ &\leq P(S_n \geq \lambda_1) + P(S_n^* \geq \lambda_2, S_n^* - S_n > \lambda_2 - \lambda_1). \end{aligned} \tag{2.17}$$

Since for all proper subset A of $\{1, 2, \cdots, n\}$ such that $S_n^* = \sum_{i \in A} X_i$ there exists a disjoint subset $B = \{1, 2, \cdots, n\}/A$ such that $S_n - S_n^* = \sum_{i \in B} X_i$, S_n^* and $S_n - S_n^*$ are PQD. Therefore it follows from (2.17) that

$$\begin{aligned} &P(S_n^* \geq \lambda_2, S_n^* - S_n > \lambda_2 - \lambda_1) \\ &\quad \leq P(S_n^* \geq \lambda_2) P(S_n^* - S_n > \lambda_2 - \lambda_1) \end{aligned} \tag{2.18}$$

according to Lehmann(1966). Now by combining (2.17), (2.18), Chebyshev's inequality, and (2.10) we have

$$\begin{aligned} P(S_n^* \geq \lambda_2) &\leq P(S_n \geq \lambda_1) + P(S_n^* \geq \lambda_2) E((S_n^* - S_n)^2)/(\lambda_2 - \lambda_1)^2 \\ &\leq P(S_n \geq \lambda_1) + P(S_n^* \geq \lambda_2) E(S_n^2)/(\lambda_2 - \lambda_1)^2, \end{aligned}$$

which immediately yields (2.12). Finally, we may obtain (2.13) by taking $\lambda_i = \alpha_i s_n$ and adding to (2.12) the analogous inequality obtained after replacing all X_i's by their negatives(which are also necessarily LPQD). □

3. A SLLN for LPQD sequence

THEOREM 3.1. *Let $\{X_j : j \geq 1\}$ be a sequence of linearly positive quadrant dependent random variables with $EX_j = 0$, $EX_j^2 < \infty$. Assume*

$$\sum_{j=1}^{\infty} j^{-2} Cov(X_j, S_j) < \infty. \tag{3.1}$$

Then as $n \to \infty$, $n^{-1} S_n \to 0$ almost surely.

PROOF. We will use the method in the proof of Theorem 2 of Birkel [1]. Let $\epsilon > 0$ be given. Then by Chebyshev's inequality

$$\begin{aligned} &\sum_{n=1}^{\infty} P\{2^{-n}|S_{2^n}| \geq \epsilon\} \\ &\leq \epsilon^{-2} \sum_{n=1}^{\infty} 4^{-n} \operatorname{Var}(S_{2^n}) \\ &\leq 2\epsilon^{-2} \sum_{j=1}^{\infty} \left(\sum_{n:2^n \geq j} 4^{-n} \operatorname{Cov}(X_j, S_j) \right) \\ &\leq \frac{8}{3} \epsilon^{-2} \sum_{j=1}^{\infty} j^{-2} \operatorname{Cov}(X_j, S_j) < \infty \end{aligned} \tag{*}$$

Thus the Borel-Cantelli lemma implies that, as $n \to \infty$, $2^{-n} S_{2^n} \to 0$ almost surely.

Now by standard arguments it suffices to prove that, as $n \to \infty$,

$$2^{-n} \max_{2^n < k \leq 2^{n+1}} |S_k - S_{2^n}| \to 0 \text{ almost surely.} \tag{3.2}$$

Let $n \geq 1$ be given. Using Chebyshev's inequality and applying Corollary 2.3 to the random variables $X_{2^n+1}, X_{2^n+2}, \cdots, X_{2^{n+1}}$ we obtain

$$\begin{aligned} &P\{2^{-n} \max_{2^n < k \leq 2^{n+1}} (S_k - S_{2^n}) \geq \epsilon\} \\ &\leq \epsilon^{-2} 4^{-n} E \left(\max_{2^n < k \leq 2^{n+1}} (S_k - S_{2^n}) \right)^2 \\ &\leq \epsilon^{-2} 4^{-n} \mathrm{Var}(S_{2^{n+1}} - S_{2^n}) \\ &\leq \epsilon^{-2} 4^{-n} \mathrm{Var}(S_{2^{n+1}}), \end{aligned} \tag{3.3}$$

since the X_j's are nonnegatively correlated. Replace the random variables X_j by their negatives(which are also LPQD). We get the analogous inequality

$$P\{2^{-n} \max_{2^n < k \leq 2^{n+1}} -(S_k - S_{2^n}) \geq \epsilon\} \leq \epsilon^{-2} 4^{-n} \mathrm{Var}(S_{2^{n+1}}). \tag{3.4}$$

(3.3) and (3.4) imply

$$P\{2^{-n} \max_{2^n < k \leq 2^{n+1}} |S_k - S_{2^n}| \geq \epsilon\} \leq 2\epsilon^{-2} 4^{-n} \mathrm{Var}(S_{2^{n+1}}),$$

and hence

$$\sum_{n=1}^{\infty} P\{2^{-n} \max_{2^n < k \leq 2^{n+1}} |S_k - S_{2^n}| \geq \epsilon\} \leq 8\epsilon^{-2} \sum_{n=1}^{\infty} 4^{-(n+1)} \mathrm{Var}(S_{2^{n+1}}) < \infty,$$

according to the consideration above(*). Again applying the Borel-Cantelli lemma, we obtain (3.2) which completes the proof of Theorem 3.1. □

REMARK. Theorem 3.1 shows that the strong law of large numbers of Birkel(1989)(Theorem 2 of [1]) for associated sequences still holds for LPQD processes.

ACKNOWLEDGMENT. The authors would like to thank the refree for helpful comments.

References

[1] Birkel, T., *A note on the strong law of large numbers for positively dependent random variables*, Statist. Probab. Lett. **7** (1989), 17-20.

[2] Esary, J., Proschan, F. and Walkup, D., *Association of random variables with applications*, Ann. Math. Statist. **38** (1967), 1466-1474.

[3] Lehmann, E. L., *Some concepts of dependence*, Ann. Math. Statist. **37** (1966), 1137-1153.

[4] Newman, C. M., *Asymptotic independence and limit theorems for positively and negatively dependent random variables*, in: Y. L. Tong, ed., Inequalities in Statistics and Probability. I. M. S., Hayward, CA (1984), 127-140.

[5] Newman, C. M. and Wright, A. L., *Associated random variables and martingale inequalities*, Z. Wahrsch. Verw. Geb. **59** (1982), 361-371.

Department of Statistics
Won Kwang University
Iksan 570-749, Korea