Comm. Korean Math. Soc. **13** (1998), No. 1, pp. 1–6

ON LEFT REGULAR *po*–SEMIGROUPS

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ABSTRACT. The paper refers to ordered semigroups in which $x^2(x \in S)$ are left ideal elements. We mainly show that this *po*–semigroup is left regular if and only if S is a union of left simple subsemigroups of S.

1. Introduction

The concept of left regular *poe*-semigroups has been introduced in [2] and extends the concept of left regular *po*-semigroups not having the greatest element "e" in [1]. Recently, Kehayopulu [3] showed that: a *poe*–semigroup S is left regular if and only if S is a union of left simple subsemigroups of S. Now we consider a *po*–semigroup which does not necessarily have a greatest element "e".

In this paper, the main result is that: a *po*–semigroup S in which every $x^2(x \in S)$ is a left ideal element, is left regular if and only if S is a union of left simple subsemigroups of S.

A *po-semigroup*(ordered semigroup) is an ordered set $(S, \leq)$ which is a semigroup such that for $a, b \in S$

$$a \leq b \Longrightarrow xa \leq xb \text{ and } ax \leq bx$$

for all $x \in S$.

DEFINITION 1. ([4]). Let S be a *po*-semigroup and $\emptyset \neq I \subseteq S$. I is called a *left ideal* of S if

1) $SI \subseteq I$.
2) $a \in I, S \ni b \leq a \Rightarrow b \in I$.

Received April 30, 1997. Revised November 3, 1997.

1991 Mathematics Subject Classification: 03G25, 06F35.

Key words and phrases: *poe*–semigroup, *po*–semigroup, left ideal, left regular, semiprime, left simple, right congruence.

This note was supported in part by the Basic Science Research Institute Program, Ministry of Education, Korea, 1997, Project No. BSRI-97-14111.

DEFINITION 2. ([5]). A *po*-semigroup S is called *left(right) regular* if for every $a \in S$ there exists $x \in S$ such that

$$a \leq xa^2 \text{ (resp. } a \leq a^2x).$$

DEFINITION 3. ([5]). Let S be a *po*-semigroup. A subset T of S is called *semiprime* if for $a \in S$

$$a^2 \in T \Longrightarrow a \in T.$$

DEFINITION 4. ([3]). An element t of a *po*-semigroup S is called *semi- prime* if $a^2 \leq t$ for some $a \in S$ implies that $a \leq t$.

DEFINITION 5. ([5]). Let S be a *po*-semigroup. A subsemigroup T of S is called *left simple* if for every left ideal L of T we have $L = T$.

Notation. Let S be a *po*-semigroup. For $H \subseteq S$,

$$(H] := \{t \in S \mid t \leq h \text{ for some } h \in H\}.$$

We denote by $L(x)$ the left ideal of S generated by $x (x \in S)$. For a *po*-semigroup S one can easily prove that

$$\begin{aligned} L(x) &= \{t \in S : t \leq x \text{ or } t \leq ax \text{ for some } a \in S\} \\ &= (x \cup Sx]. \end{aligned}$$

We define a relation "$\mathcal{L}$" on S as follows:

$$a\mathcal{L}b \Longleftrightarrow L(a) = L(b).$$

DEFINITION 6. (CF. [2]). An element t of S is called a *left ideal element* if $xt \leq t$ for all $x \in S$.

Kehayopulu [2, 4, 5] introduced a left ideal element in a *poe*−semigroup as follows: An element t of S is called a *left ideal element* if $et \leq t$ for the greatest element "e" of S.

LEMMA ([5]). *Let S be a po-semigroup. Then $\mathcal{L}$ is a right congruence on S i.e. it is an equivalence relation on S such that $a\mathcal{L}b \Longrightarrow ac\mathcal{L}bc$ for all $c \in S$.*

2. Main Theorems

THEOREM 1. *Let S be a po-semigroup. Then the following are equivalent :*

1) *S if left regular.*
2) *$L(a) \subseteq L(a^2)$ for all $a \in S$.*
3) *$a\mathcal{L}a^2$ for all $a \in S$.*

PROOF. 1) $\Longrightarrow$ 2). Let S be left regular and $a \in S$. If $t \in L(a)$, then

$$t \leq a \quad \text{or} \quad t \leq xa$$

for some $x \in S$. Since S is left regular, $a \leq ya^2$ for some $y \in S$.

If $t \leq a$, then $t \leq a \leq ya^2$ for some $y \in S$.

If $t \leq xa$, then $t \leq xa \leq (xy)a^2$ for some $y \in S$.

In any case, $t \leq za^2 \in Sa^2 \subseteq (a^2 \cup Sa^2] = L(a^2)$ for some $z \in S$. Since $L(a^2)$ is a left ideal, $t \in L(a^2)$. Hence $L(a) \subseteq L(a^2)$.

2) $\Longrightarrow$ 3). Let $a \in S$ and $t \in L(a^2)$. Then $t \leq a^2$ or $t \leq xa^2$ for some $x \in S$. In any case, $t \leq za \in Sa \subseteq (a \cup Sa] = L(a)$ for some $z \in S$. Since $L(a)$ is a left ideal, $t \in L(a)$. Hence $L(a^2) \subseteq L(a)$, and so $L(a) = L(a^2)$. Therefore $a\mathcal{L}a^2$.

