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STRONG-MAX CYCLIC SUBMODULES

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ABSTRACT. In this paper we define CR(completely reachable), MICR(minimal cyclic refinement) and MACR(maximal cyclic refinement)-Modules. We have obtained equivalent statements for minimal cyclic submodule and maximal cyclic submodule. Also, we have obtained necessary and sufficient conditions for a module M with MICR to be cyclic or strongly cyclic.

1. Introduction

In this paper we characterize the minimal and the maximal cyclic submodules of an arbitrary module M. Also we give some characterizations of classes of modules, that is to say, strongly cyclic, CR(completely reachable), strong CR. In order to do these we introduce S(m), C(m), MICR(minimal cyclic refinement) and MACR(maximal cyclic refinement) where $S(m)$ is the source set of $m \in M$ and $C(m) = \{0, q \in M : mR = qR\}$.

From now on, we assume that a ring R has an identity 1 and a right R-module $M \neq \{0\}$. We have defined *strongly cyclic module* in Park [1] but we shall restate it here. $\{0\}$ will be denoted 0.

DEFINITION 1. (1) M is *strongly cyclic* if $M \neq 0$ and $M = mR$ for any $m(\neq 0) \in M$ (or $\forall m(\neq 0), q \in M$, $q = ma$ for some $a \in R$).

(2) M is *cyclic* if $M = mR$ for some $m \in M$.

(3) mR is *a minimal cyclic submodule* if $mR \neq 0$ and $\forall q \in M$, $0 \subsetneq qR \subset mR \Longrightarrow qR = mR$.

(4) mR is *a maximal cyclic submodule* if $mR \neq M$ and $\forall q \in M$, $mR \subset qR \subsetneq M \Longrightarrow qR = mR$.

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(5) $H \subset M$ is a *strongly cyclic subset of* M if $\forall m(\neq 0), q \in H$, $q = ma$ for some $a \in R$.

The proof of the following Lemma is quite straightforward.

LEMMA 1. *$H \leq M$ is minimal submodule of $M \iff H \leq M$ is strongly cyclic submodule.*

DEFINITION 2. Let $m(\neq 0) \in M$.

(1) $S(m) = \{0, q \in M : m = qa \ for \ some \ a \in R\}$ is called the *source set* of $m \in M$.

(2) $m \in M$ is *completely reachable* in M if $M = S(m)$

(3) $C(m) = \{0, q \in M : mR = qR\}$

(4) M is a *CR-module*(module with a completely reachable element) if $M = S(n)$ for some $n(\neq 0) \in M$

LEMMA 2. *Let M be a right R-module. Then we have the following statements :*

(1) $S(m) \subset S(ma)$ for any $a \in R$ and $q \in S(m) \Longrightarrow S(q) \subset S(m)$.

(2) $M = S(m) \iff m \in \bigcap_{q(\neq 0) \in M} qR \neq \emptyset$.

(3) $\bigcap_{m(\neq 0) \in M} mR$ is a strongly cyclic submodule of M if M is a CR-module.

Proof. (1) and (2) are trivial. For (3), we shall show $\bigcap_{m(\neq 0) \in M} mR = qR$ for every $q(\neq 0) \in \bigcap_{m(\neq 0) \in M} mR$. We note that $q \in mR$ for all $m(\neq 0)$ *in* M and hence $q = mb$ for some $b \in R$. We let $t \in qR$. Then $t = qa$ for some $a \in R$. This implies $t = (mb)a = m(ba) \in mR$ for all $m(\neq 0)$ *in* M. Hence $t \in \bigcap_{m(\neq 0) \in M} mR$. The converse is trvial. □

We define new terminologies.

DEFINITION 3. Let $m(\neq 0) \in M$.

(1) $S(m)$ is *minimal set* if $\forall q \in M$, $0 \subsetneq S(q) \subset S(m) \Longrightarrow S(q) = S(m)$.

(2) $S(m)$ is *maximal set* if $\forall q \in M$, $S(m) \subset S(q) \subsetneq M \Longrightarrow S(q) = S(m)$.

