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Two Perspectives in Developing a Visualization Environment

Sang Sook Koh*

Visualization has become a hot topic of conversation among mathematics educators (NCTM, 1989, 1991). With the ever-increasing wealth of visualization technologies such as the Geometer's Sketchpad, Cabri Geometry, Mathematix, and Theorist, creating an environment that is rich in visualization possibilities is easier than ever before. As mathematics educators, we may often put the students in these environment with little consideration of their thought processes. There has not been enough discussion linking what occurs inside the student's mind during their sessions in visual environments to cognitive theories in general. The purpose of this paper is to show how constructivism and information processing explain different parts of mathematics visualization.

After defining visualization, I will discuss both of the cognitive aspects in their generality, then show how they might be applied to the process of visualization. By showing this application, I will point out the parts of visualization that each aspect clarifies. Afterwards, I will narrow the discussion to the idea of concept images. Then I will put the pieces together to form a bigger picture and a better understanding of visualization. Finally, I will conclude with an example of a visualization environment created with these models as guidelines.

Visualization

Visualization is more than perception. Students may see a great deal but understand very little. Zimmerman and Cunningham (1991) state, "If mathematics is the science of patterns, it is natural to try to find the most effective ways to visualize these patterns and to learn to use visualization creatively as a tool for understanding. This is the essence of mathematical visualization." (p. 3) Visualization also has an interpretation component in that visualization is the student's ability to translate and interpret information across both symbolic forms and visual representations.

Visualization as Constructivism

To understand what constructivism has to say about visualization, we must first understand constructivism as an epistemology. Constructivism

* 한국 교원대 강사

differs from most epistemologies in two basic respects. While most epistemologies consider "true knowledge" to be independent of the knowing subject, constructivism is based on the ideas that knowledge is actively built up by the student. Also, most epistemologies contend a fully structured and knowable world "exists" and it is the business of the students to discover what that structure is. Constructivism claims the function of cognition is adaptive, tending towards fit or viability. Constructivism relies heavily on the idea that cognition serves the student's organization of the experiential world (von Glasserfeld, 1990).

Next, we should understand constructivism as a mechanism of learning. Reflective abstraction, repeated experience, and knowledge hierarchies make up the basic moving parts of the mechanism of constructivism. Reflective abstraction is the act of abstracting commonalities and rules from experiences. Reflective abstraction "enables [the student] to avoid disagreeable situations and, to some extent, to generate agreeable ones" (von Glasserfeld, 1990, p. 24). Repeated experience is fairly self evident. Since no two situations are ever completely alike, regularity is achieved by disregarding some of the differences (von Glasserfeld, 1990). I characterize knowledge hierarchies as the way we choose which differences to disregard.

If we take a constructivist view of visualization, we must surely prepare discovery-based lessons to allow the students to construct their own meanings. Thomas (1992) has three suggestions that help us in interpreting a constructivist view of visualization. Repeated experience is evident in his first suggestion, that teachers should introduce topics thoroughly and use exploration activities to give students the feel of the problem. Multiple representations allow students to decide which information is relevant and which can be disregarded. Reflective abstraction takes time and is clearly the intent behind Thomas's second suggestion; that teachers must not force the pace of a discovery lesson through visualization. Reflective abstraction also shows up in his final suggestion, that teachers require homework that "may consist of a half hour of careful reflection the night before on the part of your students" (Thomas, 1992, p. 271). These activities also allow teachers to monitor the student's knowledge hierarchies ensuring that important features are not being disregarded.

