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KUMMER CONGRUENCE FOR q-BERNOULLI NUMBERS

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0. Introduction

Let $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ denote the ring of p-adic rational integers, the field of p-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$, which may be regarded as the p-adic analogue of the complex numbers respectively.

Let v_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = p^{-1}$ When one talks of q-extension, q is variously considered as an indeterminate, $q \in \mathbb{C}_p$. If $q = 1 + t \in \mathbb{C}_p$, we normally assume $|t|_p < p^{-\frac{1}{p-1}}$. We shall further suppose that $v_p(t) > \frac{1}{p-1}$, so that $q^x = \exp(x \log_p q)$ for $|x|_p \leq 1$ where $\log_p$ is p-adic Iwasawa logarithm [5].

Let d be a fixed positive integer and let $X = \varprojlim_N (\mathbb{Z}/dp^N\mathbb{Z})$, where the map from $\mathbb{Z}/dp^M\mathbb{Z}$ to $\mathbb{Z}/dp^N\mathbb{Z}$ for $M \geq N$ is a reduction mod dp^N. Let $a + dp^N\mathbb{Z}_p = \{x \in X \mid x \equiv a \pmod{dp^N}\}$. Without the loss of generality, we may always take to choose a, so that $0 \leq a \leq dp^N - 1$. Also,

$$\mathbb{Z}_p = \bigcup_{0 \leq x < p} (x + p\mathbb{Z}_p) \qquad \text{(disjoint union)},$$

$$a + dp^N\mathbb{Z}_p = \bigcup_{0 \leq b < p} [(a + bdp^N) + dp^{N+1}\mathbb{Z}_p] \qquad \text{(disjoint union)}.$$

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In this paper we will give a new relation between Bernoulli numbers and $A_{m,n}$ satisfying Euler summation formula. In section 1 we will treat the application of p-adic Haar measure $\mu_{Haar}(x+p^N\mathbb{Z}_p)=\frac{1}{p^N}, 0\le x<p^N$. In section 2 we will investigate some congruences of Bernoulli numbers. In section 3 we will construct a new formula of the p-adic q-L-function for q-Bernoulli numbers which is slightly different from Carlitz q-Bernoulli numbers.

1. p-adic distribution

The usual Bernoulli numbers are defined by

$$\sum_{n=0}^{\infty} B_n \frac{t^n}{n!} = \frac{t}{e^t-1},$$

which can be written symbolically as $e^{Bt}=\frac{t}{e^t-1}$, interpreted to mean of which B^k must be replaced by B_k. This relation can also be written $e^{(B+1)t}-e^{Bt}=t$, or if we equate powers of t,

$$B_0=1, \quad (B+1)^k-B^k=\begin{cases} 1, & \text{if } k=1, \\ 0, & \text{if } k>1. \end{cases}$$

The Bernoulli polynomials $B_k(x)$ are defined by

$$\frac{te^{xt}}{e^t-1}=\sum_{k=0}^{\infty} B_k(x)\frac{t^k}{k!}=e^{B(x)t}.$$

We can easily find the following relation:

$$B_k(x)=\sum_{i=0}^{k}\binom{k}{i}B_i x^{k-i}, \quad B_k(0)=B_k. \tag{1.1}$$

The above Bernoulli polynomials are closely related to the p-adic distributions.

DEFINITION 1.1 [5]. A *p-adic distribution* μ on X is a $\mathbb{Q}_p$-linear vector space homomorphism from $\mathbb{Q}_p$-vector space of locally constant functions on X to $\mathbb{Q}_p$. If $f : X \to \mathbb{Q}_p$ is locally constant, instead of writing $\mu(f)$ for the value of μ at f, we usually write $\int f\mu$.

We now give simple examples of p-adic distributions.

(1) The Haar distribution μ_{Haar}. Define

$$\mu_{Haar}(a + p^N\mathbb{Z}_p) := \frac{1}{p^N}.$$

This extends to the unique distribution (up to a constant multiple) on $\mathbb{Z}_p$, since

$$\sum_{b=0}^{p-1} \mu_{Haar}(a + bp^N + p^{N+1}\mathbb{Z}_p) = \sum_{b=0}^{p-1} \frac{1}{p^{N+1}} = \frac{1}{p^N}$$
$$= \mu_{Haar}(a + p^N\mathbb{Z}_p).$$

(2) The Dirac distribution μ_α concentrated at $\alpha \in \mathbb{Z}_p$ (α is fixed). Define

$$\mu_\alpha(U) := \begin{cases} 1, & \text{if } \alpha \in U, \\ 0, & \text{otherwise,} \end{cases}$$

where U is a compact open subset of $\mathbb{Z}_p$

(3) The Mazur distribution μ_{Mazur}. Define

$$\mu_{Mazur}(a + p^N\mathbb{Z}_p) := \frac{a}{p^N} - \frac{1}{2}.$$

(4) The k-th Bernoulli distribution. Define

$$\mu_{B,k}(a + p^N\mathbb{Z}_p) := p^{N(k-1)} B_k\left(\frac{a}{p^N}\right).$$

This extends to a distribution on $\mathbb{Z}_p$.

By (1.1), the first few $B_k(x)$ give us the following distributions:

$$\mu_{B,0}(a+p^N\mathbb{Z}_p)=\frac{1}{p^N}=\mu_{Haar},$$
$$\mu_{B,1}(a+p^N\mathbb{Z}_p)=\frac{a}{p^N}-\frac{1}{2}=\mu_{Mazur},$$
$$\mu_{B,2}(a+p^N\mathbb{Z}_p)=p^N\left(\frac{a^2}{p^{2N}}-\frac{a}{p^N}+\frac{1}{6}\right),$$
$$\mu_{B,3}(a+p^N\mathbb{Z}_p)=p^{2N}\left(\frac{a^3}{p^{3N}}-\frac{3}{2}\frac{a^2}{p^{2N}}+\frac{1}{2}\frac{a}{p^N}\right),$$

A slightly different Carlitz q-Bernoulli numbers $\beta_k(q)$ and q-Bernoulli polynomials $\beta_k(x;q)$ are defined by

$$\frac{t}{qe^t-1}=e^{\beta(q)t} \qquad \text{and} \qquad \frac{te^{xt}}{qe^t-1}=e^{\beta(x;q)t}.$$

For any positive integer m and $k\geq 0$, we can find

$$\beta_k(x;q)=m^{k-1}\sum_{i=0}^{m-1}q^i\beta_k\left(\frac{x+i}{m};q^m\right).$$

(5) The k-th q-Bernoulli distribution (see [4]). Define

$$\mu_{\beta,k}(a+dp^N\mathbb{Z}_p):=(dp^N)^{k-1}q^a\beta_k\left(\frac{a}{dp^N};q^{dp^N}\right).$$

This uniquely extends to a dstribution on $\mathbb{Z}_p$.

(6) The k-th poly-Bernoulli distribution (see [7]). Define

$$\mu_m^{(k)}(a+fp^N\mathbb{Z}_p):=\frac{Li_k(1-e^{-t})}{Li_k(1-e^{-fp^Nt})}(fp^N)^mB_m^{(k)}\left(\frac{a}{fp^N}\right).$$

This uniquely extends to a dstribution on $\mathbb{Z}_p$.

Now, let μ_0 be denoted by

$$\mu_0(x + p^N \mathbb{Z}_p) := \mu_{Haar}(x + p^N \mathbb{Z}_p) = \frac{1}{p^N}, \quad 0 \le x < p^N.$$

Then μ_0 uniquely extends to the distribution on $\mathbb{Z}_p$.

Let $C(\mathbb{Z}_p, \mathbb{C}_p)$ and $UD(\mathbb{Z}_p, \mathbb{C}_p)$ denote the space of all continuous functions and the space of all uniformly differentiable functions on $\mathbb{Z}_p$ with values in $\mathbb{C}_p$ respectively.

For $f \in UD(\mathbb{Z}_p, \mathbb{C}_p)$, we have an integral $I_0(f)$ with respect to the so called invariant measure μ_0;

$$\begin{aligned} I_0(f) &:= \int_{\mathbb{Z}_p} f(x) d\mu_0(x) \\ &= \lim_{N \to \infty} \sum_{x=0}^{p^N - 1} f(x) \mu_0(x + p^N \mathbb{Z}_p) \\ &= \lim_{N \to \infty} \sum_{x=0}^{p^N - 1} f(x) \frac{1}{p^N}. \end{aligned}$$

LEMMA 1.2. *For $f \in UD(\mathbb{Z}_p, \mathbb{C}_p)$, we have*

$$I_0(f_1) = I_0(f) + f'(0)$$

where $f_1(x) = f(x+1)$.

