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ON SOME CR-SUBMANIFOLDS OF $(n-1)$ CR-DIMENSION IN A COMPLEX PROJECTIVE SPACE

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ABSTRACT. The purpose of this paper is to give some characterizations of n-dimensional CR-submanifolds of $(n-1)$ CR-dimension immersed in a complex projective space $CP^{(n+p)/2}$ with Fubini-Study metric and we study an n-dimensional compact, orientable, minimal CR-submanifold of $(n-1)$ CR-dimension in $CP^{(n+p)/2}$.

1. Introduction

Let M be a connected real n-dimensional submanifold of real codimension p of a complex manifold $\overline{M}$ with complex structure J. If the maximal J-invariant subspace $JT_xM \cap T_xM$ of T_xM has constant dimension for any x in M, then M is called a *CR-submanifold* and the constant is called the *CR-dimension* of M [8]. Now let M be an n-dimensional CR-submanifold of $(n-1)$ CR-dimension of $\overline{M}$. Then M admidts an induced almost contact structure [11, 14, 15]. A typical example of an n-dimensional CR-submanifold of $(n-1)$ CR-dimension is a real hypersurface. When the ambient manifold $\overline{M}$ is a complex projective space, real hypersurfaces are investigated by many authors [2, 7, 9, 10, 15, 16] in connection with the shape operator and the induced almost contact structure.

Recently, from these results, n-dimensional CR-submanifolds of $(n-1)$ CR-dimension in a complex projective space $CP^{\frac{n+p}{2}}$ also have been

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investigated by several authors [3, 6, 11]. Especially, by using the Erbacher's reduction theorem [4], Okumura and Vanhecke [11] proved the following :

THEOREM A. *Let M be a real submanifold of $CP^{\frac{n+p}{2}}$ with maximal holomorphic subspace of dimension $n-1$. If the almost contact metric structure of M is normal and the normal vector field ξ_1 is parallel with respect to the normal connection, then $\pi^{-1}(M)$ is locally a product of $M_1 \times M_2$ where M_1 and M_2 belong to some odd-dimensional spheres (π is the Hopf fibration $S^{n+p+1}(1) \to CP^{\frac{n+p}{2}}$).*

The purpose of the present paper is to give another characterization of CR-submanifolds of $(n-1)$ CR-dimension immersed in $CP^{\frac{n+p}{2}}$ by using the following integral formula due to Yano [16, 17] :

$$\int_M \{Ric(X,X) + \frac{1}{2}\|\mathcal{L}_X g\|^2 - \|\nabla X\|^2 - (divX)^2\} * 1 = 0, \tag{1.1}$$

where X is an arbitrary tangent vector field to M, $\mathcal{L}_X$ the Lie derivative with respect to X, ∇ the Riemannian connection induced on M, $*1$ the volume element of M and $\|Y\|$ the length of a vector field Y with respect to the Riemannian metric on M.

As an application of the integral formula (1.1), we will prove

THEOREM 1. *Let M be an n-dimensional compact, orientable, minimal CR-submanifold of $(n-1)$ CR-dimension in $CP^{\frac{n+p}{2}}$. If the normal vector field ξ_1 is parallel with respect to the normal connection, then*

$$\int_M (trA_1^2) * 1 \geqq (n-1)Vol(M). \tag{1.2}$$

In particular, the equality holds good only when $\pi^{-1}(M)$ is locally a product of $M_1 \times M_2$ where M_1 and M_2 belong to some odd-dimensional spheres.

Next, by using the Green's theorem for a global function defined on M (for details, see (3.12)), we will prove

THEOREM 2. *Let M be a 3-dimensional compact, orientable, minimal CR-submanifold of 2 CR-dimension in $CP^{\frac{3+p}{2}}$. If the normal vector field ξ_1 is parallel with respect to the normal connection and if*

$$\sum_{\alpha=1}^{p}\{tr(A_1A_\alpha)\}^2 \leqq 4,$$

then $\pi^{-1}(M)$ is locally a product of $M_1 \times M_2$ where M_1 and M_2 belong to some odd-dimensional spheres.

In section 2 we derive a series of useful formulas for n-dimensional CR-submanifolds of $(n-1)$ CR-dimension in a complex projective space $CP^{\frac{n+p}{2}}$ for later use. Finally, in section 3 we will give the complete proof of the theorems above.

