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NORM ESTIMATE FOR A CERTAIN MAXIMAL OPERATOR

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ABSTRACT. A condition for a certain maximal operator to be of strong type (p,p) is characterized in terms of Carleson measure.

1. Introduction

Given a function f in $\mathbb{R}^n$, we define a function Mf in

$$\bar{\mathbb{R}}^{n+1}_+ = \{(x,t) : x \in \mathbb{R}^n, t \geq 0\}$$

by setting

$$Mf(x,t) = \sup\left\{ \frac{1}{|Q|} \int_Q |f(y)|dy : x \in Q \text{ and sidelength}(Q) \geq t \right\}.$$

It is well known that this maximal operator M controls Poisson integral defined by

$$P(f)(x,t) = C_n \int_{R^n} \frac{t}{(|x-y|^2+t^2)^{\frac{n+1}{2}}} f(y)\,dy,$$

for $x \in \mathbb{R}^n$ and $t > 0$, where C_n is the constant given by

$$C_n = \left(\int_{R^n} \frac{dx}{(|x|^2+1)^{\frac{n+1}{2}}} \right)^{-1}.$$

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Given a positive Borel measure μ in $\bar{\mathbb{R}}^{n+1}_+$, we define a function $\mathcal{N}_\mu$ in $\mathbb{R}^n$ by setting

$$\mathcal{N}_\mu(x) = \sup_{x \in Q} \frac{\mu(\widetilde{Q})}{|Q|},$$

where the sup is taken over all cubes containing x and for a cube Q

$$\widetilde{Q} = \{(x,t) \in \bar{\mathbb{R}}^{n+1}_+ : x \in Q \text{ and } 0 \leq t \leq \text{ sidelength}(Q)\}.$$

It is well known that

1) [4] For every p with $1 < p < \infty$, there is a constant C_p such that for every f and every μ

$$\left[\int_{\bar{\mathbb{R}}^{n+1}_+} [Mf(x,t)]^p d\mu(x,t)\right]^{\frac{1}{p}} \leq C_p \left[\int_{\mathbb{R}^n} |f(x)|^p N_\mu(x) dx\right]^{\frac{1}{p}}.$$

2) [1] For every p with $1 < p < \infty$, M is bounded from $L^p(\mathbb{R}^n, dx)$ into $L^p(\bar{\mathbb{R}}^{n+1}_+, d\mu)$ if and only if μ satisfies the so-called *Carleson condition*;

$$\mu(\widetilde{Q}) \leq C|Q| \quad \text{for each cube in } \mathbb{R}^n.$$

3) [3] For every p with $1 < p < \infty$, M is bounded from the weighted space $L^p(\mathbb{R}^n, v(x)dx)$ into $L^p(\bar{\mathbb{R}}^{n+1}_+, \mu)$ if the following condition is satisfied

$$\sup_{x \in Q} \frac{\mu(\widetilde{Q})}{|Q|} \leq cv(x) \ a.e. \ x.$$

In this paper we generalize these results in terms of spaces of homogeneous type.

2. Preliminaries

Before proving the main theorem, some definitions and facts will be introduced.

DEFINITION 2.1 [2]. Let X be a topological space. Assume d is a *pseudo-distance* on X, i.e., d is a nonnegative function defined on $X \times X$ satisfying

i) $d(x,x) = 0$; $d(x,y) > 0$ if $x \neq y$;

ii) $d(x,y) = d(y,x)$;

iii) $d(x,z) \leq K[d(x,y) + d(y,z)]$, where K is some fixed constant.
Assume further that

iv) the balls $B(x,r) = \{y \in X : d(x,y) < r\}$ form a basis of open neighborhoods at $x \in X$ and that

v) μ is a positive Borel measure on X such that $0 < \mu(B(x,2r)) \leq A\mu(B(x,r)) < \infty$, where A is some fixed constant.

Then the triple (X,d,μ) is called a space of homogeneous type.

NOTE. Properties (iii) and (v) will be referred to as the *triangle inequality* and the *doubling property*, respectively. Note that (v) is equivalent to the following condition: For every $c > 0$, there exists a constant $A_c < \infty$ such that $\mu(B(x,cr)) \leq A_c\mu(B(x,r))$.