LEMMA 3. *Let $m, n(\neq 0) \in M$. Then the following statements hold:*
(1) $mR \subset nR \Longleftrightarrow S(m) \supset S(n)$
(2) $mR = nR \Longleftrightarrow S(m) = S(n)$.
(3) mR *is minimal[max]* $\Longleftrightarrow S(m)$ *is maximal[min] set.*
(4) mR *is strongly cyclic* $\Longleftrightarrow mR$ *is minimal.*
(5) $S(m)$ *is strongly cyclic set* $\Longleftrightarrow S(m)$ *is minimal set*

Proof. For (1), ($\Rightarrow$) we let $t \in S(n)$. Then $n = tb$ for some $b \in R$. But $m = m1 \in mR \subset nR$ and then $m = nc$ for some $c \in R$. This implies $m = nc = (tb)c = t(bc)$. Hence $t \in S(m)$. ($\Leftarrow$) we can prove it in the same way. For (2), we can prove it like (1). (3) comes from (1) and (4) comes from Lemma 1. For (5), ($\Rightarrow$) we let $0 \subsetneq S(q) \subset S(m)$, $\forall q(\neq 0) \in M$. To prove $S(q) \supset S(m)$ we let $p(\neq 0) \in S(m)$. We note that $q \in S(m)$. Since $S(m)$ is strongly cyclic, we have $q = pa$ for some $a \in R$ and hence $p \in S(q)$. ($\Leftarrow$) also it is trivial. □

LEMMA 4. *Let M be a right R-module and $m(\neq 0) \in M$. Then we have the following statements :*
(1) $C(m)$ *is a strongly cyclic subset of* M.
(2) $C(m) = C(n) \Longleftrightarrow$ (i) $ma = n$ *and* $nb = m$ *for some* $a, b \in R$
$\Longleftrightarrow$ (ii) $S(m) = S(n)$
(3) *Let D_m be a strongly cyclic subset of M with $m \in D_m$. Then $D_m \subset C(m)$.*
(4) $C(n) = nR \cap S(n)$ *for any* $n(\neq 0) \in M$.
(5) $\bigcap_{m(\neq 0) \in M} C(m) = \{0\}$

Proof. For (1), let $p(\neq 0), t \in C(m)$. Then $mR = pR$ and $mR = tR$. From this we have $pa = t$ for some $a \in R$.

For (2)(i), ($\Rightarrow$) : trivial. ($\Leftarrow$) : To show $C(m) \subset C(n)$ we let $t \in C(m)$. Then $mR = tR$. This implies $S(m) = S(t)$. But $n = ma \Rightarrow S(m) \subset S(n)$ and $m = nb \Rightarrow S(n) \subset S(m)$. This means $S(m) = S(n)$. Hence we have $S(n) = S(t)$ and then $nR = tR$. i.e., $t \in C(n)$. Similarly, we can prove the converse.

For (2)(ii), ($\Rightarrow$) : To show $S(m) \subset S(n)$ we let $t \in S(m)$. Then $m = tc$ for some $c \in R$. But $ma = n$ for some $a \in R$. Hence we have $n = ma = (tc)a = t(ca)$. This means $t \in S(n)$. Similarly, it is easy to show the converse. ($\Leftarrow$) : It is trivial.

For (3), let $p(\neq 0) \in D_m$. Then we have $m = pa$ and $p = mb$ for some $a, b \in R$. From (2) we have $p \in C(p) = C(m)$.

For (4), $C(n) \subset nR \cap S(n)$: Let $t \in C(n)$. Then $nR = tR$. From this we have $n = ta$ for some $a \in R$. Hence $t \in S(n)$ and then $t \in nR \cap S(n)$. Also, it is easy to check the converse.

For (5), let $p(\neq 0) \in C(m) \cap C(q)$. Then we have $mR = pR$ and $qR = pR$. To show $C(m) \subset C(q)$ we let $t \in C(m)$. Then $mR = tR$. From this we have $tR = qR$. Hence $t \in C(q)$. Similarly, it is trivial to show $C(m) \supset C(q)$. This means that we have shown $p \in C(m) \cap C(q) \Longrightarrow C(m) = C(q)$. □

2. Characterizations of minimal and maximal cyclic submodules in a module M

THEOREM 5. *Suppose that M is not cyclic and let $m(\neq 0) \in M$. Then the following assertions are equivalent :*

(1) *mR is a maximal cyclic submodule of M ;*

(2) $S(m) = C(m)$;

(3) $S(m) \cap S(q) \neq 0, \forall q(\neq 0) \in M \Longrightarrow S(m) \subset S(q)$;

(4) $S(m) \subset mR$;

(5) *$S(m)$ is a strongly cyclic subset of M ;*

(6) $C(m) \cap qR \neq 0, \forall q(\neq 0) \in M \Longrightarrow q \in C(m)$;

(7) *$m = qa$ for some $a \in R$, $\forall q(\neq 0) \in M \Longrightarrow C(m) = C(q)$;*

(8) $S(m) \cap qR \neq 0, \forall q(\neq 0) \in M \Longrightarrow mR = qR$.

