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GENUS DISTRIBUTIONS FOR BOUQUETS OF DIPOLES

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ABSTRACT. We compute genus distributions for bouquets of dipoles by using the method concerning the cycle structure of permutations in the symmetric group. From this, we can deduce that every bouquet of dipoles is upper embeddable. We find a formula for computing the embedding polynomials for bouquets of dipoles.

1. Introduction

Let G be a finite connected graph with vertex set $V(G)$ and edge set $E(G)$. It might have loops at a vertex and multiple edges between two vertices. By regarding the vertices of G as 0-cells and the edges of G as 1-cells, the graph G can be identified with a finite 1-dimensional CW-complex in the Euclidean 3-space $\mathbb{R}^3$. We associate two oppositely directed edges e^+ and e^- to each edge e of G, and denote by $D(G)$ the set of all directed edges of G. For each $v \in V(G)$, let $N(v)$ denote the set of all edges in $D(G)$ starting at v and call it the neighborhood of v.

A surface means a compact connected 2-manifold without boundary. An *embedding* of a graph G into a surface $\mathbb{S}$ is a topological embedding $i : G \to \mathbb{S}$. An embedding $i : G \to \mathbb{S}$ of G into a surface $\mathbb{S}$ is called a *2-cell embedding* if every component of $\mathbb{S} - i(G)$, called a *region*, is homeomorphic to an open disk. A region of an embedding $i : G \to \mathbb{S}$ is said to be k-sided if the length of the walk in G corresponding to the boundary walk of the region is k.

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Two 2-cell embeddings $i : G \to \mathbb{S}$ and $j : G \to \mathbb{S}$ of G into an oriented surface $\mathbb{S}$ are said to be *equivalent* if there is an orientation-preserving homeomorphism $h : \mathbb{S} \to \mathbb{S}$ such that $h \circ i = j$.

Throughout this paper, we assume that every surface is oriented, all embeddings of graphs into surfaces are 2-cell embeddings and the number of embeddings of G into a surface $\mathbb{S}$ means the number of equivalence classes of embeddings of G into $\mathbb{S}$.

A *rotation system* ρ for G is an assignment of a cycle permutation $\rho(v)$ on $N(v)$ to each vertex v in G. Notice that for a rotation system ρ for G $\rho(v)$ can be viewed as a permutation on the directed edge set $D(G)$ which fixes each element in $D(G) - N(v)$.

It is known [4] that the equivalence classes of embeddings of G into surfaces are in one to one correspondence with the rotation systems for G. It implies that the total number of embeddings of G into surfaces is equal to $\prod_{v \in V(G)}(|N(v)| - 1)!$. Moreover the number of regions of the corresponding embedding to a rotation system ρ, denoted by $r(G, \rho)$, is equal to the number of the cycles in the representation of $(\prod_{v \in V(G)} \rho(v))\beta$ as the product of disjoint cycles, where β is the full involution on $D(G)$ that takes each directed edge to its inverse (Theorem 2.1 in [4]). Given a rotation system ρ for G, we denote by $g(G, \rho)$ the genus of the surface in the embedding of G corresponding to ρ. Thus, by Euler's equation, $2 - 2g(G, \rho) = |V(G)| - |E(G)| + r(G, \rho)$. For each nonnegative integer m, let $g_m(G)$ denote the number of embeddings of G into the surface of genus m. The *genus distribution* of the graph G is defined to be the sequence $\{g_0(G), g_1(G), \cdots, g_m(G), \cdots\}$. It is clear that there are only a finite number of nonzero terms in the sequence. The maximum (minimum) value of m that $g_m(G)$ is nonzero is called the *maximum* (*minimum*) *genus* of G, and is denoted $\gamma_M(G)$ $(\gamma(G))$.