DEFINITION 2.4. For $f \in L^1_{loc}(X,d\mu)$ and $x \in X, t \geq 0$ set

$$\mathcal{M}f(x,t) = \sup\left\{\frac{1}{\mu(B(y,s))}\int_{B(y,s)} |f|d\mu : x \in B(y,s) \text{ and } \quad s \geq t\right\}.$$

Assume for each $x \in X$, we are given a set $\Omega_x \subset X \times [0,\infty)$ and let Ω denote the family $\{\Omega_x\}_{x\in X}$.

REMARK. To avoid the technical difficuluties, we assume that $\cup_{r>0}\widehat{B}\,(x,r) = X \times [0,\infty)$.

DEFINITION 2.5. For $U \subset X$, $\widehat{U} = (\cup_{y\notin U}\Omega_y)^c$ is called the tent of U.

DEFINITION 2.6. Given a positive Borel measure ν on $X \times [0,\infty)$, define

$$\mathcal{N}\nu(x) = \sup_{x\in B}\frac{\nu(\widehat{B})}{\mu(B)}$$

for $x \in X$, where the sup is taken over all balls containing x.

DEFINITION 2.7. For $x \in X$, $r \geq 0$, and $\alpha > 0$ define

$$\mathcal{S}(x,r) = \{x_o \in X : \Omega_{x_o}(r) \cap B(x, \alpha r) \neq \phi\}$$

where

$$\Omega_{x_o}(r) = \{x \in X : (x,r) \in \Omega_{x_o}\}$$

is the cross section of Ω_{x_o} of height r.

NOTE. We will assume that $\mathcal{S}(x,r)$ is measurable for each x and r.

DEFINITION 2.8. An operator T defined in $L^p(d\mu)$ with values in the class of measurable functions defined on a measure space (Y,ν) is said to be of weak type (p,q), $1 \leq p \leq \infty, 1 \leq q \leq \infty$, if there is a constant $A(p,q)$ so that

$$\nu(\{x : |T(f)(x)| > \alpha\}) \leq A(p,q) \left(\frac{\|f\|_p}{\alpha}\right)^q$$

for all $\alpha > 0$.

Finally we need the following covering lemma:

LEMMA 2.1 [2]. *Let E be a bounded subset of X i.e. E is contained in some ball. Let $\gamma(x)$ be a positive number for each $x \in E$. Then there is a (finite or infinite) sequence of disjoint balls $B(x_i,\gamma(x_i))$, $x_i \in E$, such that the balls $B(x_i,4K\gamma(x_i))$ cover E, where K is the constant in the triangle inequality. Furthermore, every $x \in E$ is contained in some ball $B(x_i,4K\gamma(x_i))$ satisfying $\gamma(x) \leq 2\gamma(x_i)$.*

3. Results

THEOREM 3.1. *Let (X,d,μ) be a spaces of homogeneous type. Assume that*

(i) *if $t < r$ and $(x,t) \in \Omega_y$, then $(x,r) \in \Omega_y$.*

Assume also that

(ii) *there is a constant $C > 0$ so that $\nu(\widehat{\mathcal{S}}(x,r)) \leq C\nu(\widehat{B}(x,r))$ for all $x \in X$ and $r > 0$.*

Then $\mathcal{M}$ is bounded from $L^p(X, \mathcal{N}\nu d\mu)$ into $L^p(X \times [0,\infty), d\nu)$ for $1 < p < \infty$, that is, there is a constant C_p so that

$$\left[\int_{X\times[0,\infty)} [\mathcal{M}f(x,t)]^p \, d\nu(x,t)\right]^{\frac{1}{p}} \leq C_p\left[\int_X |f(x)|^p \, \mathcal{N}\nu(x)d\mu(x)\right]^{\frac{1}{p}}$$

holds for every $f \in L^p(X, \mathcal{N}\nu d\mu)$.

Proof. The idea of proof is based on Sueiro [5]. First we prove that the maximal operator $\mathcal{M}$ is of weak type (1,1). Fix $\alpha > 0$. Set

$$E_\alpha = \{(x,t) : \mathcal{M}f(x,t) > \alpha\} \tag{1}$$

and

$$E'_\alpha = \left\{x \in X : \sup_{r>0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f| d\mu > \alpha\right\}. \tag{2}$$

For each $x \in E'_\alpha$ let

$$\gamma(x) = \sup\left\{r > 0 : \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f| \, d\mu \; > \alpha\right\}.$$