"A correct constructivist approach implies the creation of a problematic environment that would elicit the student's adequate constructive endeavors."(Fiscbein, 1990, p. 8). Nowhere is this more evident than in the computer environment of the Geometer' s Sketchpad. This environment gives students the tools to geometrically construct answers to very difficult problems. This package shows the vitality of dynamic visualization software. For example, a student is allowed to construct an object on a given triangle and then to watch the phenomena through a range of different triangles. By moving one of the original vertices, the student experiences the concept as it applies to acute, right, and obtuse triangles. A knowledge hierarchy of motion is created and students quite often see what facets are important and which can be ignored. Fischbein considered instructional programs as restricting to the student. He says that programs like GSP represent only the initial level of a constructive instructional program. I see a giant step forward in the field of mathematical visualization in the classroom. I feel GSP is a tool that allows students to create a myriad of problems and solutions and valuable visualizations. So fischbein's comments are a challenge for developers to create technology that is less restrictive to students and even richer in instructional possibilities.

If the function of cognition is adaptive, tending towards fit or viability as constructivism suggests, then as educators, we must prepare experiences that point out several different aspects of a given topic. In the visualization classroom, this may mean several representations of the same mathematics concept. We must allow time for the students to make the pieces fit and to monitor their knowledge hierarchies.

Visualization as Information Processing

The main features of the information-processing system are collectively called the system architecture (Farnham-Diggory, 1992). The system architecture consists of a sensory register, working memory and long-term memory. The sensory register is the direct link from the human mind to the outside world. The working memory can be considered the executive system of the mind. The working memory continually scans and encodes new information from the sensory register. It also processes existing information. Finally, the working memory utilizes its connection to the long-term memory through storing and retrieving information. The long-term memory can be seen as the filing cabinet of the mind. It is often described in terms of networks and association, but is always given the job of storing information. This system has been used to describe several basic functions of cognition. The part of the information processing model that deals most directly with visualization is the interaction of the working memory and the long-term memory, that is, the store and retrieval knowledge.

For the purpose of this section, knowledge will be characterized in two forms: declarative and procedural (Anderson, 1983). Consider further refinement of these, by the way the information is encoded, into verbal and non-verbal systems (Paivio, 1986). The non-verbal system is often referred to as the imaginal system. Greeno (1989) characterizes the interplay of these systems as he connects symbolic and manipulative perspectives. The ability to talk about visualization as a manipulative perspective is what is so attractive from my point of view. It is this multiple perspective view that allows me to talk about concepts. I will define a concept to be a cluster of declarative and procedural knowledge that is formed from the symbolic and manipulative representations of a given topic. The act of encoding and decoding a visual representation and translating the representation to a symbolic perspective from interactions with manipulative environments takes place as the creation of a concept. Information processing deals directly with the issue of interpretation in visualization. This occurs through the schemas and concept images.

Schemas and Concept Images

The formation of mathematical concepts occurs via two steps, abstraction and classification (Skemp, 1987). Abstraction is the act of determining the crucial similarities among a group of objects. Classification uses the similarities to determine if a new object belongs with the aforementioned group. The abstraction that has been created is called a concept. Skemp calls the structure that connects these concepts a schema. The mental construct that is developed by a student to understand a concept can be referred to as a concept image. It consists of all the schemas as well as "all the mental pictures and associated properties and processes" of a given concept (Tall & Vinner, 1981). The introduction of graphs, animations, and other visual representations is of the utmost importance because this is visualization's contribution to the concept image. As educators, we must be wary of the criterial attributes that we intend students to form and how those differ from the criterial attributes that were actually formed. If a picture is worth a thousand words, then visualizations allow us to convey the wrong idea quickly. Cunningham (1991) suggests that teachers must determine that which is most critical and remove conflicting information to keep the development of misconceptions at a minimum. Material should also be presented in a logical and connected fashion so proper connections are established in the long-term memory. Finally, students should be allowed to expand on past ideas without the current information becoming confusing or overwhelming. The visual representations are meant to ease understanding, not to confuse the issue.

Creating a Visualization Environment

From the preceding arguments, five ideas for creating a visualization environment become evident. First, as teachers, we must allow time for reflective abstraction. We must give the students several attempts at relatively similar problems, thus providing repeated experiences. We must also monitor our students as they create knowledge hierarchies. Also, we must provide logical and connected visual representations. Finally, we can talk about visualizations as a manipulative environment that serves to expand the symbolic perspective.