Duke [1] showed that G has an embedding into the surface of genus m, *i.e.*, $g_m(G)$ is nonzero if and only if $\gamma(G) \le m \le \gamma_M(G)$. By Euler's equation, $\gamma_M(G) \le \lfloor \frac{\beta(G)}{2} \rfloor$, where $\beta(G) = |E(G)| - |V(G)| + 1$, $|X|$ denotes the cardinality of a set X and $\lfloor r \rfloor$ is the greatest integer less than or equal to a real number r.

Xuong [9] proved that every connected graph G satisfies the equation $\gamma_M(G) = \frac{1}{2}(\beta(G) - \xi(G))$, where $\xi(G) = \min \xi_o(G - T)$, the minimum being taken over all spanning trees T of G, and $\xi_o(G - T)$ denotes the number of components of the subgraph $G - T$ of odd size. If $\gamma_M(G) = \lfloor \frac{\beta(G)}{2} \rfloor$, G is said to be *upper embeddable*.

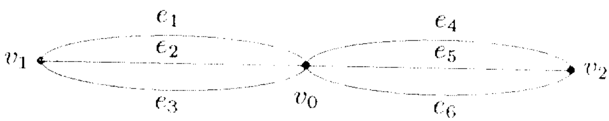

FIGURE 1. Bouquet of two 3-dipoles

There are several papers about graph embeddings that are concerned with the determination of $\gamma(G)$ or $\gamma_M(G)$ for special graphs G (see [10]). Furst *et al.* [2] have computed genus distributions for ladders and cobblestone paths. And then Gross *et al.* [4] have computed the genus distributions for bouquets of circles and asked for the genus distributions for other interesting graphs. Kwak and Lee [7] and Rieper [8] have computed the genus distributions for the dipoles independently.

Now, we introduce a polynomial related to regions of embeddings of G. For each integer $j > 0$, the exponent of variable z_j is the number of j-sided regions in the embedding, and the sum of these monomials, taken over all embeddings, is called the *embedding polynomial* of G, which is denoted $i[G](z_j)$. Gross and Furst [3] observed that the genus distributions can be calculated from the embedding polynomials and they studied the embedding polynomials for bar-amalgamations.

In this paper, we compute the genus distributions for bouquets of dipoles by using the method concerning the cycle structure of permutations in the symmetric group. From this, we can deduce that every bouquet of dipoles is upper embeddable. We find a formula for computing the embedding polynomials for bouquets of dipoles.

2. Rotation systems for bouquets of dipoles

The p-dipole D_p is the graph which consists of two vertices joined by p edges. Let $B_{n;p}$ be the graph defined as follows: The vertex set $V(B_{n;p})$ consists of $n+1$ vertices, say v_0, v_1, $\cdots$ v_n, and for each $j = 1, 2, \cdots, n$, there are exactly p edges, say $e_{(j-1)p+1}, \cdots, e_{jp}$, between v_0 and v_j. We call it the *bouquet of n p-dipoles*. For convenience, let $e_k{}^+$ be the directed edge from v_0 to $v_{1+\lfloor\frac{k-1}{p}\rfloor}$ for each $k = 1, \cdots, np$. Then its inverse e_k^- is the directed edge from $v_{1+\lfloor\frac{k-1}{p}\rfloor}$ to v_0 for each $k = 1, \cdots, np$.

Now, the following lemma comes from $|V(B_{n;p})| = n+1$, $|E(B_{n;p})| = np$ and the discussions in section 1.

LEMMA 2.1. *Let ρ be any rotation system for $B_{n;p}$. Then*

(1) $g(B_{n;p},\rho) = \dfrac{1}{2}\{np - n + 1 - r(B_{n;p},\rho)\}$

(2) $r(B_{n;p},\rho)$ *is equal to the number of cycles of the representation of* $\rho(v_0)\rho(v_1)\cdots\rho(v_n)\,\beta$ *as the product of disjoint cycles, where* $\beta = (e_1^+\, e_1^-)(e_2^+\, e_2^-)\cdots(e_{np}^+\, e_{np}^-)$.