Proof. (1) $\Rightarrow$ (2) : We shall show $S(m) \subset C(m)$. Let $q \in S(m)$. Then $S(q) \subset S(m) \Rightarrow qR \supset mR$. Hence $qR = mR$ and then $q \in C(m)$. $S(m) \supset C(m)$ comes from Lemma 4(4).

(2) $\Rightarrow$ (3) : Let $t \in S(m) \cap S(q)$. Then we have $t \in S(m)$ and $t \in S(q)$. This implies $S(t) \subset S(m)$ and $S(t) \subset S(q)$. But $t \in C(m)$. This means $S(m) = S(t)$. Hence $S(m) = S(t) \subset S(q)$.

(3) $\Rightarrow$ (4) : Let $q(\neq 0) \in S(m)$. Then we have $S(q) \subset S(m)$ and then $S(q) \cap S(m) \neq 0$. From assumption we have $S(m) \subset S(q)$. Hence $S(m) = S(q)$. From this $q = ma$ for some $a \in R$ and hence $q \in mR$.

(4) $\Rightarrow$ (5) : Let $p, q(\neq 0) \in S(m) \subset mR$. Then we have $S(q) \subset S(m)$ and $S(q) \subset S(m)$. Also $p = ma$ and $q = mb$ hold for some $a, b \in R$. This means that $m \in S(p) \Rightarrow S(m) \subset S(p)$ and $m \in S(q) \Rightarrow S(m) \subset$

$S(q)$. Hence $S(m) = S(p) = S(q)$. This shows that $p = qc$ for some $c \in R$.

(5) $\Rightarrow$ (1) : Let $mR \subset qR \subsetneq M$ for $q \in M$. Then from Lemma 3(1) we have $S(m) \supset S(q)$. Since $S(m)$ is strongly cyclic, $q = ma$ holds for some $a \in R$. This implies $m \in S(q)$ and then $S(m) \subset S(q)$. Hence we have $S(m) = S(q)$. This means $mR = qR$ from Lemma 3(2).

(1) $\Rightarrow$ (6) : Let $p \in C(m) \cap qR \neq 0$. Then we have $mR = pR$ and $pR \subset qR$. This means $mR \subset qR$ and from assumption $mR = qR$ holds. Hence $q \in C(m)$. (6) $\Rightarrow$ (1) : From Lemma 4(4) it is trivial.

(1) $\Leftrightarrow$ (7) : It is trivial.

(1) $\Rightarrow$ (8) : Let $p \in S(m) \cap qR$. Then we have $p \in S(m)$ and $p \in qR$. This implies that $m = pa$ for some $a \in R$ and $pR \subset qR$. Also we have $mR \subset pR$. Since mR is maximal, we have $mR = qR$.

(8) $\Rightarrow$ (1) : Let $mR \subset qR$. Then from Lemma 4(4) we have $S(m) \cap qR \neq 0$. Hence $mR = qR$ □

THEOREM 6. *Let $m(\neq 0) \in M$. Then the following conditions are equivalent :*

(1) *mR is minimal ;*

(2) *$C(m) = mR$;*

(3) *$C(m)$ is a submodule of M ;*

(4) *$C(m) \cap S(q) \neq 0, \forall q(\neq 0) \in M \Longrightarrow S(m) = S(q)$;*

(5) *$mR \subset S(m)$;*

(6) *$\forall a \in R \ \exists \ b \in R : mab = m$;*

(7) *$mR \cap qR \neq 0, \forall q(\neq 0) \in M \Longrightarrow mR \subset qR$;*

Proof. (2) $\Leftrightarrow$ (3) : It is trivial. (1) $\Rightarrow$ (3) : (i) let $q \in C(m)$ and $a \in R$. Then we have $mR = qR$. But $qaR \subset qR = mR$. Since mR is minimal, $qaR = mR$ holds. Hence $qa \in C(m)$. (ii) to show $(C(m), +)$ is a subgroup of M we let $p, q \in C(m)$. Then $mR = pR$ and $mR = qR$ hold. From this for every $a \in R$ we have $mb = pa$ and $mc = qa$ for some $b, c \in R$. This implies $(p - q)a = m(b - c) \in mR$. Hence $(p - q)R \subset mR$ holds. Since mR is minimal, we have $(p - q)R = mR$. This means $(p - q) \in C(m)$.

(3) $\Rightarrow$ (4) : Let $p \in C(m) \cap S(q)$. Then $p \in C(m)$ and $p \in S(m)$. From this we have $q = pa \in C(m)$ for some $a \in R$. Hence $mR = qR$ holds.

(4) $\Rightarrow$ (5) : $S(m) \subset S(ma)$ holds for $a \in R$. From Lemma 4(4) we have $C(m) \cap S(ma) \neq 0$. Hence $ma \in S(m)$.