3) $\Longrightarrow$ 1). Assume that $a\mathcal{L}a^2$ for all $a \in S$. Then $a \in L(a) = L(a^2) = (a^2 \cup Sa^2]$. Thus

$$a \leq a^2 \quad \text{or} \quad a \leq xa^2$$

for some $x \in S$. In any case, we can get $a \leq ta^2$ for some $t \in S$. Thus a is left regular, and so S is left regular. □

EXAMPLE. The ordered semigroup $S = \{a, b, c, d, f\}$ defined by the multiplication and the order below:

·	a	b	c	d	f
a	b	b	d	d	d
b	b	b	d	d	d
c	d	d	c	d	c
d	d	d	d	d	d
f	d	d	c	d	c

$$\leq := \{(a,a),(b,b),(c,c),(d,d),(f,f),(a,b),(d,b),(d,c),(f,c)\}.$$

Then S is a po−semigroup which has no the greatest element "e". Then for any $y \in S$,

$$\begin{aligned} ya^2 &= yb = b \text{ or } d \leq b = a^2 \\ yb^2 &= yb = b \text{ or } d \leq b = b^2, \\ yc^2 &= yc = c \text{ or } d \leq c = c^2, \\ yd^2 &= yd = d \leq d = d^2, \\ yf^2 &= yc = c \text{ or } d \leq c = f^2 \end{aligned}$$

Thus every x^2 ($x \in S$) and d are left ideal elements but a, b, c, f are not left ideal elements.

THEOREM 2. *Let S be a po-semigroup in which every x^2 ($x \in S$) is a left ideal element. The following are equivalent:*

(1) *S is left regular.*
(2) *Every left ideal element of S is semiprime.*
(3) *Every left ideal of S is semiprime.*

PROOF. 1) $\Longrightarrow$ 2). Let t be a left ideal element of S and $a^2 \leq t$ for $a \in S$. Since S is left regular, $a \leq xa^2 \leq xt \leq t$ for some $x \in S$. Thus t is semiprime.

2) $\Longrightarrow$ 3). Let L be a left ideal of S and $a^2 \in L$ for $a \in S$. Since $a^2 \leq a^2$ and a^2 is a left ideal element, we have $a \leq a^2$. By Definition 1, $a \in L$. Hence L is semiprime.

3) $\Longrightarrow$ 1). Let $a \in S$. Since a left ideal $L(a^2)$ is semiprime and $a^2 \in L(a^2)$, we have $a \in L(a^2)$. Thus $L(a) \subseteq L(a^2)$. By 2) $\Rightarrow$ 1) of Theorem 1, S is left regular. □

THEOREM 3. *Let S be a po-semigroup in which every $x^2(x \in S)$ is a left ideal element. Then S is left regular if and only if there exists a family $\{S_\alpha | \alpha \in Y\}$ of left simple subsemigroups of S such that $S = \bigcup\{S_\alpha | \alpha \in Y\}$.*

PROOF. Let S be left regular. We denote by $(x)_{\mathcal{L}}$ the $\mathcal{L}$-class of S containing $x(x \in S)$. $(x)_{\mathcal{L}}$ is a left simple subsemigroup of S for all $x \in S$. Indeed: Since $x \in (x)_{\mathcal{L}}$, $(x)_{\mathcal{L}}$ is not empty.

Let $a, b \in (x)_{\mathcal{L}}$. Then $a\mathcal{L}x$ and $x\mathcal{L}b$. Since $\mathcal{L}$ is a right congruence, we have $ab\mathcal{L}xb$ and $xb\mathcal{L}b^2$. Since S is left regular, $b^2\mathcal{L}b$ by Theorem 1. Therefore $ab\mathcal{L}b$. Hence $ab \in (b)_{\mathcal{L}} = (x)_{\mathcal{L}}$ so $(x)_{\mathcal{L}}$ is a subsemigroup of S.

Let L be a left ideal of $(x)_{\mathcal{L}}$ and $z \in L$. If $y \in (x)_{\mathcal{L}}$, then $z \in L \subseteq (x)_{\mathcal{L}} = (y)_{\mathcal{L}}$. Since S is left regular, by Theorem 1 $y \in L(y) = L(z) = L(z^2)$. Then $y \leq z^2$ or $y \leq tz^2$ for some $t \in S$. If $y \leq z^2$, then $y \leq z^2 \in (x)_{\mathcal{L}}L \subseteq L$ since L is a left ideal of $(x)_{\mathcal{L}}$. If $y \leq tz^2$, then $y \leq z^2$ since every $x^2 (x \in S)$ is a left ideal element.
In any cases, $y \in L$ and so $L = (x)_{\mathcal{L}}$ since L is a left ideal of $(x)_{\mathcal{L}}$. Hence every $(x)_{\mathcal{L}}$ is a left simple subsemigroup of S. Therefore $S = \bigcup\{(x)_{\mathcal{L}} | x \in S\}$.

Conversely, suppose every S_α is a left simple subsemigroup of S and let $S = \bigcup\{S_\alpha | \alpha \in Y\}$. If L is a left ideal of S and $a^2 \in L$ for $a \in S$, then $a \in S_\alpha$ for some $\alpha \in Y$. Consider a subset $L \cap S_\alpha$ of S. Since S_α is a subsemigroup of S, $a^2 \in S_\alpha$. Thus $L \cap S_\alpha \neq \emptyset$. Furthermore

$$S_\alpha(L \cap S_\alpha) \subseteq S_\alpha L \cap {S_\alpha}^2 \subseteq SL \cap S_\alpha \subseteq L \cap S_\alpha.$$

Let $x \in L \cap S_\alpha$ and $x \geq y \in S_\alpha$. Since $x \in L$ and $y \leq x$, $y \in L$. Thus $y \in L \cap S_\alpha$. Hence $L \cap S_\alpha$ is a left ideal of S_α. Since S_α is left simple, we have $L \cap S_\alpha = S_\alpha$. Thus $a \in L$. It follows that L is semiprime. By Theorem 2, S is left regular. □

ACKNOWLEDGEMENT. The authors thank for the referee's kind comments.

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