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A STRUCTURE THEOREM FOR p-HYPONORMAL CONTRACTIONS

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ABSTRACT. In this paper we prove a structure theorem for p-hyponormal contractions and also give an example of a p-hyponormal operator which is not *-paranormal.

1. Introduction

Let $\mathcal{H}$ be a complex separable Hilbert space and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. For an operator T, let $\sigma(T), \sigma_p(T)$ and $\sigma_{ap}(T)$ denote the spectrum, the point spectrum and the approximate point spectrum of T, respectively. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *p-hyponormal* if $(T^*T)^p \geq (TT^*)^p$ for $0 < p$. If $p = 1$, then T is hyponormal and if $p = \frac{1}{2}$, then T is semi-hyponormal. It is well known that a p-hyponormal operator is a q-hyponormal operator for $q \leq p$ by Löwner's Theorem(see [10]). But the converse is not true in general. Hyponormal operators have been studied by many authors(see [4], [10] and [12]). D. Xia [13] studied the basic properties of semi-hyponormal operators, and he also gave an example of semi-hyponormal but not hyponormal. The p-hyponormal operator was introduced by A. Aluthge(see [1]). A. Aluthge, M. Chō, H. Huruya, H. Jin, M. Fujii, C. Himeji, A. Matsumoto, R. Nakamote and H. Watanabe studied some properties of p-hyponormal operators([1], [5], [6], [8] and [9]). This paper is a continuation of these papers. In section 2, we give a structure theorem for p-hyponormal contractions for $0 < p \leq 1$. In section 3, we give an example of a p-hyponormal operator which is not *-paranormal.

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2. Results

Throughout this paper, $0 < p \leq 1$. The following theorem due to McCarthy [11] is an operator variant of the Hölder Inequality.

HÖLDER-MCCARTHY THEOREM. *Let A be a positive operator on $\mathcal{H}$. Then the following inequalities hold:*

(1) $(A^r x, x) \leq ||x||^{2(1-r)}(Ax, x)^r$ *for* $x \in \mathcal{H}$ *if* $0 < r \leq 1$.
(2) $(A^r x, x) \geq ||x||^{2(1-r)}(Ax, x)^r$ *for* $x \in \mathcal{H}$ *if* $r \geq 1$.

In this section we will discuss some properties of p-hyponormal operators. For this, we need the following lemma.

LEMMA 1. *Let T be a p–hyponormal contraction on $\mathcal{H}$ and let*

$$\mathcal{N}_p = \{x \in \mathcal{H} | (TT^*)^p x = x\}.$$

Then

(1) *the vectors in $\mathcal{N}_p$ are fixed under $(T^*T)^p$, and*
(2) *$\mathcal{N}_p$ is an invariant subspace for T.*

PROOF. (1) Note that $||(T^*T)^{\frac{p}{2}}|| \leq ||T^*T||^{\frac{p}{2}}$ by Hölder-McCarthy Theorem. Let $x \in \mathcal{N}_p$. Then, since T is a p-hyponormal contraction, we have

$$\begin{aligned}
&||(T^*T)^p x - Ix||^2 \\
=&||(T^*T)^p x||^2 - 2((T^*T)^p x, x) + (x, x) \\
=&||(T^*T)^p x||^2 - 2||(T^*T)^{\frac{p}{2}} x||^2 + ((TT^*)^p x, x) \\
\leq&||(T^*T)^{\frac{p}{2}}||^2 ||(T^*T)^{\frac{p}{2}} x||^2 - 2||(T^*T)^{\frac{p}{2}} x||^2 + ||(T^*T)^{\frac{p}{2}} x||^2 \\
\leq&0.
\end{aligned}$$

Thus, $(T^*T)^p x = x$ for all $x \in \mathcal{N}_p$.

(2) First, since $\mathcal{N}_p$ is the null space of the operator $(TT^*)^p - I$, $\mathcal{N}_p$ is a subspace of $\mathcal{H}$.

Secondly, we will prove that $||Tx||^2 = ||(TT^*)^{\frac{p}{2}} Tx||^2$ for all $x \in \mathcal{N}_p$, using the well known identity on the polar decomposition:

$$U|T|^r U^* = |T^*|^r$$

where $T = U|T|$ is a polar decomposition of T and $r > 0$. Indeed,

$$\begin{aligned}(Tx, Tx) &= (T(T^*T)^p x, Tx) \\ &= (T^*(TT^*)^p Tx, x) \\ &= ((TT^*)^p Tx, Tx),\end{aligned}$$

for all $x \in \mathcal{N}_p$.

Finally, for all $x \in \mathcal{N}_p$, we have

$$\begin{aligned}||(TT^*)^p Tx - Tx||^2 &= ||(TT^*)^p Tx||^2 - 2((TT^*)^p Tx, Tx) + ||Tx||^2 \\ &\le ||(TT^*)^{\frac{p}{2}}||^2 ||(TT^*)^{\frac{p}{2}} Tx||^2 - ||(TT^*)^{\frac{p}{2}} Tx||^2 \\ &\le 0.\end{aligned}$$

Hence, $(TT^*)^p Tx = Tx$ and $Tx \in \mathcal{N}_p$. □

We now have a structure theorem for p-hyponormal contractions:

THEOREM 2. *If T is a p–hyponormal contraction on $\mathcal{H}$, there exists a subspace $\mathcal{M}_p$ of $\mathcal{H}$ such that*

(1) *the vectors in $\mathcal{M}_p$ are fixed under TT^* and T^*T, and*
(2) *$\mathcal{M}_p$ is an invariant subspace for T.*

In particular, if $\mathcal{M}_p$ is nonzero, the restriction $T|\mathcal{M}_p$ is unitary.