Proof.

$$\begin{aligned}
I_0(f_1) &= \lim_{N\to\infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f_1(x) \\
&= \lim_{N\to\infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x+1) \\
&= \lim_{N\to\infty} \frac{1}{p^N} \sum_{x=1}^{p^N} f(x) \\
&= \lim_{N\to\infty} \frac{1}{p^N} \left\{ \sum_{x=0}^{p^N-1} f(x) + f(p^N) - f(0) \right\} \\
&= I_0(f) + f'(0).
\end{aligned}$$

REMARK. If $f \in UD(\mathbb{Z}_p, \mathbb{C}_p)$ and $n \geq 1$, then

$$I_0(f_n) = I_0(f) + \sum_{k=0}^{n-1} f'(k).$$

LEMMA 1.3 (WITT'S FORMULA). *For $n \in \mathbb{Z}_{\geq 0}$, we have*

$$B_n = \int_{\mathbb{Z}_p} x^n d\mu_0(x),$$

where $\mu_0(x + p^N\mathbb{Z}_p) = \frac{1}{p^N}$.

Proof. In Lemma 1.2, we take $f(x) = e^{tx} \in UD(\mathbb{Z}_p, \mathbb{C}_p), t \in \mathbb{Z}_p$. Then

$$e^t I_0(e^{tx}) - I_0(e^{tx}) = t$$

and

$$I_0(e^{tx}) = \frac{t}{e^t - 1} = e^{Bt} = \sum_{n=0}^{\infty} \frac{B_n}{n!} t^n.$$

On the other hand,

$$\begin{aligned} I_0(e^{tx}) &= \int_{\mathbb{Z}_p} e^{tx} d\mu_0(x) \\ &= \int_{\mathbb{Z}_p} \sum_{n=0}^{\infty} \frac{x^n}{n!} t^n d\mu_0(x) \\ &= \sum_{n=0}^{\infty} \frac{\int_{\mathbb{Z}_p} x^n d\mu_0(x)}{n!} t^n \\ &= \sum_{n=0}^{\infty} \frac{I_0(x^n)}{n!} t^n. \end{aligned}$$

By comparing the coefficients of both sides, we have

$$B_n = I_0(x^n) = \int_{\mathbb{Z}_p} x^n d\mu_0(x).$$

The following proposition is proved in [4]. We give another proof here, however, which depends only upon Lemma 1.2 and Lemma 1.3.

Let C_{p^n} be the cyclic group consisting of all p^n-th roots of unity in $\mathbb{C}_p$ for all $n \geq 0$ and $\mathbf{T}_p$ be the direct limit of C_{p^n} with respect to the natural homomorphisms, hence $\mathbf{T}_p$ is the union of all C_{p^n} with discrete topology. By Lemma 1.2, if $f(x) = q^x e^{xt} \in UD(\mathbb{Z}_p, \mathbb{C}_p)$, then

$$I_0(q^x e^{xt}) = \frac{\ln q + t}{qe^t - 1}.$$

For $q \in \mathbf{T}_p$, we have

$$(qe^t - 1) I_0(q^x e^{xt}) = t.$$

Hence for $q \in \mathbf{T}_p$,

$$I_0(q^x e^{xt}) = \frac{t}{qe^t - 1}.$$

Therefore we obtain the following;

PROPOSITION 1.4. *We have the Witt's formula*

$$\beta_n(q) = \int_{\mathbb{Z}_p} q^x x^n d\mu_0(x) \quad \textit{for} \quad q \in \mathbf{T}_p,\ n \geq 0,$$

where $\beta_n(q)$ *is defined by* $\frac{t}{qe^t - 1} = e^{\beta(q)t}$.

In section 3 we consider the properties of the q-Bernoulli numbers $\beta_n(q)$ defined above, and using the measure $\mu_{\alpha,k;q}$ we will investigate Kummer congruence and construct the p-adic q-L-function for the q-Bernoulli numbers.

2. Congurence of Bernoulli numbers

In this section using the Haar measure in section 1, we construct the congruence of Bernoulli numbers on p-adic number field, and the congruence formula (see Proposition 2.2) contains the von Staudt-Clausen theorem for $i = 1$. We also treat the congruence of the usual Bernoulli numbers for the type of generalized Euler formula.

Let $\mathbb{Z}_p^\times$ be the group of p-adic units, and let $1 + p\mathbb{Z}_p$ is the subgroup of $\mathbb{Z}_p^\times$ consisting of all elements of the form $1 + pa$, $a \in \mathbb{Z}_p$. Let C_p be the cyclic group of order $p - 1$ consisting of $(p-1)$-th roots of unity in $\mathbb{Q}_p$. Each x in $\mathbb{Z}_p^\times$ can be uniquely written in the form

$$x = \omega(x) < x >$$

where $\omega(x)$ and $< x >$ denote the projections of x on C_p and $1 + p\mathbb{Z}_p$ respectively.

We see easily that if $p > 2$, then

$$\omega(x) = \lim_{n\to\infty} x^{p^n} \quad \text{and} \quad < x >^{p-1} = 1 + pq_x, \qquad q_x \in \mathbb{Z}_p.$$

PROPOSITION 2.1. *For any $x \in \mathbb{Z}_p^\times$*

$$\omega(x) = x(1+pq_x)^{\frac{1}{1-p}}.$$

In particular,

$$\sum_{x=1}^{p^n}{}^* x^m (1+pq_x)^{\frac{m}{1-p}} = \begin{cases} 0, & \text{if } p-1 \nmid m, \\ p^{n-1}(p-1), & \text{if } p-1 \mid m, \end{cases}$$

where $*$ denotes to take sum over all integers prime to p in given ranges.

Proof. Since $x^{p^n} = x(1+pq_x)^{1+\cdots+p^{n-1}}$, we have

$$\omega(x)^{p^n-1} = x^{p^n-1}(1+pq_x)^{\frac{p^n-1}{1-p}}.$$

Hence $\omega(x) = x(1+pq_x)^{\frac{1}{1-p}}$.

Put $pB_{p-1} = -1 + p\alpha_p$ with a rational number α_p which becomes a p-adic integer by the von Staudt-Clausen theorem.

PROPOSITION 2.2. *For any odd prime $p \geq 5$ and $i \geq 1$ we have*

$$B_{i(p-1)} \equiv 1 - \frac{1}{p} + i(\alpha_p - 1) \pmod{p}.$$

Proof. From Lemma 1.3 and with $n = i(p-1)$, we obtain

$$\begin{aligned}(1-p^{i(p-1)-1})B_{i(p-1)} &= \lim_{N\to\infty} \frac{1}{p^N} \sum_{i=1}^{p^N}{}^* x^{i(p-1)} \\ &= \int_{\mathbb{Z}_p^\times} x^{i(p-1)} d\mu_0(x).\end{aligned}$$

By the von Staudt-Clausen theorem, $pB_{i(p-1)} \in \mathbb{Z}_p$, and clearly

$$B_{i(p-1)} \equiv \int_{\mathbb{Z}_p^\times} x^{i(p-1)} d\mu_0(x) \pmod{p}.$$

Now, we define the Fermat quotient q_x by $x^{p-1} = 1 + pq_x$ for any integer $x \in \mathbb{Z}_p^\times$, that is, $(x,p) = 1$. Then we have

$$x^{i(p-1)} \equiv 1 + ipq_x \pmod p.$$

Hence by using the above congruence, we have

$$\begin{aligned}
\int_{\mathbb{Z}_p^\times} x^{i(p-1)} d\mu_0(x) &\equiv \int_{\mathbb{Z}_p^\times} d\mu_0(x) + ip \int_{\mathbb{Z}_p^\times} q_x d\mu_0(x) \pmod p \\
&\equiv 1 - \frac{1}{p} + ip \int_{\mathbb{Z}_p^\times} \frac{x^{p-1}-1}{p} d\mu_0(x) \pmod p \\
&\equiv 1 - \frac{1}{p} + i \int_{\mathbb{Z}_p^\times} x^{p-1} d\mu_0(x) - i \int_{\mathbb{Z}_p^\times} d\mu_0(x) \pmod p \\
&\equiv 1 - \frac{1}{p} + i \left(\int_{\mathbb{Z}_p^\times} x^{p-1} d\mu_0(x) + \frac{1}{p} - 1 \right) \pmod p \\
&\equiv 1 - \frac{1}{p} + i \left(B_{p-1} - p^{p-2} B_{p-1} + \frac{1}{p} - 1 \right) \pmod p \\
&\equiv 1 - \frac{1}{p} + i \left(B_{p-1} + \frac{1}{p} - 1 \right) \pmod p \\
&\equiv 1 - \frac{1}{p} + i(\alpha_p - 1) \pmod p.
\end{aligned}$$

The Euler-Maclaurin summation formula involving the Bernoulli number B_m and the Euler formula involving the Bernoulli number B_m (see Corollary 2.4 and Lemma 2.5) are well known. Now we consider them.