With these five ideas in mind, I have constructed a visualization environment to teach ellipses utilizing several technologies. The main unit is in the form of hyper-text markup language (html) documents or "pages" for short. The lessons use the Netscape 4.5 Web Browser to access this information. The small exercises within the unit are programs prepared using the Geometer's Sketchpad (GSP) as well as the program I deviced in CD Rom.

The first page the students encounter is an overview of the unit. Successive pages take the students through the construction of the ellipses, the geometry inherent in the ellipse, and the algebraic manipulations of the formulas of the ellipses. At each level, students are given the links to software packages and encouraged to explore.

Repeated experience was the main concept driving the creation of this visualization unit. On each page, students are encouraged to repeat the investigations with each of various ellipses. These functions are a good platform because the objects are usually related in their axes and there is a degree of increase difficulty between each of preceding windows. So. it seemed logical to describe the behavior of ellipse, let the students recreate my exploration for themselves, and leave problem solving for the initial self-constructed exploration available to the students. They would then progress through the remaining other graphs drawing out the similarities. Repeated experience also is present as students are allowed to view my demonstrations either by constructing them from as minimal set of given circumstances, or view a basic model (Figure 1), or view the abstract model (Figure 2) complete with color to accentuate certain notions and animations to enhance certain ideas. Reflective abstraction becomes evident due to the hypermedia aspect of the internet browser software. Students are encouraged to move through the unit pages, but

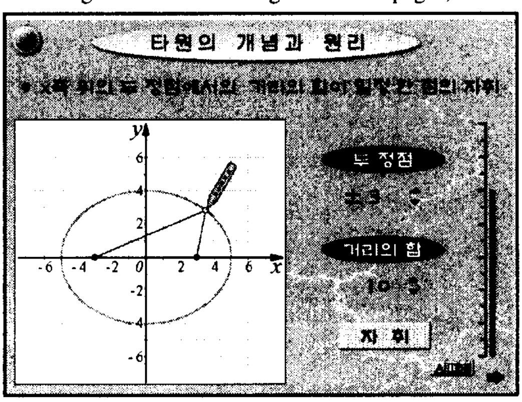

Figure 1

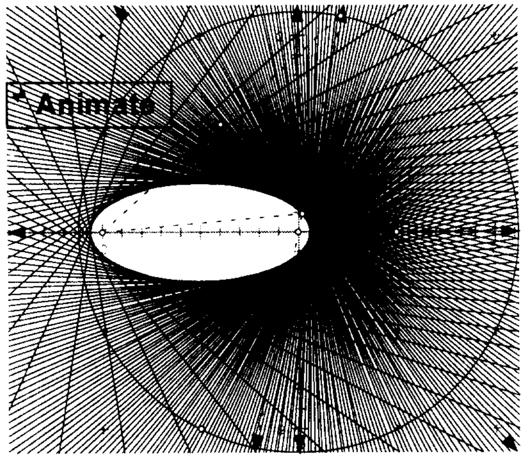

Figure 2

do so at their own pace. The depth of investigation is also left partially to the discretion of the student. So, a students who requires time and depth of exploration is not penalized with the use of this visualization unit. Instead, the student is given the time and the flexibility to pursue the knowledge to its fullest.

Knowledge hierarchies manifest themselves in different ways through the use of this unit. The close connection to GSP and its inherent abilities allows students to manipulate and concern. The connections made between the various representations (GSP and the CD Rom) during the units were selected to clarify the relation of several different aspects of ellipses to the previous material covered in geometry and algebra. The rule of thumb for switching between software representations and different content ideas has been to switch one while holding the other constant. For example, I chose to use the CD Rom first because the model that I wanted to display as the foundation of ellipses (see Figure 1) was relatively easy to construct using the geometry software. Consistently using the CD Rom, I then shifted the content focus to graphs with formulas (equations). Then, for a different view of graphing, I switched software package to the GSP.