2. Preliminaries

Let M be an n-dimensional Riemannian manifold isometrically immersed in a complex space form $M^{\frac{n+p}{2}}(c)$ and denote by $(J,\overline{g})$ the Kähler structure on $M^{\frac{n+p}{2}}(c)$. For x of M we denote by T_xM and $T_xM^\perp$ the tangent space and normal space of M at x, respectively.

From now on we assume that M is an n-dimensional CR-submanifold of $(n-1)$ CR-dimension, that is,

$$\dim(JT_xM \cap T_xM) = n-1.$$

This implies that $dimM$ is odd [3, 11].

Note that the definition of CR-submanifold of $(n-1)$ CR-dimension meets the definition of CR-submanifold in the sense of Bejancu [1].

Furthermore, our hypothesis implies that there exists a unit vector field ξ_1 normal to M such that $JTM \subset TM \oplus Span\{\xi_1\}$. Hence, for any tangent vector field X and for a local orthonormal basis $\{\xi_\alpha, \alpha = 1,\ldots,p\}$ of normal vectors to M, we have the following decomposition in tangental and normal components :

$$JX = FX + u^1(X)\xi_1, \tag{2.1}$$

$$J\xi_\alpha = -U_\alpha + P\xi_\alpha, \quad \alpha = 1,\ldots,p. \tag{2.2}$$

Then it is easily seen that F and P are skew-symmetric endomorphisms acting on T_xM and $T_xM^\perp$, respectively. Moreover, the Hermitian property of J implies

$$g(FU_\alpha, X) = -u^1(X)\overline{g}(\xi_1, P\xi_\alpha), \tag{2.3}$$

$$g(U_\alpha, U_\beta) = \delta_{\alpha\beta} - \overline{g}(P\xi_\alpha, P\xi_\beta). \tag{2.4}$$

From $\overline{g}(JX, \xi_\alpha) = -\overline{g}(X, J\xi_\alpha)$, we get

$$g(X, U_\alpha) = u^1(X)\delta_{1\alpha} \tag{2.5}$$

and hence

$$g(U_1, X) = u^1(X), \quad U_\alpha = 0, \quad \alpha = 2, \dots, p. \tag{2.6}$$

Next, applying J to (2.1) and using (2.2) and (2.6) yield

$$F^2X = -X + u^1(X)U_1, \quad u^1(X)P\xi_1 = -u^1(FX)\xi_1. \tag{2.7}$$

Since P is skew-symmetric, (2.3) and the second equation of (2.7) give

$$u^1(FX) = 0, \quad P\xi_1 = 0, \quad FU_1 = 0. \tag{2.8}$$

So, (2.2) may be written in the form

$$J\xi_1 = -U_1, \quad J\xi_\alpha = P\xi_\alpha, \quad \alpha = 2, \dots, p \tag{2.9}$$

and further, we may put

$$P\xi_\alpha = \sum_{\beta=2}^{p} P_{\alpha\beta}\xi_\beta, \quad \alpha = 2, \dots, p,$$

where $(P_{\alpha\beta})$ is a skew-symmetric matrix which satisfies

$$\sum_\beta P_{\alpha\beta}P_{\beta\gamma} = -\delta_{\alpha\gamma}. \tag{2.11}$$

These results imply that (F, U_1, u^1, g) defines an almost contact metric structure on (M, g) [15].

Now, let $\overline{\nabla}$ and ∇ denote the Levi-Civita connection on $\overline{M}$ and M, respectively and denote by D the normal connection induced from $\overline{\nabla}$ in the normal bundle $TM^{\perp}$ of M. The Gauss and Weingarten equations are

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.12}$$

$$\overline{\nabla}_X \xi_\alpha = -A_\alpha X + D_X \xi_\alpha, \quad \alpha = 1, \ldots, p \tag{2.13}$$

for any tangent vectors X, Y to M. Here h denotes the second fundamental form and A_α is the shape operator corresponding to ξ_α. They are related by

$$h(X, Y) = \sum_{\alpha=1}^{p} g(A_\alpha X, Y)\xi_\alpha.$$

Furthermore, putting

$$D_X \xi_\alpha = \sum_{\beta=1}^{p} s_{\alpha\beta}(X)\xi_\beta, \tag{2.14}$$

it follows that $(s_{\alpha\beta})$ is the skew-symmetric matrix of connection forms of D. Finally, if the ambient space $\overline{M}$ is of constant holomorphic sectional curvature 4, the Gauss, Codazzi, Ricci equations and Ricci curvature are respectively given as follows [3, 11] :