LEMMA 2.3 [1]. *For any rational integer $m, n \geq 1$*

$$(m+1)S_m(n) = (n+B)^{m+1} - B_{m+1},$$

where $S_k(n) = 1^k + 2^k + \cdots + (n-1)^k$.

COROLLARY 2.4. *For any rational integer $m, n \geq 1$ and $B_0 = 1$*

$$(m+1)S_m(n) = \sum_{k=0}^{m} \binom{m+1}{k} B_k n^{m+1-k}.$$

LEMMA 2.5 [1]. *For any rational integer $m \geq 4$*

$$\sum_{k=2}^{m-2} \binom{m}{k} B_k B_{m-k} = -(m+1)B_m.$$

LEMMA 2.6 (VON STAUDT-CLAUSEN) [8]. *Let n be even and positive. Then*

$$B_n + \sum_{(p-1)|n} \frac{1}{p} \in \mathbb{Z},$$

where the sum is over those primes p such that $p-1$ divides n (in particular, 2 and 3 appear in the denominator of Bernoulli numbers). Consequently, pB_n is p-integral for all n and all p.

DEFINITION 2.7. For $m, n \geq 1$ we define the sum

$$A_{m,n} = \frac{1}{n} \sum_{k=1}^{n} \binom{n}{k} (-1)^k B_{m+k} B_{n-k}$$

The effective of $A_{m,n}$ is the form of generalized Euler formula (see Lemma 2.5) in the p-adic cases. Here we easily see that $A_{m,n} = A_{n-1,m+1}$. In particular, since $A_{m+p-1,p} = A_{p-1,m+p}$ with odd m modulo odd prime $p \geq 5$, we obtain congruence for Bernoulli numbers which can be regarded as a refinement of the ordinary Kummer's congruence.

Here the aim of us is to investigate several types of the usual congruences of Bernoulli numbers not using the concept of distribution or measure but using the generalized Euler formula. The congruences obtained in this way are the generalization of the well known congruences of Bernoulli numbers.

By the definition 2.7 we obtain

$$A_{m+p-1,p} = -B_{m+p}B_{p-1} + \frac{1}{p}\binom{p}{r}(-1)^{p-r}B_{i(p-1)}B_r + \sum_{\substack{k=2 \\ k\neq p-r}}^{p} \frac{1}{p}\binom{p}{k}(-1)^k B_{m+p-1+k}B_{p-k}, \qquad (2.1)$$

where $m = (p-1)\left[\frac{m}{p-1}\right] + r - 1,\ 2 \leq r < p-1$ and $i = \left[\frac{m}{p-1}\right] + 2$. Here $[\cdot]$ means the greatest integer function.

LEMMA 2.8. *For $0 < k < p$ we have*

$$\binom{p}{k} \equiv (-1)^{k-1}\frac{1}{k}p + (-1)^k\frac{1}{k}L_{k-1}p^2 \pmod{p^3},$$

where $L_{k-1} = 1 + \frac{1}{2} + \cdots + \frac{1}{k-1}$.

Proof. It suffices to prove that

$$\begin{aligned}
\frac{1}{p}\binom{p}{k} &= \frac{1}{p}\frac{p(p-1)(p-2)\cdots(p-k+1)}{k!} \\
&= \frac{(-1)^{k-1}}{k}\frac{(1-p)(2-p)\cdots(k-1-p)}{(k-1)!} \\
&= \frac{(-1)^{k-1}}{k}\prod_{n=1}^{k-1}\left(\frac{n-p}{n}\right) \\
&\equiv \frac{(-1)^{k-1}}{k}\left(1 - \sum_{n=1}^{k-1}\frac{p}{n}\right) \pmod{p^2} \\
&= \frac{(-1)^{k-1}}{k}(1 - pL_{k-1}) \pmod{p^2}.
\end{aligned}$$

Hence we obtain

$$\binom{p}{k} \equiv (-1)^{k-1}\frac{1}{k}p + (-1)^k\frac{1}{k}L_{k-1}p^2 \pmod{p^3}.$$

By using Lemma 2.8 and (2.1) we have

$$\begin{aligned} A_{m+p-1,p} \equiv & -B_{m+p}B_{p-1} + B_{\imath(p-1)}\frac{B_r}{r} - pL_{r-1}B_{\imath(p-1)}\frac{B_r}{r} \\ & - \frac{1}{p}B_{m+p-1+p} \\ & + \sum_{\substack{k=2 \\ k\neq p-r}}^{p-2} \frac{1}{p}\binom{p}{k}(-1)^k B_{m+p-1+k}B_{p-k} \pmod{p}. \end{aligned} \tag{2.2}$$

Next we transform the last term in (2.2). We know that if $n \equiv m \not\equiv 0 \pmod{p-1}$, then

$$\frac{B_m}{m} \equiv \frac{B_n}{n} \pmod{p} \qquad \textbf{(Kummer's congruence)}.$$

Also we have $m+p-1+k \equiv m+k \pmod{p-1}$. Hence, we obtain the congruence

$$B_{m+p-1+k} \equiv B_{m+k} - \frac{B_{m+k}}{m+k} \pmod{p}.$$

In this congruence, the last term of (2.2) implies that

$$\begin{aligned} & \sum_{\substack{k=2 \\ k\neq p-r}}^{p-2} \frac{1}{p}\binom{p}{k}(-1)^k B_{m+p-1+k}B_{p-k} \\ & \equiv \sum_{k=1}^{p} \frac{1}{p}\binom{p}{k}(-1)^k B_{p-k}B_{m+k} - \frac{1}{p}\binom{p}{r}(-1)^{p-r}B_rB_{p-r+m} \\ & \qquad + B_{p-1}B_{m+1} - \frac{(-1)^p}{p}B_{m+p} - (-1)^{p-1}B_1B_{m+p-1} \\ & \qquad - \sum_{\substack{k=2 \\ k\neq p-r}}^{p-2} \frac{1}{p}\binom{p}{k}(-1)^k \frac{B_{m+k}}{m+k}B_{p-k} \pmod{p} \qquad (2.3) \\ & \equiv A_{m,p} + \frac{1}{p}\binom{p}{r}B_{p-r+m}B_r + B_{m+1}B_{p-1} + \frac{B_{m+p}}{p} \\ & \qquad + \sum_{\substack{k=2 \\ k\neq p-r}}^{p-2} \frac{1}{p}\binom{p}{k}\frac{B_{m+k}}{m+k}B_{p-k} \pmod{p}. \end{aligned}$$

For a positive integer $r \geq 2$ we define the sum

$$A^{(r)} = \sum_{k=2}^{r-2} \frac{B_k}{k} \frac{B_{r-k}}{r-k} \quad \text{and} \quad C_p^{(r)} = \sum_{k=r+2}^{p-2} \frac{B_k}{k} \frac{B_{p-1+r-k}}{p-1+r-k}.$$

Then we obtain the following.

PROPOSITION 2.9. *We have the congruences*

$$\begin{aligned} A_{m+p-1,p} \equiv A_{m,p} &- \frac{1}{p}(B_{m+2p-1} - 2B_{m+p} + B_{m+1}) \\ &- \alpha_p(B_{m+p} - B_{m+1}) + (\alpha_p - 1)\frac{B_r}{r} - A^{(r)} - C_p^{(r)} \pmod{p}, \end{aligned}$$

where $\alpha_p = (1 + pB_{p-1})/p$.

Proof. By (2.2) and (2.3) we have

$$\begin{aligned} A_{m+p-1,p} \equiv & - B_{m+p}B_{p-1} + B_{\imath(p-1)}\frac{B_r}{r} - pL_{r-1}B_{\imath(p-1)}\frac{B_r}{r} \\ & - \frac{1}{p}B_{m+p-1+p} + A_{m,p} + \frac{1}{p}\binom{p}{r}B_{(\imath-1)(p-1)}B_r \\ & + B_{m+1}B_{p-1} + \frac{1}{p}B_{m+p} \\ & + \sum_{\substack{k=2 \\ k \neq p-r}}^{p-2} \frac{1}{p}\binom{p}{k}(-1)^k \frac{B_{m+k}}{m+k}B_{p-k} \pmod{p}. \end{aligned}$$

We note that $B_{p-1} = -\frac{1}{p} + \alpha_p$ implies the followings;

(i) $-B_{m+p}B_{p-1} = -B_{m+p}\left(-\frac{1}{p} + \alpha_p\right) = \frac{B_{m+p}}{p} - \alpha_p B_{m+p}$.