For each rotation system ρ for $B_{n;p}$, let

$$\rho(v_0) = (e_1^+\, e_{k_2}^+\cdots e_{k_{np}}^+),\quad \rho(v_j) = (e_{(j-1)p+1}^-\, e_{l_{j_2}}^-\cdots e_{l_{j_p}}^-)$$

and define $\bar\rho : V(B_{n;p}) \to S_{np}$ by

$$\bar\rho(v_0) = (1\, k_2\cdots k_{np}),\quad \bar\rho(v_j) = ((j-1)p+1\; l_{j_2}\cdots l_{j_p}),$$

where $1 \le j \le n$, $\{k_2, k_3, \cdots, k_{np}\} = \{2, 3, \cdots, np\}$, $\{l_{j_2}, l_{j_3}, \cdots, l_{j_p}\} = \{(j-1)p+2, (j-1)p+3, \cdots, jp\}$ and S_{np} is the symmetric group on $\{1, 2, \cdots, np\}$. Notice that for two rotation systems ρ and ρ' for $B_{n;p}$, $\rho = \rho'$ iff $\bar\rho = \bar\rho'$. For convenience, we let $\hat\rho = \bar\rho(v_0)\bar\rho(v_1)\cdots\bar\rho(v_n) \in S_{np}$.

For each integer $1 \le k \le np$, let $\mathrm{j}_k : S_{np} \to \mathbb{N}\cup\{0\}$ be the map defined by $\mathrm{j}_k(\tau)$ is the number of k-cycles occurring in the representation of $\tau \in S_{np}$ as the product of disjoint cycles.

LEMMA 2.2. *For each rotation system ρ for $B_{n;p}$,*

$$r(B_{n;p},\rho) = \sum_{k=1}^{np} \mathrm{j}_k(\hat\rho).$$

Proof. We observe that any cycle occurring in the representation of $\rho(v_0)\rho(v_1)\cdots\rho(v_n)\beta$ as the product of disjoint cycles is of the form

$$(e_{m_1}^+\, e_{m_2}^-\, e_{m_3}^+\cdots e_{m_{2l-1}}^+\, e_{m_{2l}}^-),$$

where $1 \le m_{2k-1}, m_{2k} \le np$ and $m_{2k-1} = \hat\rho^{k-1}(m_1)$ for each $k = 1, \cdots, l$. Let $\mathfrak{A}$ be the set of cycles in $\rho(v_0)\rho(v_1)\cdots\rho(v_n)\beta$ and $\mathfrak{B}$ the set of cycles in $\hat\rho$. Now, we define a map $\Psi : \mathfrak{A} \to \mathfrak{B}$ by

$$\Psi((e_{m_1}^+\, e_{m_2}^-\, e_{m_3}^+\cdots e_{m_{2l-1}}^+\, e_{m_{2l}}^-)) = (m_1\, m_3\cdots m_{2l-1}).$$

Then Ψ is a well-defined bijection from $\mathfrak{A}$ onto $\mathfrak{B}$. Hence, by Lemma 2.1(2), we have the lemma. □

3. The number of embeddings of bouquets of dipoles

For a nonnegative integer k and an element $\zeta \in S_{np}$ with $\mathrm{j}_p(\zeta) = n$, let

$$F^{(p)}_{\zeta;k}(n) = \left\{ \sigma \in S_{np} \;\middle|\; \sum_{l=1}^{np} \mathrm{j}_l(\sigma\zeta) = k,\ \mathrm{j}_{np}(\sigma) = 1 \right\}.$$

We note that $\left|F^{(p)}_{\zeta;k}(n)\right| = \left|F^{(p)}_{\zeta';k}(n)\right|$ for any $\zeta, \zeta' \in S_{np}$ with $\mathrm{j}_p(\zeta) = n$ and $\mathrm{j}_p(\zeta') = n$. We shall denote this number $f^{(p)}_k(n)$.