$$\begin{aligned} R(X, Y)Z =& g(Y, Z)X - g(X, Z)Y + g(FY, Z)FX \\ & - g(FX, Z)FY - 2g(FX, Y)FZ \\ & + \sum_{\alpha} g(A_\alpha Y, Z)A_\alpha X - \sum_{\alpha} g(A_\alpha X, Z)A_\alpha Y, \end{aligned} \tag{2.15}$$

$$\begin{aligned} & (\nabla_X A_1)Y - (\nabla_Y A_1)X \\ & = g(X, U_1)FY - g(Y, U_1)FX - 2g(FX, Y)U_1, \end{aligned} \tag{2.16}$$

$$\overline{g}(R^{\perp}(X, Y)\xi_\alpha, \xi_1) = g([A_1, A_\alpha]X, Y), \quad \alpha = 2, \ldots, p, \tag{2.17}$$

$$(2.18)\qquad \begin{aligned} Ric(X,Y) =&(n+2)g(X,Y) - 3u^1(X)u^1(Y) \\ &+ \sum_\alpha (trA_\alpha) g(A_\alpha X, Y) - \sum_\alpha g(A_\alpha^2 X, Y) \end{aligned}$$

for any tangent vector fields X, Y, Z to M. R denotes the Riemannian curvature tensor of M. $R^\perp$ is the curvature tensor of the normal connection D.

3. Proof of Theorems

In this section, we consider the case of a complex projective space $\overline{M} = CP^{\frac{n+p}{2}}$ and $\overline{\nabla} J = 0$. Then by differentiating (2.1) and (2.2) covariantly and comparing the tangential and normal parts, we have

$$(3.1)\qquad (\nabla_Y F)X = u^1(X)A_1 Y - g(A_1 X, Y)U_1,$$

$$(3.2)\qquad (\nabla_Y u^1)(X) = g(FA_1 Y, X),$$

$$(3.3)\qquad \nabla_X U_1 = FA_1 X,$$

$$(3.4)\qquad g(A_\alpha U_1, X) = -\sum_{\beta=2}^{p} s_{1\beta}(X) P_{\beta\alpha}, \quad \alpha = 2, \ldots, p$$

for any tangent vector fields X, Y to M.

On the other hand, the almost contact metric structure (F, U_1, u^1, g) is said to be *normal* if the tensor field S defined by

$$(3.5)\qquad \begin{aligned} S(X,Y) =&[FX, FY] - F[FX, Y] - F[X, FY] \\ &+ F^2[X,Y] + 2du^1(X,Y)U_1 \end{aligned}$$

vanishes identically [11, 18]. By using (2.7), (2.8), (3.1), (3.2) and (3.5), we can easily prove the following lemma.

LEMMA 3.1 [11]. *Let M be an n-dimensional CR-submanifold of $(n-1)$ CR-dimension in a complex space form. If the normal vector field ξ_1 is parallel with respect to the normal connection. Then the following conditions are equivalent to each other :*

1. *the induced almost contact metric structure (F, U_1, u^1, g) is normal,*
2. $A_1 F = F A_1$.

PROOF OF THEOREM 1. Putting $X = U_1$ in (1.1) gives

$$\int_M \{Ric(U_1, U_1) + \frac{1}{2}\|\mathcal{L}_{U_1} g\|^2 - \|\nabla U_1\|^2 - (divU_1)^2\} * 1 = 0. \tag{3.6}$$

On the other hand, since ξ_1 is parallel with respect to the normal connection D, from (2.14) and (3.4) we have

$$A_\alpha U_1 = 0, \quad \alpha = 2, \ldots, p. \tag{3.7}$$

The Ricci equation (2.18), together with (3.7), yields

$$Ric(U_1, U_1) = n - 1 + (trA_1)g(A_1U_1, U_1) - g(A_1^2 U_1, U_1). \tag{3.8}$$

From (3.3) it follows that

$$divU_1 = tr(FA_1) = 0. \tag{3.9}$$

We have from (3.3)

$$\begin{aligned}(\mathcal{L}_{U_1} g)(X, Y) &= g(\nabla_X U_1, Y) + g(\nabla_Y U_1, X) \\ &= g((FA_1 - A_1F)X, Y).\end{aligned} \tag{3.10}$$