(ii) $B_{m+1}B_{p-1} = B_{m+1}\left(-\frac{1}{p} + \alpha_p\right) = -\frac{B_{m+1}}{p} + \alpha_p B_{m+1}$.

(iii) Since $\binom{p-1}{r} \equiv (-1)^r \pmod{p}$,

$$\begin{aligned} & \frac{1}{p}\binom{p}{r}B_{(\imath-1)(p-1)}B_r + B_{\imath(p-1)}\frac{B_r}{r} \\ \equiv & -\frac{B_r}{r}(-1)^r B_{(\imath-1)(p-1)} + B_{\imath(p-1)}\frac{B_r}{r} \pmod{p} \\ \equiv & -\frac{B_r}{r}B_{(\imath-1)(p-1)} + B_{\imath(p-1)}\frac{B_r}{r} \pmod{p}. \end{aligned}$$

It follows that

$$-\frac{B_r}{r}B_{(i-1)(p-1)} + B_{i(p-1)}\frac{B_r}{r}$$

$$\equiv -\frac{B_r}{r}\left(1-\frac{1}{p}+i(\alpha_p-1)-(\alpha_p-1)\right)+\frac{B_r}{r}\left(1-\frac{1}{p}+i(\alpha_p-1)\right) \pmod p$$

$$\equiv (\alpha_p-1)\frac{B_r}{r} \pmod p,$$

since $B_{i(p-1)} \equiv 1-\frac{1}{p}+i(\alpha_p-1) \pmod p$. Thus, we obtain

$$\frac{1}{p}\binom{p}{r}B_{(i-1)(p-1)}B_r + B_{i(p-1)}\frac{B_r}{r} \equiv (\alpha_p-1)\frac{B_r}{r} \pmod p.$$

(iv) $\frac{1}{p}\binom{p}{k} = \frac{1}{p}\binom{p}{p-k} = \frac{1}{p-k}\binom{p-1}{p-k-1} \equiv \frac{(-1)^{p-k-1}}{p-k} \pmod p$.

By (iv), we easily see that

$$\sum_{\substack{k=2\\k\neq p-r}}^{p-2}\frac{1}{p}\binom{p}{k}\frac{B_{m+k}}{m+k}B_{p-k} \equiv -\sum_{\substack{k=2\\k\neq p-r}}^{p-2}\frac{(-1)^{p-k}}{p-k}\frac{B_{m+k}}{m+k}B_{p-k} \pmod p$$

$$\equiv -\sum_{\substack{k=2\\k\neq p-r}}^{p-2}\frac{B_{p-k}}{p-k}\frac{B_{m+k}}{m+k} \pmod p$$

$$\equiv -A^{(r)} - C_p^{(r)} \pmod p.$$

By using (i), (ii), (iii) and (iv), we obtain the congruence

$$A_{m+p-1,p} \equiv A_{m,p} - \frac{1}{p}(B_{m+2p-1}-2B_{m+p}+B_{m+1})$$

$$-\alpha_p(B_{m+p}-B_{m+1}) + (\alpha_p-1)\frac{B_r}{r} - A^{(r)} - C_p^{(r)} \pmod p.$$

By the Kummer's congruence, since $m+2p-1 \equiv m+p \pmod{p-1}$ and $m+p \equiv m+1 \pmod{p-1}$,

$$\frac{B_{m+2p-1}}{m+2p-1} \equiv \frac{B_{m+p}}{m+p} \equiv \frac{B_{m+1}}{m+1} \pmod p,$$

and so we have

$$\frac{B_{m+2p-1}}{m+2p-1} - 2\frac{B_{m+p}}{m+p} + \frac{B_{m+1}}{m+1} \equiv 0 \pmod p. \tag{2.4}$$

We note $m = (p-1)\left[\frac{m}{p-1}\right] + r - 1$ implies that by the Kummer's congruence

$$\frac{B_{m+1}}{m+1} \equiv \frac{B_r}{r} \pmod p.$$

By using this congruence, we see

$$\frac{B_{m+p}}{m+p} \equiv \frac{B_{m+1}}{m+1} \pmod p$$

and

$$B_{m+p} - B_{m+1} + \frac{B_r}{r} \equiv 0 \pmod p. \tag{2.5}$$

We can now prove the following, which corresponds to congruence for the generalized Euler formula.

THEOREM 2.10. *For any even r with $2 \le r < p-1$ we have*

$$\begin{aligned} A_{m+p-1,p} \equiv A_{m,p} &+ \frac{2}{p}\left(\frac{B_{m+p}}{m+p} - \frac{B_{m+1}}{m+1}\right) \\ &+ (2\alpha_p - 1)\frac{B_r}{r} - A^{(r)} - C_p^{(r)} \pmod p. \end{aligned}$$

Proof. By using (2.4) and (2.5), we see that

$$\begin{aligned} B_{m+2p-1} &\equiv \left(2\frac{B_{m+p}}{m+p} - \frac{B_{m+1}}{m+1}\right)(m+2p-1) \pmod p \\ &\equiv 2B_{m+p} - B_{m+1} - 2\frac{B_{m+p}}{m+p} + \frac{2B_{m+1}}{m+1} \pmod p. \end{aligned}$$

And $B_{m+p} - B_{m+1} \equiv -\frac{B_r}{r} \pmod{p}$. Hence, using Proposition 2.9 and the above results

$$\begin{aligned}
A_{m+p-1,p} &\equiv A_{m,p} - \frac{1}{p}(B_{m+2p-1} - 2B_{m+p} + B_{m+1}) - \alpha_p(B_{m+p} \\
&\qquad - B_{m+1}) + (\alpha_p - 1)\frac{B_r}{r} - A^{(r)} - C_p^{(r)} \pmod{p} \\
&\equiv A_{m,p} - \frac{1}{p}(2B_{m+p} - B_{m+1} - 2\frac{B_{m+p}}{m+p} + \frac{2B_{m+1}}{m+1} - 2B_{m+p} \\
&\qquad + B_{m+1}) - \alpha_p\left(-\frac{B_r}{r}\right) + (\alpha_p - 1)\frac{B_r}{r} - A^{(r)} - C_p^{(r)} \\
&\qquad \pmod{p} \\
&\equiv A_{m,p} + \frac{2}{p}\left(\frac{B_{m+p}}{m+p} - \frac{B_{m+1}}{m+1}\right) + (2\alpha_p - 1)\frac{B_r}{r} \\
&\qquad - A^{(r)} - C_p^{(r)} \pmod{p}.
\end{aligned}$$

We put $m = (p-1)(i-2) + r - 1$ and $j = i - 2 \geq 0$, r is even with $2 \leq r < p-1$. By Proposition 2.2 and Lemma 2.8, we obtain

$$\begin{aligned}
A_{r-1+j(p-1),p} \equiv -\frac{1}{p}\left(B_{r+(j+1)(p-1)} - B_{r+j(p-1)} + \frac{B_r}{r}\right) \\
+ ((2j+1-r)\alpha_p - j + L_{r-1})\frac{B_r}{r} \\
+ \sum_{\substack{k=2 \\ k\neq r}}^{p-2} \frac{B_k}{k} B_{r-k+(j+1)(p-1)} \pmod{p}.
\end{aligned} \tag{2.6}$$

Indeed, by the Definition 2.7, we have

$$\begin{aligned}
A_{r-1+j(p-1),p} = -B_{r-1+j(p-1)+1}B_{p-1} + \frac{1}{p}(-1)^p B_{r-1+j(p-1)+p} \\
+ \frac{1}{p}\binom{p}{r}(-1)^{p-r} B_{r-1+j(p-1)+p-r} B_r \\
+ (-1)^{p-1} B_{r-1+j(p-1)+p-1} B_1 \\
+ \sum_{\substack{k=2 \\ k\neq p-r}}^{p-2} \frac{1}{p}\binom{p}{k}(-1)^k B_{r-1+j(p-1)+k} B_{p-k}.
\end{aligned}$$