(5) $\Rightarrow$ (6) : It is trivial. (6) $\Rightarrow$ (1) : Let $ma, mc \in mR$. For $a \in R$ $\exists b \in R : mab = m$. From this $mc = (mab)c = ma(bc)$.

(1) $\Rightarrow$ (7) : Let $p \in mR \cap qR$. Then $pR \subset mR$ and $pR \subset qR$. Since mR is minimal, we have $mR \subset qR$.

(7) $\Rightarrow$ (1) : It is trivial. □

The following Theorem comes from Theorem 5 and Theorem 6. Therefore, we shall omit its proof.

THOEREM 7. *Suppose that M is not cyclic and let $m(\neq 0) \in M$. Then the following conditions are equivalent :*

(1) *mR is minimal and maxmal ;*
(2) *$S(m) = C(m)$ is a submodule of M ;*
(3) *$S(m) \cap S(q) \neq 0$, $\forall q(\neq 0) \in M \Longrightarrow S(m) = S(q)$;*
(4) *$mR = S(m)$;*
(5) *$S(m)$ is strongly cyclic submodule of M ;*
(6) *$mR \cap qR \neq 0$, $\forall q(\neq 0) \in M \Longrightarrow mR = qR$*

3. MACR-modules and MICR-modules

We introduce new terminologies.

DEFINITION 4. Let M be a right R-module.

(1) $R^{-1}(min) = \{0, m \in M : mR \ is \ minimal\}$.
(2) $R^{-1}(max) = \{0, m \in M : mR \ is \ maximal\}$.
(3) M is a *MACR(maximal cyclic refinement)* - module if $\forall m(\neq 0) \in M \ \exists \ q(\neq 0) \in R^{-1}(max) : mR \subset qR$.
(4) M is a *MICR(minimal cyclic refinement)* - module if $\forall m(\neq 0) \in M \ \exists \ q(\neq 0) \in R^{-1}(min) : qR \subset mR$.
(5) M is a *strong CR*-module if $M = S(m)$ for every $m(\neq 0) \in M$.

LEMMA 8. *Let M be a right R-module. Then the following statements hold :*

(1) *Every CR-module is a $MICR$-module such that $M = S(q)$ for every $q(\neq 0) \in R^{-1}(min)$.*

(2) *mR is minimal cyclic submodule for every completely reachable element $m \in M$.*

(3) *Strongly cyclic module* $\Longrightarrow$ *Strong* CR*-module* $\Longrightarrow$ CR*-module.*

Proof. For (1), since M is CR-module, we have $M = S(m)$ for some $m(\neq 0) \in M$. To prove $S(q) \supset M$ we let $p \in M = S(m)$. Then $p \in S(p) \subset S(m)$. On the other hand, we have $q \in M = S(m)$. This implies that $S(q) \subset S(m) \Longleftrightarrow qR \supset mR$ and hence $qR = mR$ since qR is minimal. Hence it holds. To prove that M is a MICR we let $m(\neq 0) \in M$. Then $S(m) \subset S(p)$ for $p(\neq 0) \in R^{-1}(min)$ and hence $mR \supset pR$.

For (2), we let $0 \subsetneq qR \subset mR$ for $q \in R$. Then $M = S(m) \subset S(q)$. This implies $S(q) = S(m)$ and hence $qR = mR$. (3) comes from definitions. □

THEOREM 9. *Let M be a $MACR$-module. If there is a $m(\neq 0) \in M$ such that $C(m) = R^{-1}(max)$, then M is cyclic.*

Proof. Let $q \in M$. Since M is $MACR$-module, there is a $p \in R^{-1}(max)$ such that $qR \subset pR$. From this we have $q \in qR \subset pR = mR$. Hence $M = mR$. □

THEOREM 10. *Let M be a $MICR$-module. Then we have the following statements :*

(1) *M is a CR - module* $\Longleftrightarrow \exists$ *$m(\neq 0) \in M$ such that $C(m) = R^{-1}(min)$.*
(2) *M is strongly cyclic* $\Longleftrightarrow \exists$ *$m(\neq 0) \in M$ such that $S(m) = R^{-1}(min)$.*
(3) *M is strong CR* $\Longleftrightarrow$ *M is strongly cyclic.*

Proof. For (1), ($\Leftarrow$) Let $m \in M$ such that $C(m) = R^{-1}(min)$. We let $q(\neq 0) \in M$. Then $\exists$ $p(\neq 0) \in R^{-1}(min)$ such that $pR \subset qR$. From $p \in C(m)$ we have $mR = pR \subset qR$. Hence $q \in S(m)$ and then $M = S(m)$.