Then for each $x \in E'_\alpha$ we have $\gamma(x) > 0$ and

$$\frac{1}{\mu(B(x,\gamma(x)))} \int_{B(x,\gamma(x))} |f| \, d\mu \; \geq \alpha.$$

If $\gamma(x) = \infty$ for some x, we can choose a sequence $\{r_n\}$ of positive numbers such that $r_n \uparrow \infty$ and

$$\frac{\nu(\widehat{B}(x,r_n))}{\mu(B(x,r_n))} \int_{B(x,r_n)} |f| \, d\mu \; \geq \; \alpha\nu(\widehat{B}(x,r_n)).$$

Since

$$\mathcal{N}\nu(y) \; \geq \frac{\nu(\widehat{B}(x,r_n))}{\mu(B(x,r_n))},$$

for any $y \in B(x, r_n)$, we obtain

$$\frac{1}{\alpha}\int_{B(x,r_n)} |f|\ \mathcal{N}\nu d\mu \ \geq\ \nu(\widehat{B}(x, r_n)).$$

By the monotone convergence theorem,

$$\frac{1}{\alpha}\int_X |f|\ \mathcal{N}\nu d\mu \ \geq\ \nu(X \times (0, \infty)).$$

So there is nothing to prove and this observation yields that $\gamma(x)$ is assumed to be everywhere finite. Assume first that E'_α is bounded. We can apply lemma 2.1 to the balls $B(x, \gamma(x))$ to obtain a sequence of disjoint balls $B(x_i, \gamma_i)$, $\gamma_i = \gamma(x_i)$, so that

$$E'_\alpha \subset \bigcup_i B(x_i, 4K\gamma_i).$$

We want to show that

$$E_\alpha \subset \bigcup_i \widehat{S}(x_i, 2K(1+2K)\gamma_i). \tag{3}$$

If $(x, t) \in E_\alpha$, then

$$\frac{1}{\mu(B(y,r))}\int_{B(y,r)} |f|\ d\mu \ > \alpha$$

for some $B(y, r)$ such that $x \in B(y, r)$ and $r \geq t$. Thus $y \in E'_\alpha$ and $r \leq \gamma(y)$. Further, by the last part of lemma 2.1, we have $y \in B(x_i, 4K\gamma_i)$ for some i such that $r \leq \gamma(y) \leq 2\gamma_i$. Then

$$\begin{aligned} d(x_i, x) &\leq K[d(x_i, y) + d(y, x)] \\ &< K[4K\gamma_i + 2\gamma_i] \\ &= 2K(1+2K)\gamma_i. \end{aligned}$$

Hence we have $x \in B(x_i, 2K(1+2K)\gamma_i)$ and
$$t \leq \gamma(y) \leq 2\gamma_i < 2K(1+2K)\gamma_i.$$

If $(x,t) \notin \widehat{\mathcal{S}}(x_i, 2K(1+2K)\gamma_i)$ for any i, then $(x,t) \in \Omega_{y_o}$ for some $y_o \notin \mathcal{S}(x_i, 2K(1+2K)\gamma_i)$. By the assumption i) of theorem 3.1,

$$(x, 2K(1+2K)\gamma_i) \in \Omega_{y_o}.$$

So we get

$$x \in \Omega_{y_o}(2K(1+2K)\gamma_i) \cap B(x_i, 2K(1+2K)\gamma_i) \neq \phi$$

and therefore $y_o \in \mathcal{S}(x_i, 2K(1+2K)\gamma_i)$, which is a contradiction. Thus (3) holds. Hence we have

$$\begin{aligned}
\nu(E_\alpha) &\le \sum_i \nu(\widehat{\mathcal{S}}(x_i, 2K(1+2K)\gamma_i)) \\
&\le C \sum_i \nu(\widehat{B}(x_i, 2K(1+2K)\gamma_i)) \\
&= C \sum_i \frac{\nu(\widehat{B}(x_i, 2K(1+2K)\gamma_i))}{\mu(B(x_i, 2K(1+2K)\gamma_i))} \mu(B(x_i, 2K(1+2K)\gamma_i)) \\
&\le C \sum_i \frac{\nu(\widehat{B}(x_i, 2K(1+2K)\gamma_i))}{\mu(B(x_i, 2K(1+2K)\gamma_i))} \mu(B(x_i, \gamma_i)) \\
&\le \frac{C}{\alpha} \sum_i \frac{\nu(\widehat{B}(x_i, 2K(1+2K)\gamma_i))}{\mu(B(x_i, 2K(1+2K)\gamma_i))} \int_{B(x_i,\gamma_i)} |f|\, d\mu \\
&\le \frac{C}{\alpha} \int_X |f|\, \mathcal{N}\nu d\mu.
\end{aligned}$$