By contrast, for a nonnegative integer k and an element $\sigma \in S_{np}$ with $\mathrm{j}_{np}(\sigma) = 1$, let

$$E^{(p)}_{\sigma;k}(n) = \left\{ \zeta \in S_{np} \;\middle|\; \sum_{l=1}^{np} \mathrm{j}_l(\sigma\zeta) = k,\ \mathrm{j}_p(\zeta) = n \right\}.$$

We also note that $\left|E^{(p)}_{\sigma;k}(n)\right| = \left|E^{(p)}_{\sigma';k}(n)\right|$ for any $\sigma, \sigma' \in S_{np}$ with $\mathrm{j}_{np}(\sigma) = 1$ and $\mathrm{j}_{np}(\sigma') = 1$. We shall denote this number $e^{(p)}_k(n)$ according to Jackson's notation [6].

Notice that $f^{(p)}_0(n) = 0$ and $e^{(p)}_0(n) = 0$.

THEOREM 3.1. *For each nonnegative integer m, the number of embeddings of $B_{n;p}$ into the surface of genus m is*

$$\begin{aligned} g_m(B_{n;p}) &= ((p-1)!)^n\, f^{(p)}_{np-2m-n+1}(n) \\ &= ((p-1)!)^n\, p^{n-1}\,(n-1)!\, e^{(p)}_{np-2m-n+1}(n). \end{aligned}$$

Proof. By Lemmas 2.1, 2.2 and the above discussions, we have

$$\begin{aligned} g_m(B_{n;p}) &= \left|\{\rho \in \mathfrak{R} \mid r(B_{n;p}) = np - 2m - n + 1\}\right| \\ &= \left|\left\{ \rho \in \mathfrak{R} \;\middle|\; \sum_{k=1}^{np} \mathrm{j}_k(\hat{\rho}) = np - 2m - n + 1 \right\}\right| \\ &= \Bigg|\Bigg\{ (\sigma, \eta_1 \cdots \eta_n) \in S_{np} \times S_{np} \;\Bigg|\; \mathrm{j}_{np}(\sigma) = 1, \eta_j \in \mathfrak{C}_j, \\ &\qquad \sum_{l=1}^{np} \mathrm{j}_l(\sigma\eta_1 \cdots \eta_n) = np - 2m - n + 1 \Bigg\}\Bigg| \\ &= ((p-1)!)^n\, f^{(p)}_{np-2m-n+1}(n), \end{aligned}$$

where $\mathfrak{R}$ is the set of all rotation systems for $B_{n;p}$ and $\mathfrak{C}_j$ is the set of cycle permutations on $\{(j-1)p+1, \cdots, jp\}$ for each $j = 1, \cdots, n$.

Now we consider the set:

$$\Lambda = \left\{ (\sigma, \zeta) \in S_{np} \times S_{np} \;\middle|\; \sum_{l=1}^{np} \mathrm{j}_l(\sigma\zeta) = k,\ \mathrm{j}_{np}(\sigma) = 1,\ \mathrm{j}_p(\zeta) = n \right\}.$$

Since the number of $\zeta \in S_{np}$ with $\mathrm{j}_p(\zeta) = n$ is $\frac{(np)!}{n!p^n}$ and the number of $\sigma \in S_{np}$ with $\mathrm{j}_{np}(\sigma) = 1$ is $(np-1)!$.

$$\frac{(np)!}{n!p^n} f_k^{(p)}(n) = |\Lambda| = (np-1)! e_k^{(p)}(n).$$

It completes the proof. □

We note that $B_{n;2}$ is homeomorphic to the bouquet of n circles B_n, and $B_{1;p}$ is the dipole graph D_p. Since the genus distribution is topological invariant, we have the following corollary.

COROLLARY 3.2. (1) $g_m(B_n) = 2^{n-1}(n-1)!\, e^{(2)}_{n-2m+1}(n)$.
(2) $g_m(D_p) = (p-1)!\, e^{(p)}_{p-2m}(1)$.