And using (2.7) we get

$$\|\nabla U_1\|^2 = trA_1^2 - g(A_1^2 U_1, U_1). \tag{3.11}$$

Since M is minimal, $trA_\alpha = 0$, $\alpha = 1, \ldots, p$. Therefore substituting (3.8), (3.9) and (3.11) into (3.6), we obtain

$$\int_M \{\frac{1}{2}\|\mathcal{L}_{U_1} g\|^2 + (n-1) - trA_1^2\} * 1 = 0.$$

Thus we have the inequality (1.2). Now we assume that the equality of (1.2) holds good. Then the hypothesis implies $\|\mathcal{L}_{U_1} g\|^2 = 0$ and consequently $A_1F = FA_1$ because of (3.10). Combining Theorem A, Lemma 3.1 and $A_1F = FA_1$, we have the required result of Theorem 1. □

From Theorem 1, we have immediately

COROLLARY 3. *Let M be an n-dimensional compact, orientable, minimal CR-submanifold of $(n-1)$ CR-dimension in $CP^{\frac{n+p}{2}}$. If the normal vector field ξ_1 is parallel with respect to the normal connection and $n-1 \geqq trA_1^2$, then $\pi^{-1}(M)$ is locally a product of $M_1 \times M_2$ where M_1 and M_2 belong to some odd-dimensional spheres.*

PROOF OF THEOREM 2. First of all, we consider the following equation :

$$\frac{1}{2}\Delta\|A_1\|^2 = (\Delta A_1)A_1 + \|\nabla A_1\|^2, \tag{3.12}$$

where Δ denotes the Laplacian operator and $\|A_1\|^2 = trA_1^2$.
Putting

$$(\overset{*}{\nabla}_X A_1)Y = (\nabla_X A_1)Y + g(Y, U_1)FX + g(FX, Y)U_1,$$

we have [6]

$$\|\nabla A_1\|^2 = \|\overset{*}{\nabla} A_1\|^2 + 2(n-1). \tag{3.13}$$

Substituting (3.13) into (3.12), we have

$$\frac{1}{2}\Delta\|A_1\|^2 = (\Delta A_1)A_1 + \|\overset{*}{\nabla} A_1\|^2 + 2(n-1). \tag{3.14}$$

On the other hand, since ξ_1 is parallel with respect to the normal connection D, from (2.17) we have

$$g(A_1A_\alpha X, Y) = g(A_\alpha A_1 X, Y), \quad \alpha = 2, \ldots, p. \tag{3.15}$$

From (2.7) and (3.10), we get

$$\|\mathcal{L}_{U_1} g\|^2 = 2\{tr(FA_1)^2 + trA_1^2 - g(A_1^2U_1, U_1)\}. \tag{3.16}$$

Moreover, since M is minimal, we can easily see that

$$(\Delta A_1)A_1 = (n-3)trA_1^2 + 3\|\mathcal{L}_{U_1} g\|^2 - \sum_{\alpha=1}^{p}\{tr(A_1A_\alpha)\}^2 \tag{3.17}$$

with the help of the Ricci identity, (2.8), (2.15), (2.16), (2.18), (3.1), (3.3), (3.15) and (3.16). Therefore by (3.14) and (3.17) we have

$$\frac{1}{2}\Delta\|A_1\|^2 =(n-3)trA_1^2 + 3\|\mathcal{L}_{U_1}g\|^2 + [2(n-1) - \sum_{\alpha=1}^{p}\{tr(A_1A_\alpha)\}^2] + \|\overset{*}{\nabla}A_1\|^2.$$

Hence we have

$$\int_M [(n-3)trA_1^2 + 3\|\mathcal{L}_{U_1}g\|^2 + \{2(n-1) - \sum_{\alpha=1}^{p}(tr(A_1A_\alpha))^2\} + \|\overset{*}{\nabla}A_1\|^2] * 1 = 0.$$

By the hypotheses of Theorem 2, we have $\|\mathcal{L}_{U_1}g\|^2 = 0$ and hence $A_1F = FA_1$ because of (3.10). Combining Theorem A, Lemma 3.1 and $A_1F = FA_1$, we have the required result of Theorem 2. □

REMARK 1. Taking account of $A_1F = FA_1$, we can easily see that the submanifolds M_1 and M_2 in theorem 2 have dimensions 1 and 3, respectively.

REMARK 2. There does not exist such a submanifold in Theorem 2 with the conditions

$$n > 3 \quad \text{and} \quad \sum_{\alpha=1}^{p}\{tr(A_1A_\alpha)\}^2 \leqq 2(n-1).$$

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