The second inequality follows from assumption ii) of theorem 3.1, the third inequality follows from the doubling property and the last inequality follows from the disjointness of the balls $B(x_i, \gamma_i)$.If E'_α is not bounded, fix $a \in X$ and $R > 0$. Let $E''_\alpha = \{(x,t) : \mathcal{M}f(x,t) > \alpha$ and $y \in E'_\alpha \cap B(a,R)$ for some y such that $x \in B(y,r)$ and $r \ge t\}$. The above argument shows that

$$\nu(E''_\alpha) \le \frac{A}{\alpha} \int_X |f|\, \mathcal{N}\nu d\mu.$$

Applying lemma 2.1 to the balls $\{B(x,\gamma(x)) : x \in E'_\alpha \cap B(a,R)\}$ and letting $R \uparrow \infty$, we obtain the same weak type estimate as before. This proves that $\mathcal{M}$ is of weak type (1,1).

It is easy to see that $\mathcal{M}$ is of type (∞,∞). Suppose that $\mathcal{N}\nu(x) \neq 0$ for all x and $\alpha > \|f\|_{L^\infty(\mathcal{N}\nu d\mu)}$. (There is no loss of generality in doing so for if $\mathcal{N}\nu(x) = 0$ for some x, then $\nu(\widehat{B}) = 0$ for any ball B containing x. Thus $\nu(X \times [0,\infty)) = 0$, so there is nothing to prove.) We have $\int_{\{|f|>\alpha\}} \mathcal{N}\nu d\mu = 0$ and consequently $\mu\{|f| > \alpha\} = 0$. Hence $|f(x)| \leq \alpha$ almost everywhere and so we have $\mathcal{M}f \leq \alpha$. Therefore $\|\mathcal{M}f\|_{L^\infty(d\nu)} \leq \alpha$ and finally we obtain

$$\|\mathcal{M}f\|_{L^\infty(d\nu)} \leq \|f\|_{L^\infty(\mathcal{N}\nu d\mu)}.$$

It follows from the Marcinkiewicz interpolation theorem [6] that theorem 3.1 holds. The proof is complete. □

EXAMPLE. Fix $\alpha > 0$. Define $\Omega_x = \{(y,t) : d(x,y) < \alpha t\}$. Then Ω_x satisfies the hypothesis of Theorem 3.1. In fact,

$$\mathcal{S}(x,r) \subset B(x, K(1+\alpha)r)$$

and (1) is obvious. When $X = \mathbb{R}^n$ (so that $X \times [0,\infty)$ can be identified with $\bar{\mathbb{R}}^{n+1}_+$), the set Ω_x in this example is a nontangential cone in $\bar{\mathbb{R}}^{n+1}_+$ with vertex $x \in \mathbb{R}^n$ and *aperture* α.

REMARK. Let $X = \mathbb{R}^n$ and $\Omega_x = \{(y,t) : |x-y| < t\}$ be a nontangential cone in $\mathbb{R}^{n+1}_+$ with vertex $x \in \mathbb{R}^n$ and aperture 1. When $d\mu = dx$ the Lebesgue measure, we have the same result as in 1) cited in the introduction, because

$$\widetilde{B}(x, r/C) \subset \widehat{B}(x,r) \subset \widetilde{B}(x,r)$$

for some constant $C > 0$

With Theorem 3.1, we can show results similar to 2) and 3) cited in the introduction.

COROLLARY 1. *Definie Ω so that*

$$\Omega_x = \{(y,t) \in X \times [0,\infty) : d(x,y) < t\}.$$

Then $\mathcal{M}$ is bounded from $L^p(X,d\mu)$ into $L^p(X \times [0,\infty), d\nu)$, $1 < p < \infty$, if and only if there is a constant $C > 0$ so that

$$\sup_B \frac{\nu(\widehat{B})}{\mu(B)} \leq C, \tag{4}$$

where the sup is taken over all balls.