The Stiling numbers of the first kind $s_n^{(k)}$ are given by the coefficients of

$$x(x-1)(x-2)\cdots(x-n+1) = \sum_{k=0}^{n} s_n^{(k)} x^k$$

and the Stiling numbers of the second kind $S_n^{(k)}$ are given by the coefficients of

$$x^n = \sum_{k=0}^{n} k!\, S_n^{(k)} \binom{x}{k}.$$

On the other hand, a combinatorial argument shows that $s_n^{(k)} = (-1)^{n-k}c(n,k)$ and $S_n^{(k)} = p(n,k)$, where $c(n,k)$ is the number of permutations in S_n with k cycles, and $p(n,k)$ is the number of partitions of $\{1, 2, \cdots, n\}$ into k nonempty blocks. Then $s_n^{(k)}$ satisfy the following recurrence equation

$$s_n^{(k)} = -(n-1)s_{n-1}^{(k)} + s_{n-1}^{(k-1)}$$

with the conditions $s_n^{(0)} = 0, s_n^{(n)} = 1, s_n^{(1)} = (-1)^{n-1}(n-1)!$. And $S_n^{(k)}$ satisfy the following recurrence equation

$$S_n^{(k)} = kS_{n-1}^{(k)} + S_{n-1}^{(k-1)}$$

with the conditions $S_n^{(0)} = 0, S_n^{(n)} = 1, S_n^{(1)} = 1$.

Jackson [6] computed the number $e_k^{(p)}(n)$ in terms of the Stirling numbers of the first and the second kinds as follows.

LEMMA 3.3. *For positive integers n, p and k,*

$$e_k^{(p)}(n) = \frac{1}{1+np} \sum_{l=0}^{1+np-n-k} p^l \binom{n+k+l}{k} s_{1+np}^{(n+k+l)} S_{n+l}^{(n)}.$$

Now, Theorem 3.1 can be rephrased as follows.

THEOREM 3.4. *The number of embeddings of $B_{n;p}$ into the surface of genus m is*

$$g_m(B_{n;p}) = \frac{(p!)^n(n-1)!}{p(1+np)} \sum_{l=0}^{2m} p^l \binom{1+np-2m+l}{1+np-2m-n} s_{1+np}^{(1+np-2m+l)} S_{n+l}^{(n)}.$$

We notice that $e_k^{(p)}(n) \neq 0$ if and only if $1 \leq k \leq 1+np-n$. By Theorem 3.1, $g_m(B_{n;p}) \neq 0$ if and only if m is a nonnegative integer with $1 \leq np-2m-n+1 \leq 1+np-n$, or equivalently m is a nonnegative integer with $0 \leq m \leq \lfloor \frac{np-n}{2} \rfloor$. Hence, $\gamma(B_{n;p}) = 0$ and $\gamma_M(B_{n;p}) = \lfloor \frac{\beta(B_{n;p})}{2} \rfloor$. Hence we have the following.

THEOREM 3.5. *Every bouquet of dipoles is planar and upper embeddable.*

We remark that the upper embeddability of bouquet of dipoles can be also obtained from Xuong's formula.

From the recurrence equations for $s_n^{(k)}$ and $S_n^{(k)}$, we can have

$$s_n^{(n)} = 1,\ s_{n+1}^{(n)} = -\frac{1}{2}n(n+1),\ s_{n+2}^{(n)} = \frac{1}{24}n(n+1)(n+2)(3n+5),$$

$$S_n^{(n)} = 1,\ S_{n+1}^{(n)} = \frac{1}{2}n(n+1),\ S_{n+2}^{(n)} = \frac{1}{24}n(n+1)(n+2)(3n+1).$$

From this, we have the following.