Now, we see that

(i) Since $-B_{r+j(p-1)}B_{p-1} = \frac{B_{r+j(p-1)}}{p} - \alpha_p B_{r+j(p-1)}$ and $r+j(p-1) \equiv r \pmod{p-1}$, we obtain

$$\frac{B_{r+j(p-1)}}{r+j(p-1)} \equiv \frac{B_r}{r} \pmod{p},$$

so that

$$B_{r+j(p-1)} \equiv \frac{B_r}{r}(r-j) \pmod{p}$$

and

$$-B_{r+j(p-1)}B_{p-1} \equiv \frac{B_{r+j(p-1)}}{p} - \frac{B_r}{r}\alpha_p(r-j) \pmod{p}.$$

(ii) By Lemma 2.8, $(-1)^{p-r}\frac{1}{p}\binom{p}{r} \equiv \frac{1}{r}(1 - L_{r-1}p) \pmod{p^2}$, and also Proposition 2.2 implies

$$\begin{aligned}&(-1)^{p-r}\frac{1}{p}\binom{p}{r}B_{(p-1)(j+1)}\\ &\equiv \left(1 - \frac{1}{p} + (j+1)(\alpha_p - 1) + L_{r-1}\right)\frac{1}{r} \pmod{p}.\end{aligned}$$

Therefore

$$\begin{aligned}&(-1)^{p-r}\frac{1}{p}\binom{p}{r}B_{(p-1)(j+1)}B_r\\ &\equiv -\frac{1}{p}\frac{B_r}{r} + \frac{B_r}{r}(\alpha_p(j+1) - j + L_{r-1}) \pmod{p}.\end{aligned}$$

By the above results (i), (ii) and Definition 2.7, we can find that

$$\begin{aligned}A_{r-1+j(p-1),p} \equiv\ & \frac{B_{r+j(p-1)}}{p} + \frac{B_r}{r}\alpha_p(j-r) - \frac{1}{p}B_{r+(p-1)(j+1)}\\ & - \frac{1}{p}\frac{B_r}{r} + \frac{B_r}{r}(\alpha_p(j+1) - j + L_{r-1})\\ & + \sum_{\substack{k=2\\ k\neq p-r}}^{p-2} \frac{1}{p}\binom{p}{k}(-1)^k B_{r-1+j(p-1)+k}B_{p-k} \pmod{p}\end{aligned}$$

$$\equiv -\frac{1}{p}\left(B_{r+(p-1)(j+1)} - B_{r+j(p-1)} + \frac{B_r}{r}\right) + \frac{B_r}{r}\left(\alpha_p(j-r) + \alpha_p(j+1) - j + L_{r-1}\right) + \sum_{\substack{k=2\\k\neq p-r}}^{p-2} \frac{1}{p}\binom{p}{k}(-1)^k B_{r-1+j(p-1)+k}B_{p-k} \pmod p$$

$$\equiv -\frac{1}{p}\left(B_{r+(p-1)(j+1)} - B_{r+j(p-1)} + \frac{B_r}{r}\right) + \frac{B_r}{r}\left(\alpha_p(2j-r+1) - j + L_{r-1}\right) + \sum_{\substack{k=2\\k\neq r}}^{p-2} \frac{1}{p}\binom{p}{k}(-1)^{p-k} B_k B_{r-1+j(p-1)+p-k} \pmod p.$$

Here we obtain the required congruence (2 6)

Also the last term of the above formula implies that

$$\sum_{\substack{k=2\\k\neq r}}^{p-2} \frac{1}{p}\binom{p}{k}(-1)^{p-k} B_k B_{r-1+j(p-1)+p-k}$$

$$\equiv \sum_{\substack{k=2\\k\neq r}}^{p-2} \frac{1}{k}(-1)^{k-1}(-1)^{p-k} B_k B_{r+(p-1)(j+1)-k} \pmod p$$

$$\equiv \sum_{\substack{k=2\\k\neq r}}^{p-2} \frac{1}{k} B_k B_{r+(p-1)(j+1)-k} \pmod p.$$

Therefore we get

$$A_{r-1+j(p-1),p} \equiv -\frac{1}{p}\left(B_{r+(j+1)(p-1)} - B_{r+j(p-1)} + \frac{B_r}{r}\right) + \left((2j+1-r)\alpha_p - j + L_{r-1}\right)\frac{B_r}{r} + \sum_{\substack{k=2\\k\neq r}}^{p-2} \frac{B_k}{k} B_{r-k+(j+1)(p-1)} \pmod p.$$

By using the congruence, we easily obtain the following lemma.

LEMMA 2.11. *For any even r with $2 \le r < p-1$ we have*

$$\sum_{\substack{k=2\\k\neq r}}^{p-2} \frac{B_k}{k} B_{r-k+(j+1)(p-1)} \equiv \frac{r-j-2}{2} A^{(r)} + \frac{r-j-1}{2} C_p^{(r)} \pmod p. .$$

Proof. We consider first the case that

$$\begin{aligned}\sum_{\substack{k=2\\k\neq m}}^{p-2} \frac{1}{k} B_k B_{p-k+m-1} &= \sum_{k=2}^{m-2} \frac{B_k}{k} B_{p-k+m-1} + \sum_{k=m+2}^{p-2} \frac{B_k}{k} B_{p-k+m-1} \\ &= \sum_{k=2}^{m-2} \frac{B_k}{k} \frac{B_{p-1+m-k}}{p-1+m-k}(m-k+p-1) \\ &\quad + \sum_{k=m+2}^{p-2} \frac{B_k}{k} B_{p-1+m-k} \\ &\equiv \sum_{k=2}^{m-2} \frac{B_k}{k} B_{m-k} - \sum_{k=2}^{m-2} \frac{B_k}{k} \frac{B_{m-k}}{m-k} \\ &\quad + \sum_{k=m+2}^{p-2} \frac{B_k}{k} B_{p-1+m-k} \pmod p \\ &\equiv \frac{m-2}{2} A^{(m)} + \frac{m-1}{2} C_p^{(m)} \pmod p.\end{aligned}$$

By using this congruence with $m = j(p-1)+r$, we can find

$$\begin{aligned}\sum_{\substack{k=2\\k\neq r}}^{p-2} \frac{B_k}{k} B_{r-k+(j+1)(p-1)} &= \sum_{\substack{k=2\\k\neq r}}^{p-2} \frac{B_k}{k} B_{p-k+jp-j+r-1} \\ &\equiv \frac{r-j-2}{2} A^{(r)} + \frac{r-j-1}{2} C_p^{(r)} \pmod p.\end{aligned}$$

This completes the proof of our assertion.

By using Definition 2.7 and Lemma 2.11 we have the following theorem.

THEOREM 2.12. *For any $j \geq 0$, any even r with $2 \leq r < p-1$ we have*

$$\begin{aligned} A_{r-1+j(p-1),p} \equiv &-\frac{1}{p}\left(B_{r+(j+1)(p-1)} - B_{r+j(p-1)} + \frac{B_r}{r}\right) \\ &+ ((2j+1-r)\alpha_p - j + L_{r-1})\frac{B_r}{r} \\ &+ \frac{r-j-2}{2}A^{(r)} + \frac{r-j-1}{2}C_p^{(r)} \pmod p. \end{aligned}$$

REMARK. The above Theorem 2.12 is a similar to [M, Lemma 3 and 4] under the assumption $4 \leq r \leq p-3$.

The next step is to transform the numbers $A_{p-1,r+p-1}$. By Definition 2.7, we have

$$\begin{aligned} A_{p-1,r+p-1} = &\frac{1}{r+p-1}\binom{r+p-1}{r}B_{p-1+r}B_{p-1} \\ &+ \frac{1}{r+p-1}\binom{r+p-1}{p-1}B_{2(p-1)}B_r \\ &+ \frac{1}{r+p-1}\sum_{\substack{k=1 \\ k\neq r,p-1}}^{r+p-1}\binom{r+p-1}{k}B_{p-1+k}B_{r+p-1-k}. \end{aligned}$$

The last term leads to

$$\begin{aligned} &\frac{1}{r+p-1}\sum_{\substack{k=1 \\ k\neq r,p-1}}^{r+p-1}\binom{r+p-1}{k}B_{p-1+k}B_{r+p-1-k} \\ \equiv &\frac{1}{r+p-1}\sum_{\substack{k=1 \\ k\neq r,p-1}}^{r+p-1}\binom{r+p-1}{k}\left(1-\frac{1}{k}\right)B_kB_{r+p-1-k} \pmod p \\ \equiv &A_{0,r+p-1} - \frac{2}{r+p-1}\binom{r+p-1}{r}B_rB_{p-1} \\ &- \frac{1}{r+p-1}\sum_{\substack{k=1 \\ k\neq r,p-1}}^{r+p-1}\binom{r+p-1}{k}\frac{B_k}{k}B_{r+p-k-1} \pmod p. \end{aligned}$$

By the symmetry of $A_{m,n}$ in Definition 2.7, we see that $A_{m,n} = A_{n-1,m+1}$. We also have $A_{0,r+p-1} = A_{r+p-2,1} = -B_{r+p-1}$.