($\Rightarrow$) we note that $M = S(m)$ for some $m(\neq 0) \in M$. From Lemma 8(2) we have $m \in R^{-1}(min)$. It is trivial to show $C(m) = R^{-1}(min)$ from Lemma 2(2).

For (2), ($\Leftarrow$) Let $m \in M$ such that $S(m) = R^{-1}(min)$. Claim : $C(m) = S(m)$. (proof) since mR is minimal, we have $mR = C(m) \subset S(m)$. To prove $C(m) \supset S(m)$ we let $q \in S(m)$. From assumption

$\exists\ p \in R^{-1}(min)$ such that $pR \subset qR$. From this we have $S(p) \subset S(m)$ and then $mR \subset pR \subset qR$. Hence we have $mR = qR$ and $q \in C(m)$.

Combining the claim and hypothesis, we have $C(m) = S(m) = R^{-1}(min)$. Also, from (1) we have $M = S(m)$. Now we shall show that M is strongly cyclic. Let $p, q \in M = C(m)$. Then we have $mR = pR$ and $mR = qR$. This implies $p = qa$ for some $a \in R$.

($\Rightarrow$) We note that $M = S(m)$ for every $m(\neq 0) \in M$ since M is strongly cyclic. We shall show $R^{-1}(min) = M$. Let $q(\neq 0) \in M$. Then $\exists\ p \in R^{-1}(min)$ such that $pR \subset qR$. Since M is strongly cyclic, we have $M = pR$. Hence $pR = qR$ holds and $q \in R^{-1}(min)$. From this we have $S(m) = R^{-1}(min)$. □

From Theorem 6 and the above Theorem we have the following Corollary.

COROLLARY 10.1. *If M is a CR-module, then $R^{-1}(min) = C(m) = mR$.*

4. Examples

EXAMPLE 1. Let $\mathbb{Z}_3 = \{\ 0, 1, 2\ \}$. Then

(1) $\mathbb{Z}_3$ is strongly cyclic $\mathbb{Z}_3$ - module since $1\mathbb{Z}_3 = \mathbb{Z}_3$ and $2\mathbb{Z}_3 = \mathbb{Z}_3$.

(2) $1, 2 \in \mathbb{Z}_3$ are completely reachable elements of $\mathbb{Z}_3$ since $\mathbb{Z}_3 = S(1)$ and $\mathbb{Z}_3 = S(2)$.

(3) $1\mathbb{Z}_3$ and $2\mathbb{Z}_3$ are minimal submodules of $\mathbb{Z}_3$ from (2).

(4) $R^{-1}(min) = \{0, 1, 2\} = S(1) = S(2)$. Hence $\mathbb{Z}_3$ is strongly cyclic like we have mentioned in (1).

(5) $\mathbb{Z}_3$ is a MICR-module since $1\mathbb{Z}_3 \subset 1\mathbb{Z}_3$ and $2\mathbb{Z}_3 \subset 2\mathbb{Z}_3$.

EXAMPLE 2. Let $\mathbb{Z}_4 = \{\ 0, 1, 2, 3\ \}$. Then

(1) $\mathbb{Z}_4$ is not strongly cyclic $\mathbb{Z}_4$ - module since $2\mathbb{Z}_4 \neq \mathbb{Z}_4$.

(2) $2 \in \mathbb{Z}_4$ is a completely reachable element of $\mathbb{Z}_4$ but $1, 3 \in \mathbb{Z}_4$ are not completely reachable elements of $\mathbb{Z}_4$ since $\mathbb{Z}_4 = S(2)$ but $\mathbb{Z}_4 \neq S(1)$ and $\mathbb{Z}_4 \neq S(3)$.

(3) $2\mathbb{Z}_4$ is minimal submodule of $\mathbb{Z}_4$ from (2) and also a maximal submodule of $\mathbb{Z}_4$ since 2 is a prime dividing 4.

(4) $R^{-1}(min) = \{0, 2\}$.

(5) $\mathbb{Z}_4$ is a MICR-module since $2\mathbb{Z}_4 \subset 1\mathbb{Z}_4$, $2\mathbb{Z}_4 \subset 2\mathbb{Z}_4$ and $2\mathbb{Z}_4 \subset 3\mathbb{Z}_4$.

(6) $\nexists\ m(\neq 0) \in \mathbb{Z}_4$ such that $S(m) = R^{-1}(min)$ since $S(1) = \{0,1,3\}, S(2) = \{0,1,2,3\}$ and $S(3) = \{0,1,3\}$. Hence $\mathbb{Z}_4$ is not strongly cyclic like we have mentioned in (1).

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