COROLLARY 3.6. *The number of embeddings of $B_{n;p}$ into the sphere is equal to*

$$\frac{(p!)^n(n-1)!}{p(1+np)}\binom{1+np}{1+np-n}.$$

COROLLARY 3.7. *The number of embeddings of $B_{n;p}$ into the torus is equal to*

$$\frac{1}{24}(p!)^n\, n!\,(np^2-np-2)\binom{np-1}{n}.$$

The numbers $G_m(B_{n;p})$ for small m, n and p are listed in the following.

$(n;p)\backslash m$	0	1	2	3	4	total
(1;2)	1	0	0	0	0	1
(1;3)	2	2	0	0	0	4
(1;4)	6	30	0	0	0	36
(2;2)	4	2	0	0	0	6
(2;3)	36	300	144	0	0	480
(2;4)	576	22176	118944	39744	0	181440
(3;1)	1	0	0	0	0	1
(3;2)	40	80	0	0	0	120
(3;3)	1728	1728	34272	284832	0	322560

4. Embedding polynomials for bouquets of dipoles

We first give an example to be well acquainted with the concept of embedding polynomials. Let K_4 be the complete graph on four vertices. It is clear that K_4 has sixteen embeddings. A routine computation gives the following: Two of them are embedded in the sphere with four 3-sided faces, six of them are embedded in the torus with one 4-sided face and one 8-sided face, and the remained eight of them are embedded in the torus with one 3-sided face and one 9-sided face. Therefore,

$$i[K_4](z_j) = 2z_3^4 + 6z_4z_8 + 8z_3z_9.$$

Recall that regions of the embedding associated with a rotation system ρ for $B_{n;p}$ are completely determined by cycles of the permutation $\rho(v_0)\rho(v_1)\cdots\rho(v_n)\beta$. Moreover, the number of sides of a region R is equals to the length of the corresponding cycle to R in $\rho(v_0)\rho(v_1)\cdots\rho(v_n)$ β. Let $\mathfrak{R}$ denote the set of all rotation systems for $B_{n;p}$, $\mathfrak{S}_{np}$ the set of cycle permutations on $\{1,\cdots,np\}$ and $\mathfrak{C}_j$ the set of cycle permutations on $\{(j-1)p+1, \cdots, jp\}$ for each $j = 1, \cdots, n$. By the proof of Lemma 2.2, we have that the corresponding monomial to a rotation system ρ for $B_{n;p}$ is $\prod_{k=1}^{np} z_{2k}^{\mathrm{j}_k(\hat{\rho})}$. It implies that

$$i[B_{n;p}](z_j) = \sum_{\rho\in\mathfrak{R}} \prod_{k=1}^{np} z_{2k}^{\mathrm{j}_k(\hat{\rho})}.$$

It comes from the definition of rotation system that

$$\begin{aligned}\sum_{\rho\in\mathfrak{R}} \prod_{k=1}^{np} z_{2k}^{\mathrm{j}_k(\hat{\rho})} &= \sum_{(\sigma,\eta_1,\cdots,\eta_n)\in\mathfrak{S}_{np}\times\mathfrak{C}_1\times\cdots\times\mathfrak{C}_n} \prod_{k=1}^{np} z_{2k}^{\mathrm{j}_k(\sigma\eta_1\cdots\eta_n)} \\ &= \sum_{(\eta_1,\cdots,\eta_n)\in\mathfrak{C}_1\times\cdots\times\mathfrak{C}_n} \left(\sum_{\sigma\in\mathfrak{S}_{np}} \prod_{k=1}^{np} z_{2k}^{\mathrm{j}_k(\sigma\eta_1\cdots\eta_n)} \right).\end{aligned}$$

We observe that for any two elements $\zeta, \zeta' \in S_{np}$ with $\mathrm{j}_p(\zeta) = n$ and $\mathrm{j}_p(\zeta') = n$, $\sum_{\sigma\in\mathfrak{S}_{np}} \prod_{k=1}^{np} z_{2k}^{\mathrm{j}_k(\sigma\zeta)} = \sum_{\sigma\in\mathfrak{S}_{np}} \prod_{k=1}^{np} z_{2k}^{\mathrm{j}_k(\sigma\zeta')}$. By combining our discussions and the fact that $\mathrm{j}_p(\eta_1\cdots\eta_n) = n$ for any $(\eta_1, \cdots, \eta_n) \in \mathfrak{C}_1 \times \cdots \times \mathfrak{C}_n$, we have the following theorem.