On the other hand, we have

$$\begin{aligned}
&\frac{1}{r+p-1}\sum_{\substack{k=1\\k\neq r,p-1}}^{r+p-1}\binom{r+p-1}{k}\frac{B_k}{k}B_{r+p-k-1}\\
&=\frac{1}{r+p-1}\sum_{k=1}^{r-1}\binom{r+p-1}{k}\frac{B_k}{k}B_{r+p-k-1}\\
&\quad+\frac{1}{r+p-1}\sum_{\substack{k=r+1\\k\neq p-1}}^{r+p-1}\binom{r+p-1}{k}\frac{B_k}{k}B_{r+p-k-1}.
\end{aligned}\tag{2.7}$$

We note that $\binom{r+p-1}{k} \equiv \binom{r-1}{k} \pmod p$ for $1 \leq k < p$, so that

$$\begin{aligned}
&\frac{1}{r+p-1}\sum_{k=1}^{r-1}\binom{r+p-1}{k}\frac{B_k}{k}B_{r+p-k-1}\\
&\equiv\frac{1}{r-1}\sum_{k=1}^{r-1}\binom{r-1}{k}\frac{B_k}{k}B_{r+p-k-1} \pmod p.
\end{aligned}$$

By using the Kummer's congruence, since $p+r-1-k \equiv r-k \pmod{p-1}$,

$$\frac{B_{p+r-1-k}}{p+r-1-k} \equiv \frac{B_{r-k}}{r-k} \pmod p.$$

That is,

$$B_{p+r-1-k} \equiv B_{r-k} - \frac{B_{r-k}}{r-k} \pmod p.$$

Thus for $r \geq 4$ and is even, we have

$$\begin{aligned}
&\frac{1}{r+p-1}\sum_{k=1}^{r-1}\binom{r+p-1}{k}\frac{B_k}{k}B_{r+p-k-1}\\
&\equiv \frac{1}{2}B^{(r)}+\frac{1}{r-1}\frac{r+1}{r}B_r \pmod p,
\end{aligned}\tag{2.8}$$

where $B^{(r)} = \sum_{k=2}^{r-2}\binom{r}{k}\frac{B_k}{k}\frac{B_{r-k}}{r-k}$.

Next we observe the following congruence. Let m be odd with $m \geq 5$. Then we have

$$\frac{1}{m}\sum_{j=1}^{m}\binom{m+p}{j}\frac{B_j}{j}\frac{B_{m+1-j}}{m+1-j} \equiv \frac{1}{2m}B^{(m+1)} \pmod p.$$

Hence

$$\frac{1}{m}\sum_{j=1}^{m}\binom{m+p}{j}\frac{B_j}{j}\frac{B_{m+1-j}}{m+1-j} \equiv \frac{1}{2m}B^{(m+1)} \pmod p.$$

We can deduce from (2.8) that

$$\begin{aligned}
&\frac{1}{m+p}\sum_{j=1}^{m}\binom{m+p}{j}\frac{B_j}{j}B_{p+m-j} \\
\equiv\ & \frac{m+1}{m}\frac{1}{2}B^{(m+1)} + \frac{1}{m}\frac{m+2}{m+1}B_{m+1} - \frac{1}{m}\frac{1}{2}B^{(m+1)} \pmod p \qquad (2.9)\\
\equiv\ & \frac{1}{2}B^{(m+1)} + \frac{1}{m}\frac{m+2}{m+1}B_{m+1} \pmod p.
\end{aligned}$$

Now we consider the last trems of (2.7) and put $m = r-1$, then we have

$$\begin{aligned}
&\frac{1}{m+p}\sum_{\substack{j=m+2\\ j\neq p-1}}^{m+p}\binom{m+p}{j}\frac{B_j}{j}B_{p+m-j} \\
&= \frac{1}{m+p}\sum_{\substack{l=2\\ l\neq p-1-m}}^{p}\binom{m+p}{m+l}\frac{B_{m+l}}{m+l}B_{p-l} \quad (\text{put } j=m+l) \\
&\equiv \frac{1}{m}\sum_{\substack{l=2\\ l\neq p-1-m}}^{p}\binom{m+p}{p-l}\frac{B_{m+l}}{m+l}B_{p-l} \pmod p \qquad (2.10)\\
&\equiv \frac{1}{m}\sum_{\substack{k=0\\ k\neq m+1}}^{p-2}\binom{m}{k}B_k\frac{B_{m+p-k}}{m+p-k} \pmod p \quad (\text{put } k=p-l)
\end{aligned}$$

$$\equiv \frac{1}{m}\sum_{k=0}^{m}\binom{m}{k}B_k\frac{B_{m+1-k}}{m+1-k} \pmod p$$
$$\equiv \frac{1}{m}\frac{B_{m+1}}{m+1}+\frac{1}{m}\sum_{k=1}^{m}\binom{m}{k}B_k\frac{B_{m+1-k}}{m+1-k} \pmod p.$$

Setting $j=m+1-k$, we obtain the following equation

$$\begin{aligned}&\sum_{k=1}^{m}\binom{m}{k}B_k\frac{B_{m+1-k}}{m+1-k}\\&=\frac{m+1}{2}B^{(m+1)}-\sum_{j=1}^{m}\binom{m}{j}\frac{B_j}{j}B_{m+1-j}.\end{aligned} \tag{2.11}$$

Hence, (2.11) goes to

$$\begin{aligned}&\frac{1}{m}\sum_{k=1}^{m}\binom{m}{k}B_k\frac{B_{m+1-k}}{m+1-k}\\&=\frac{1}{m}\frac{m+1}{2}B^{(m+1)}-\frac{1}{m}\sum_{j=1}^{m}\binom{m}{j}\frac{B_j}{j}B_{m+1-j}\\&\equiv -\frac{1}{m}\frac{m+2}{m+1}B_{m+1} \pmod p.\end{aligned}$$

By using the above results and (2.11), we have

$$\begin{aligned}&\frac{1}{m+p}\sum_{\substack{j=m+2\\ j\neq p-1}}^{m+p}\binom{m+p}{j}\frac{B_j}{j}B_{p+m-j}\\&\equiv \frac{1}{m}\frac{B_{m+1}}{m+1}-\frac{1}{m}\frac{m+2}{m+1}B_{m+1} \pmod p\\&\equiv -\frac{1}{m}B_{m+1} \pmod p,\end{aligned} \tag{2.12}$$

and thus from (2.9) and (2.12)

$$\begin{aligned}\frac{1}{m+p} & \sum_{\substack{j=1 \\ j\neq m+1,p-1}}^{m+p} \binom{m+p}{j} \frac{B_j}{j} B_{p+m-j} \\ & \equiv \frac{1}{2}B^{(m+1)} + \frac{1}{m}\frac{m+2}{m+1}B_{m+1} - \frac{1}{m}B_{m+1} \pmod p \\ & \equiv \frac{1}{2}B^{(m+1)} + \frac{1}{m(m+1)}B_{m+1} \pmod p.\end{aligned}$$

¿From these formulas we obtain the following theorem.

THEOREM 2.13. *For even $r \geq 4$ we have*

$$\begin{aligned} & A_{p-1,r+p-1} \\ \equiv\ & \frac{1}{r+p-1}\binom{r+p-1}{r} B_{r+p-1}B_{p-1} \\ & + \frac{1}{r+p-1}\binom{r+p-1}{p-1} B_{2(p-1)}B_r - B_{r+p-1} \\ & - \frac{2}{r+p-1}\binom{r+p-1}{r} B_rB_{p-1} - \frac{1}{2}B^{(r)} - \frac{1}{r-1}\frac{1}{r}B_r \pmod p.\end{aligned}$$

3. Another q-Bernoulli numbers

If $q = 1+t \in \mathbb{C}_p$, we normally assume $|t|_p < p^{-\frac{1}{p-1}}$. We shall further suppose that $v_p(t) > \frac{1}{p-1}$, so that $q^x = \exp(x \log_p q)$ for $|x|_p \leq 1$, where $\log_p$ is p-adic Iwasawa logarithm.