Proof. The "*if*" part follows from Theorem 3.1. Conversely, suppose that $\mathcal{M}$ is bounded from $L^p(X, d\mu)$ into $L^p(X \times [0,\infty), d\nu)$, $1 < p < \infty$. Put $f = \chi_{B(y,r)}$ the characteristic function of $B(y,r)$. Then

$$\int_{X\times[0,\infty)} [\mathcal{M}f(x,t)]^p d\nu(x,t) \le C\mu(B(y,r)) \tag{5}$$
$$\le C\mu(B(y,r)).$$

On the other hand, if $(x,t) \in \widehat{B}(y,r)$, then $d(x,z) \ge t$ for any $z \notin B(y,r)$. In other words, $B(x,t) \subset B(y,r)$. Hence

$$\begin{aligned} 1 &= \frac{\mu(B(x,t) \cap B(y,r))}{\mu(B(x,t))} \\ &= \frac{1}{\mu(B(x,t))} \int_{B(x,t)} \chi_{B(y,r)} d\mu \\ &\le \mathcal{M}f(x,t) \end{aligned}$$

and so

$$1 = \chi_{\widehat{B}(y,r)}(x,t) \le \mathcal{M}f(x,t)$$

for any $(x,t) \in X \times [0,\infty)$. Thus

$$\nu(\widehat{B}(y,r)) \le \int_{X\times[0,\infty)} [\mathcal{M}f(x,t)]^p d\nu(x,t). \tag{6}$$

Combining (5) and (6), we obtain (1). The proof is complete. □

COROLLARY 2. *For $\Omega = \{\Omega_x\}$ satisfying i) and ii),*

$$\mathcal{N}\nu(x) = \sup_{x\in B} \frac{\nu(\widehat{B})}{\mu(B)} \le c\omega(x) \quad a.e. \quad x \in X$$

is sufficient for $\mathcal{M}$ to be bounded from $L^p(X, \omega d\mu)$ into $L^p(X\times[0,\infty), d\nu)$ for $1 < p < \infty$.

Let $\phi \ge 0$ and a measurable function on $(X, d\,\mu)$. Take $d\nu(x,t) = \phi(x)d\mu(x) \otimes d\delta(t)$, where δ is the unit mass concentrated at the origin in the t axis. Then $\mathcal{M}f(x,0) = Mf(x)$ and $\mathcal{N}\nu(x) = M\phi(x)$, where $Mf(x) = \sup_{x\in B} \frac{1}{\mu(B)} \int_B |f|\, d\mu$. From this and Theorem 3.1 we have following corollary.

COROLLARY 3. *Let ϕ be an nonnegative measurable function on (X, d, μ). For $1 < p < \infty$, there is a constant C such that*

$$\int_X [Mf(x)]^p \phi(x) d\mu(x) \leq C \int_X |f|^p \ M\varphi(x) d\mu(x)$$

REMARK. The crucial hypothesis in theorem 3.1 is (2). Condition (1) was included because it makes the statement of the theorem some what simpler. We now proceed to show that when studying the weak type property of $\mathcal{M}$, there is no loss of generality in requiring that (1) be satisfied. Define

$$\widetilde{\Omega}_{x_o} = \{(x,t) \in X \times (0,\infty) : (x,s) \in \Omega_{x_o} \text{ for some } s \leq t\}$$

and let $\widetilde{\mathcal{S}}(x,r)$ be defined as in Definition 2.7 but with $\widetilde{\Omega}_{x_o}$ in place of Ω_{x_o}. Note that $\mathcal{S}(x,r) \subset \widetilde{\mathcal{S}}(x,r)$ since $\Omega_{x_o} \subset \widetilde{\Omega}_{x_o}$. Thus the following result shows that theorem 3.1 can be strengthened.

THEOREM 3.3. *If there is a constant $C \geq 0$ so that $\nu(\widetilde{\mathcal{S}}(x,r)) \leq \nu(B(x,r))$ for every $(x,r) \in X \times (0,\infty)$, then $\mathcal{M}$ is bounded from $L^p(X, \mathcal{N}\nu d\mu)$ into $L^p(X \times [0,\infty), d\nu)$ for $1 < p < \infty$.*

Proof. If the assumption holds then the sets $\widetilde{\Omega}_{x_o}$ satisfy hypothesis (2) of Theorem 3.1. Since they also satisfy (1), in the proof of Theorem 3.1,

$$\{\mathcal{M}f > \alpha\} \subset \bigcup_i \widehat{\widetilde{\mathcal{S}}}(x_i, 2K(1+2K)\gamma_i).$$

Hence the proof is same as that of Theorem 3.1. □

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