Finally, considering visualization as a manipulative environment was a factor in choosing the area of ellipse equations. In my own teaching, I was frustrated teaching the material from a symbolic perspective. The students asked for a deeper understanding. In other words, they wanted more declarative and procedural knowledge. Until I found the visualization environment, I was unable to meet that need. Now, the students get a different perspective on the ellipse equations. They can take the information they derive from the GSP environment and group that with the symbolic perspective they already had. Students exposed to the visualization environment not only have an imaginal system, but by talking about the pictures they have a verbal system as well.

Conclusions

As mathematics educators, we see that looking at visualization from these perspectives is useful in forming a bigger picture of visualization and the mathematics student's mind as well as creating visualization environments. To follow a constructivist view, we must prepare discovery-based lessons that allow students to construct their own meanings. We must prepare experiences that point out several different aspects of a given topic to help in student's reflective abstraction as well as giving repeated experiences with the same information. At the same time, we are reminded that no two children will create the same knowledge from the same visual experiences. We must, therefore, make an attempt to understand the knowledge hierarchies that the students are forming as a result of their experiences in the visualization environment.

Information-processing system tells us that in order to cluster declarative and procedural knowledge from the different symbolic and manipulative perspectives, we must provide logical and connected visualization experiences. It is precisely the ability of the visualization environment to provide a manipulative perspective that makes visualization such a vibrant area for future growth. In days gone by, students were lucky to have seen the model in Figure 2 much less to have used it as a basis to begin the study of ellipse. Perhaps this glimpse has shown some of what lies beneath the surface of a visualization experience and has given some insight into the possibilities that these experiences may hold.

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시각화 환경의 개발에 대한 두 관점

고 상 숙

컴퓨터의 급속한 보급으로 시각화는 수학 교육자사이의 논의에 자주 등장하는 소재가 되었다. 우리는 다양한 소프트웨어를 사용하여 준비한 수업에 학생들로 임하게는 하지만 거의 그들의 사고 발달과정에는 관심을 갖지 못하고 있다. 이 논문은 구성주의(Constructivism)와 정보처리체계 (Information-Processing System)에 입각하여 수학의 시각화를 생각해보고 어떻게 시각화 환경을 준비해야하는지 논해보고자 하였다. 구성주의의 시각화에서는 반영적 추상(reflective abstraction), 반복되는 경험(repeated experience), 그리고 지식 위계성이 학습의 기능체계를 이루므로 발견적 학습을 통해 학생 스스로 의미를 구성할 수 있도록 Thomas (1992)의 세 가지 제안을 이용하여 수업을 준비할 수 있다. 정보처리체계에서는 지식은 서술적인 것과 과정적인 것으로 나뉘어지고, 시각적 표상을 수록하고 삭제하는 과정과 조작 가능한

(manipulative) 환경과의 상호작용으로 기호적 시각으로 표상을 변화하는 과정을 통해 개념을 습득하게된다. 시각화는 스키마와 개념상을 통해서 일어난다. 그래프, 애니메이션, 그리고 다른 시각적 표상 등은 이 개념상에 직접적 효과를 주므로 매우 중요하다. 이런 논란을 바탕으로 교사는 반영적 추상화를 위해 시간을 충분히 제공해야하고, 비슷한 문제를 가지고 여러 번 시도를 할 수 있게 하며, 학생을 잘 관찰하여 그들의 지식 위계성을 이해하고 방향을 제시하며, 논리적이고 잘 연결된 시각적 표상을 제공하고, 상징적 사관으로 확장할 수 있게 조작할 수 있는 환경에서 시각화에 대해 학생과 많은 대화를 하도록 수업을 준비해야한다. 그 한 예로 타원을 가르치기 위해 몇 가지 테크놀로지를 활용한 시각화 환경을 구성해보았다.