In [4], the q-Bernoulli numbers and q-Bernoulli polynomials are defined by

$$\frac{t}{qe^t - 1} = e^{\beta(q)t} \qquad \text{and} \qquad \frac{te^{xt}}{qe^t - 1} = e^{\beta(x;q)t}.$$

This relation can also be written $qe^{(\beta(q)+1)t} - e^{\beta(q)t} = t$, or if we equate powers of t,

$$\beta_0 = 1, \quad q(\beta+1)^n - \beta_n = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{if } n > 1, \end{cases}$$

with the usual convention of replacing $\beta^n(q)$ by β_n. We can easily find the following relation:

$$\beta_n(x;q) = \sum_{i=0}^{n} \binom{n}{i} \beta_i x^{n-i}, \quad \beta_n(0;q) = \beta_n. \tag{3.1}$$

REMARK. If $q \neq 1$, then for $n \geq 1$

$$\frac{\beta_n(q)}{n} = \frac{q^{-1}}{1-q^{-1}} H_{n-1}(q^{-1}),$$

where $H_{n-1}(q^{-1})$ means the $(n-1)$-th Euler numbers. If $q = 1$, then

$$\beta_n(q) = B_n,$$

where B_n is the usual Bernoulli number.

Let $q \in \mathbb{C}_p$ with $|1-q|_p < p^{-\frac{1}{p-1}}$ and d be a fixed positive integer and let p be a fixed odd prime number. We set

$$\begin{aligned} X &= \varprojlim_N \mathbb{Z}/dp^N\mathbb{Z}, \\ X^* &= \bigcup_{\substack{0<a<dp \\ (a,p)=1}} a + dp\mathbb{Z}_p, \\ a + dp^N\mathbb{Z}_p &= \{x \in X \mid x \equiv a \pmod{dp^N}\}, \end{aligned} \tag{3.2}$$

where $0 \leq a < dp^N$.

LEMMA 3.1 [4]. (1) *For any rational integer $m \geq 1$ and $k \geq 0$,*

$$\beta_k(x;q) = m^{k-1} \sum_{i=0}^{m-1} q^i \beta_k\left(\frac{x+i}{m}; q^m\right).$$

(2) *Let $q \in \mathbb{C}_p$. For any positive integer N, k and d, let $\mu_{\beta,k} := \mu_{\beta,k;q}$ be defined by*

$$\mu_{\beta,k}(a + dp^N\mathbb{Z}_p) = (dp^N)^{k-1} q^a \beta_k\left(\frac{a}{dp^N}; q^{dp^N}\right).$$

Then $\mu_{\beta,k}$ extends uniquely to distribution on X.

The first few $\beta_k(x;q)$ give us the following distributions:

$$\mu_{\beta,0}(a+dp^N\mathbb{Z}_p) = q^a \frac{1}{dp^N},$$
$$\mu_{\beta,1}(a+dp^N\mathbb{Z}_p) = q^a\left(\frac{a}{dp^N}-1\right),$$
$$\mu_{\beta,2}(a+dp^N\mathbb{Z}_p) = q^a dp^N\left(\frac{a^2}{(dp^N)^2}-2\frac{a}{dp^N}+\frac{q^{dp^N}}{q^{dp^N}-1}\right),$$

and so on.

Let $\alpha \in X^*, \alpha \neq 1$ and $k \geq 1$. For compact-open $U \subset X$, $\mu_{\alpha,k,q}$ is defined by

$$\mu_{\alpha,k;q}(U) := \mu_{\beta,k;q}(U) - \alpha^{-k}\mu_{\beta,k;q}(\alpha U). \tag{3.3}$$

If $\alpha \in X^*$, we denote $\{\alpha\}_N$ the rational integers between 0 and dp^N-1 which are congruent to $\alpha \mod dp^N$.

PROPOSITION 3.2. *$|\mu_{\alpha,1,q}(U)|_p \leq 1$ for all compact-open $U \subset X$.*

Proof. Applying (3.3) with $k=1$, we obtain

$$\begin{aligned}\mu_{\alpha,1;q}(a+dp^N\mathbb{Z}_p) &= q^a\left(\frac{a}{dp^N}-1\right) - \frac{1}{\alpha}q^a\left(\frac{\{\alpha a\}_N}{dp^N}-1\right)\\ &= q^a\left(\left(\frac{1}{\alpha}-1\right)+\frac{a}{dp^N}-\frac{1}{\alpha}\left(\frac{\alpha a}{dp^N}-\left[\frac{\alpha a}{dp^N}\right]\right)\right)\\ &= q^a\left(\left(\frac{1}{\alpha}-1\right)+\frac{1}{\alpha}\left[\frac{\alpha a}{dp^N}\right]\right),\end{aligned}$$

where $[\cdot]$ means the greatest integer function. Notice that $q^a\left(\left(\frac{1}{\alpha}-1\right)\right.$ $\left.+\frac{1}{\alpha}\left[\frac{\alpha a}{dp^N}\right]\right) \in \mathbb{Z}_p$, since $\frac{1}{\alpha} \in \mathbb{Z}_p$ and $\left[\frac{\alpha a}{dp^N}\right] \in \mathbb{Z}$. On the other hand, since every compact-open U is a finite disjoint union of intervals $a+dp^N\mathbb{Z}_p$, we may conclude that $|\mu_{\alpha,1,q}(U)|_p \leq \max|\mu_{\alpha,1;q}(a+dp^N\mathbb{Z}_p)|_p \leq 1$.

We will give a relation between $\mu_{\alpha,k;q}$ and $\mu_{\alpha,1;q}$.

THEOREM 3.3. *Let d_k be the least common denominator of the coefficients of $\beta_k(x;q)$. Then*

$$d_k\mu_{\alpha,k,q}(a+dp^N\mathbb{Z}_p) \equiv d_k k a^{k-1} q^a \left(\left(\frac{1}{\alpha}-1\right)+\frac{1}{\alpha}\left[\frac{\alpha a}{dp^N}\right]\right) \pmod{p^N}$$
$$\equiv d_k k a^{k-1}\mu_{\alpha,1;q}(a+dp^N\mathbb{Z}_p) \pmod{p^N},$$

where both sides of this congruence lie in $\mathbb{Z}_p$.

Proof. By using (3.3) we obtain

$$\begin{aligned}
& d_k\mu_{\alpha,k;q}(a+dp^N\mathbb{Z}_p)\\
&= d_k\left(\mu_{\beta,k;q}(a+dp^N\mathbb{Z}_p)-\alpha^{-k}\mu_{\beta,k;q}(\alpha(a+dp^N\mathbb{Z}_p))\right)\\
&= d_k\left((dp^N)^{k-1}q^a\beta_k\left(\frac{a}{dp^N};q^{dp^N}\right)-\alpha^{-k}(dp^N)^{k-1}q^a\beta_k\left(\frac{\{\alpha a\}_N}{dp^N};q^{dp^N}\right)\right)\\
&= d_k q^a (dp^N)^{k-1}\left(\beta_k\left(\frac{a}{dp^N};q^{dp^N}\right)-\alpha^{-k}\beta_k\left(\frac{\{\alpha a\}_N}{dp^N};q^{dp^N}\right)\right).
\end{aligned}$$

Equation (3.1) gives

$$\begin{aligned}
& d_k\mu_{\alpha,k;q}(a+dp^N\mathbb{Z}_p)\\
&= d_k q^a (dp^N)^{k-1}\Bigg(\sum_{i=0}^{k}\binom{k}{i}\beta_i(q^{dp^N})\left(\frac{a}{dp^N}\right)^{k-i}\\
&\qquad -\alpha^{-k}\sum_{i=0}^{k}\binom{k}{i}\beta_i(q^{dp^N})\left(\frac{\{\alpha a\}_N}{dp^N}\right)^{k-i}\Bigg)\\
&\equiv d_k q^a (dp^N)^{k-1}\Bigg(\left(\left(\frac{a}{dp^N}\right)^{k}-k\left(\frac{a}{dp^N}\right)^{k-1}\right)\\
&\qquad -\alpha^{-k}\left(\left(\frac{\{\alpha a\}_N}{dp^N}\right)^{k}-k\left(\frac{\{\alpha a\}_N}{dp^N}\right)^{k-1}\right)\Bigg) \pmod{p^N}\\
&= d_k q^a\Bigg(\frac{a^k}{dp^N}-\alpha^{-k}\left(\frac{(\alpha a)^k}{dp^N}-k(\alpha a)^{k-1}\left[\frac{\alpha a}{dp^N}\right]+\cdots\right)\\
&\qquad -k\left(a^{k-1}-\alpha^{-k}\left((\alpha a)^{k-1}-(k-1)(\alpha a)^{k-2}dp^N\left[\frac{\alpha a}{dp^N}\right]+\cdots\right)\right)\Bigg)\\
&\equiv d_k q^a\left(\frac{a^k}{dp^N}-\alpha^{-k}\left(\frac{(\alpha a)^k}{dp^N}-k(\alpha a)^{k-1}\left[\frac{\alpha a}{dp^N}\right]\right)\right)
\end{aligned}$$

$$-k\left(a^{k-1}-\alpha^{-k}(\alpha a)^{k-1}\right)\Big) \pmod{p^N}$$

$$\equiv d_k k a^{k-1} q^a \left(\left(\frac{1}{\alpha}-1\right)+\frac{1}{\alpha}\left[\frac{\alpha a}{dp^N}\right]\right) \pmod{p^N}.$$

This completes the proof of our assertion.