PROOF. It is sufficient to prove that $p = \frac{1}{2^k}$ for some $k \in \mathbf{N}$. Suppose that $p = \frac{1}{2^k}$ for some $k \in \mathbf{N}$. Put $\mathcal{M}_p := \mathcal{N}_p$ in Lemma 1. Then, for every $x \in \mathcal{M}_p$, we have

$$TT^*x = \underbrace{(TT^*)^p (TT^*)^p \cdots (TT^*)^p}_{2^k \text{ times}} x = x$$

and

$$T^*Tx = \underbrace{(T^*T)^p (T^*T)^p \cdots (T^*T)^p}_{2^k \text{ times}} x = x.$$

Furthermore, for every $x \in \mathcal{M}_p$, we have

$$||Tx||^2 = (T^*Tx, x) = (x, x) = ||x||^2,$$

and

$$||Tx|| = ||x|| = ||T^*x||.$$

Hence the restriction of T to $\mathcal{M}_p$ is unitary. □

COROLLARY 3. *Every p–hyponormal operator T has an approximate proper value μ such that $||T|| = |\mu|$.*

PROOF. Without loss of generality, we may assume that $||T|| = 1$. Since $(TT^*)^p$ is positive and $||(TT^*)^p|| = 1$, 1 is an approximate proper value for $(TT^*)^p$(see [4]). Since the property of p-hyponormal is preserved under *-isomorphism, we may assume, after Berberian-Quiqley extension, that $1 \in \sigma_p((TT^*)^p)$(see [3]). Let $\mathcal{M}_p$ be as in Theorem 2. Then $\mathcal{M}_p$ is a nontrivial invariant subspace, and

$$T = \begin{pmatrix} T|\mathcal{M}_p & * \\ 0 & * \end{pmatrix}.$$

Since $T|\mathcal{M}_p$ is unitary, $T|\mathcal{M}_p$ has an approximate proper value μ such that $|\mu| = 1$. Let $\{x_n\}$ be a sequence of unit vectors in $\mathcal{M}_p$ such that $||(T|\mathcal{M}_p - \mu I)x_n|| \to 0$. Then $\mu \in \sigma_{ap}(T)$ and $||T|| = 1 = |\mu|$ since $||(T|\mathcal{M}_p - \mu I)x_n|| = ||(T - \mu I)x_n||$. □

COROLLARY 4. *The only quasinilpotent p-hyponormal operator is zero.*

PROOF. Since T is p-hyponormal, there exists $\mu \in \sigma(T)$ such that $|\mu| = ||T||$. For every $n \in \mathbf{N}$,

$$||T||^n = |\mu|^n = |\mu^n| \leq ||T^n|| \leq ||T||^n.$$

Since T is generalized nilpotent, we have $T = 0$. □

COROLLARY 5. *If T is p-hyponormal, then*

$$||T|| = lub\{|(Tx, x)| : ||x|| = 1\}.$$

3. Example

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be **-paranormal* if $||T^*x||^2 \leq ||T^2x||$ for every unit vector x in $\mathcal{H}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *paranormal* if $||Tx||^2 \leq ||T^2x||$ for every unit vector x in $\mathcal{H}$. It is well known that every p-hyponormal is paranormal(see [2] and [9]). But the inclusive relation of the p-hyponormality and the *-paranormality is not known. In this section, we give an example which is not true for the relation (*) as follow:

$$\begin{array}{ccccc} \text{hyponormal} & \longrightarrow & \text{p}-\text{hyponormal} & \longrightarrow & \text{paranormal} \\ & \searrow & \big\downarrow (*) & & \\ & & * - \text{paranormal} & & \end{array}$$

The idea of below example comes from M. Chō and H. Jin(see [6]). Let $\mathcal{H}$ be a 2-dimensional Hilbert space and let $\mathcal{K}$ be the direct sum of countably many copies of $\mathcal{H}$. i. e., $x \in \mathcal{K}$ means

$$x = (\cdots, x_{-1}, \boxed{x_0}, x_1, \cdots)$$

with $\sum ||x_i||^2 < \infty$ and $x_i \in \mathcal{H}$. Let $\{P_n | n = 0, \pm 1, \pm 2, \cdots\}$ be a sequence of positive operators on $\mathcal{H}$ such that the sequence $\{||P_n||\}$ of norms is bounded. Then the equations

$$(Px)_n := P_n x_n$$

define an operator P on $\mathcal{K}$. If U is the shift defined by $(Ux)_n := x_{n-1}$, then U is an operator on $\mathcal{K}$. Let $T = UP$. Then

$$(T^*Tx)_n = P_n^2 x_n, \quad (TT^*x)_n = P_{n-1}^2 x_n.$$

EXAMPLE. Let C and D be 2×2 matrices defined by

$$C = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}.$$

Then C, D and $D - C$ are positive. Let $\{P_n\}$ be a sequence of positive 2×2-matrices defind by

$$P_n = \begin{cases} C & (n \leq 0) \\ D & (n > 0). \end{cases}$$

Then $((T^*T)^{\frac{1}{2}}x)_n = P_n x_n$ and $((TT^*)^{\frac{1}{2}}x)_n = P_{n-1}x_n$. Since $D \geq C$, we have T is semi-hyponormal. Let $\tilde{x} = (\cdots, 0, \boxed{0}, x_1, 0, \cdots)$ and $x_1 = (\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. Then $||\tilde{x}|| = ||x_1|| = 1$,

$$T^2\tilde{x} = D^2x_1 = \begin{pmatrix} 41 & 40 \\ 40 & 41 \end{pmatrix} \begin{pmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix},$$

and

$$T^*\tilde{x} = Cx_1 = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{-2}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix}.$$

Thus we have T is not *-paranormal.

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