THEOREM 4.1. *The embedding polynomial for a bouquet of dipoles $B_{n;p}$ is*

$$i[B_{n;p}](z_j) = ((p-1)!)^n \sum_{\sigma\in\mathfrak{S}_{np}} \prod_{k=1}^{np} z_{2k}^{\mathrm{j}_k(\sigma\alpha)},$$

where $\alpha = (1 \cdots p)(p+1 \cdots 2p)\cdots((n-1)p+1 \cdots np) \in S_{np}$.

REMARK 4.2. If we convert the multivariate embedding polynomial in $z_2, z_4, \cdots, z_{2np}$ into a univariate polynomial in z by dropping all subscripts, then the coefficient of the term $z^{np-2m-n+1}$ is $g_m(B_{n;p})$.

COROLLARY 4.3. *The embedding polynomial for a dipole D_p is*

$$i[D_p](z_j) = ((p-1)!) \sum_{\sigma\in\mathfrak{S}_p} \prod_{k=1}^{p} z_{2k}^{\mathrm{j}_k(\sigma\alpha)},$$

where $\alpha = (1 \cdots p) \in S_p$.

COROLLARY 4.4. *The embedding polynomial for a bouquet of n circles B_n is*

$$i[B_n](z_j) = \sum_{\sigma\in\mathfrak{S}_{2n}} \prod_{k=1}^{2n} z_k^{\mathrm{j}_k(\sigma\alpha)},$$

where $\alpha = (1\,2)\cdots(2n-1\,2n) \in S_{2n}$.

EXAMPLE 4.5. The embedding polynomial for the bouquet of two circles B_2 is

$$4z_1^2 z_2 + 2z_4.$$

The embedding polynomial for the dipole D_4 is

$$6z_2^4 + 24z_2z_6 + 6z_4^2.$$

The embedding polynomial for the bouquet of two 3-dipoles $B_{2;3}$ is

$$36z_2^4z_4 + 144z_2z_4z_6 + 144z_2^2z_8 + 12z_4^3 + 144z_{12}.$$

References

[1] R. A. Duke, *The Genus, Regional number, and Betti number of a Graph*, Canadian J. Math. **18** (1966), 817-822.

[2] M. Furst, J. L. Gross and R. Statman, *Genus Distributions for two classes of graphs*, J. Combin. Theory, Ser. B **46** (1989), 22-36.

[3] J. L. Gross and M. L. Furst, *Hierarchy for Imbedding Distribution Invariants of a Graph*, J. Graph Theory **11** (1987), 205-220.

[4] J. L. Gross, D. P. Robbins and T. W. Tucker, *Genus Distributions for Bouquets of Circles*, J. Combin. Theory, Ser. B **47** (1989), 292-306.

[5] J. L. Gross and T. W. Tucker, *Topological Graph Theory*, John Wiley and Sons, New York, 1987.

[6] D. M. Jackson, *Counting cycles in permutations by group characters*, with an application to topological problem, Trans. Amer. Math. Soc., **299** (1987), 785-801.

[7] J. H. Kwak and J. Lee, *Genus polynomials of Dipoles*, Kyungpook Math. Journal, **33** (1993), 115-125.

[8] R. G. Rieper, *Ph. D. Thesis*, Western Michigan University, 1987.

[9] N. H. Xuong, *How to determine the maximum genus of a graph*, J. Combin. Theory, Ser. B **26** (1979), 217-225.

[10] A. T. White, *Graphs, Groups and Surfaces*, North-Holland, Amsterdam, 1984.

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