It is not hard to show that any open subset which is compact is a finite union of compact-open sets of the form $a + dp^N\mathbb{Z}_p$.

DEFINITION 3.4 [6]. An $\mathbb{C}_p$-valued *measure* μ on X is a finitely additive bounded map from the set of compact-open $U \subset X$ to $\mathbb{C}_p$.

COROLLARY 3.5. *$\mu_{\alpha,k;q}$ is a measure for all $k = 1, 2, \cdots$ and any $\alpha \in X^*, \alpha \neq 1$.*

Proof. We must show that $\mu_{\alpha,k;q}(a + dp^N\mathbb{Z}_p)$ is bounded. But by Theorem 3.3, we have

$$\begin{aligned}
|\mu_{\alpha,k;q}(a+dp^N\mathbb{Z}_p)|_p &= \left|\frac{xp^N}{d_k} + ka^{k-1}\mu_{\alpha,1;q}(a+dp^N\mathbb{Z}_p)\right|_p \\
&\quad (\text{ for some } x \in \mathbb{Z}) \\
&\le \max\left\{\left|\frac{xp^N}{d_k}\right|_p, |ka^{k-1}\mu_{\alpha,1;q}(a+dp^N\mathbb{Z}_p)|_p\right\} \\
&\le \max\left\{\left|\frac{1}{d_k}\right|_p, |ka^{k-1}\mu_{\alpha,1;q}(a+dp^N\mathbb{Z}_p)|_p\right\} \\
&< \infty,
\end{aligned}$$

since d_k is fixed.

COROLLARY 3.6. *Let $f : X \to X$ be the function $f(x) = x^{k-1}$, k is a fixed positive integer. Then for all compact-open $U \subset X$,*

$$\int_U 1\, d\mu_{\alpha,k;q}(x) = k\int_U f\, d\mu_{\alpha,1;q}(x).$$

Proof. It follows from Theorem 3.3.

Let χ be a primitive Dirichlet character with conductor d, where d is a positive integer. Define

$$\beta_{m,\chi}(q) = \sum_{a=0}^{d-1} q^a \chi(a) d^{m-1} \beta_m \left(\frac{a}{d}; q^d\right). \tag{3.4}$$

We express the q-Bernoulli numbers as an integral over X, by using the distribution $\mu_{\beta,k}(x)$.

PROPOSITION 3.7.

(1) $\int_X \chi(x)\, d\mu_{\beta,k}(x) = \beta_{k,\chi}(q)$.
(2) $\int_{pX} \chi(x)\, d\mu_{\beta,k}(x) = \chi(p)p^{k-1}\beta_{k,\chi}(q^p)$.
(3) $\int_X \chi(x)\, d\mu_{\beta,k}(\alpha x) = \chi\left(\frac{1}{\alpha}\right)\beta_{k,\chi}(q)$.
(4) $\int_{pX} \chi(x)\, d\mu_{\beta,k}(\alpha x) = \chi\left(\frac{p}{\alpha}\right)p^{k-1}\beta_{k,\chi}(q^p)$.

Proof. It follows immediately from (3.4) and Lemma 3.1.

¿From the definition of $\mu_{\alpha,k;q}$ in (3.3),

$$\int_{X^*} \chi(x)\, d\mu_{\alpha,k,q}(x) = \int_{X^*} \chi(x)\, d\mu_{\beta,k}(x) - \alpha^{-k} \int_{X^*} \chi(x)\, d\mu_{\beta,k}(\alpha x).$$

Now, using Proposition 3.7,

$$\begin{aligned} \int_{X^*} \chi(x)\, d\mu_{\beta,k}(x) &= \int_X \chi(x)\, d\mu_{\beta,k}(x) - \int_{pX} \chi(x)\, d\mu_{\beta,k}(x) \\ &= \beta_{k,\chi}(q) - \chi(p)p^{k-1}\beta_{k,\chi}(q^p) \end{aligned} \tag{3.5}$$

and

$$\int_{X^*} \chi(x)\, d\mu_{\beta,k}(\alpha x) = \chi\left(\frac{1}{\alpha}\right)\beta_{k,\chi}(q) - \chi\left(\frac{p}{\alpha}\right)p^{k-1}\beta_{k,\chi}(q^p). \tag{3.6}$$

By the above results (3.5) and (3.6), we can find that

$$\int_{X^*} \chi(x)\, d\mu_{\alpha,k;q}(x) \\ = \left(1-\alpha^{-k}\chi\left(\frac{1}{\alpha}\right)\right)(\beta_{k,\chi}(q)-\chi(p)p^{k-1}\beta_{k,\chi}(q^p)). \tag{3.7}$$

Finally, we define

$$< x >:= \frac{x}{\omega(x)},$$

where ω is the first kind Teichmüller character $< x >^{p^n} \equiv 1 \pmod{p^n}$. Put $\chi_k = \chi\omega^{-k}$. By Corollary 3.6, we have

$$\int_{X^*} \chi_k(x)\, d\mu_{\alpha,k;q}(x) = \int_{X^*} \chi_k(x) k x^{k-1}\, d\mu_{\alpha,1;q}(x) \\ = \int_{X^*} \chi_1(x) < x >^{k-1} k\, d\mu_{\alpha,1;q}(x)$$

If $k_1 \equiv k_2 \pmod{p^N}$, then

$$\left(1-\alpha^{-k_1}\chi_{k_1}\left(\frac{1}{\alpha}\right)\right)(\beta_{k_1,\chi_{k_1}}(q)-\chi_{k_1}(p)p^{k_1-1}\beta_{k_1,\chi_{k_1}}(q^p)) \\ = \int_{X^*} \chi_{k_1}(x)\, d\mu_{\alpha,k_1;q}(x) \\ = \int_{X^*} \chi_1(x) < x >^{k_1-1} k_1\, d\mu_{\alpha,1;q}(x) \\ \equiv \int_{X^*} \chi_1(x) < x >^{k_2-1} k_2\, d\mu_{\alpha,1;q}(x) \pmod{p^N} \\ = \int_{X^*} \chi_{k_2}(x)\, d\mu_{\alpha,k_2,q}(x) \\ = \left(1-\alpha^{-k_2}\chi_{k_2}\left(\frac{1}{\alpha}\right)\right)(\beta_{k_2,\chi_{k_2}}(q)-\chi_{k_2}(p)p^{k_2-1}\beta_{k_2,\chi_{k_2}}(q^p)).$$

Therefore, we obtain the following theorems.

THEOREM 3.8 (KUMMER CONGRUENCE FOR THE q-BERNOULLI NUMBERS). *If $k_1 \equiv k_2 \pmod{p^N}$, then for any $\alpha \in X^*, \alpha \neq 1$,.*

$$\left(1-\alpha^{-k_1}\chi_{k_1}\left(\frac{1}{\alpha}\right)\right)(\beta_{k_1,\chi_{k_1}}(q)-\chi_{k_1}(p)p^{k_1-1}\beta_{k_1,\chi_{k_1}}(q^p))$$
$$\equiv\left(1-\alpha^{-k_2}\chi_{k_2}\left(\frac{1}{\alpha}\right)\right)(\beta_{k_2,\chi_{k_2}}(q)-\chi_{k_2}(p)p^{k_2-1}\beta_{k_2,\chi_{k_2}}(q^p)) \pmod{p^N}.$$

THEOREM 3.9 (p-ADIC q-L-FUNCTION). *For $\alpha \in X^*, \alpha \neq 1$, the p-adic q-L-function*

$$L_{p;q}(s,\chi) \stackrel{\text{def}}{=} \frac{1}{s-1}\int_{X^*} < x >^{-s} \chi_1(x)(1-s)\, d\mu_{\alpha,1;q}(x), \quad s \in \mathbb{Z}_p,$$

interpolates the values

$$-\frac{1}{k}\left(1-\alpha^{-k}\chi_k\left(\frac{1}{\alpha}\right)\right)(\beta_{k,\chi_k}(q)-\chi_k(p)p^{k-1}\beta_{k,\chi_k}(q^p))$$

when $s = 1-k$.

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